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# ON THE SIMPLICITY OF SOME CUNTZ-PIMSNER ALGEBRAS

PAUL S. MUHLY* and BARUCH SOLEL†

**Abstract.**

Conditions are given on a $C^*$-correspondence $E$ over a $C^*$-algebra that guarentee that the associated Cuntz-Pimsner algebra $\mathcal{O}(E)$ is simple. Our findings generalize earlier results of Cuntz [5], Paschke [14], and Boyd, Keswani, and Raeburn [2].

## 1. Introduction.

Throughout this note, $A$ will denote a fixed $C^*$-algebra and $E$ will denote a $C^*$-correspondence over $A$. This means, first of all, that $E$ is a (right) Hilbert module over $A$ in the sense of Paschke [13] and Rieffel [17]. We shall follow the toolkit of Lance [10] for notation and terminology about Hilbert $C^*$-modules and we shall write the $A$-valued inner product on $E$ as $\langle \cdot, \cdot \rangle$. Also, we shall write $\mathscr{L}(E)$ for the $C^*$-algebra of bounded, linear, *adjointable* operators on $E$ and we shall view $E$ as a left module over $\mathscr{L}(E)$. To say that $E$ is a *correspondence* over $A$, or a *correspondence from $A$ to $A$*, means that in addition to $E$ being a Hilbert $C^*$-module over $A$, $E$ is an essential left module over $A$ with the action given by elements from $\mathscr{L}(E)$. We shall emphasize this by saying that we are given a homomorphism $\varphi : A \mapsto \mathscr{L}(E)$ and make $\varphi$ explicit in our formulas. In particular, we shall write $\varphi(a)\xi$ for what otherwise would be written simply as $a\xi$.

Given a $C^*$-correspondence $E$ over $A$, one can build an analogue of Fock space over $E$, $\mathscr{F}(E)$. From this, one may build an analogue, $\mathscr{T}(E)$, of the Toeplitz algebra and take a quotient to obtain the Cuntz-Pimsner algebra $\mathcal{O}(E)$, which is a generalization of the well known Cuntz algebras $\mathcal{O}_n$, $n \geq 1$. The details are spelled out in [16] and in our paper [11]. However, we recall

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here the basics in order to establish notation and to expose results that we shall need.

From $E$ one can form the $n$-fold tensor product *over* $A$ of $E$ with itself, obtaining, for each non-negative integer $n$, a new correspondence $E^{\otimes n}$. The correspondence $E^{\otimes 0}$ is just $A$ with its usual right Hilbert $A$-module structure and left action given by left multiplication. We write $\varphi_0(a)b = ab$. Of course $E^{\otimes 1}$ is just $E$, and $\varphi_1 := \varphi$. For $E^{\otimes 2}$ the inner product is given by the formula

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_2, \varphi(\langle \xi_1, \eta_1 \rangle)\eta_2 \rangle,$$

the right action is given by the obvious formula, as is the left action, but we denote it by $\varphi_2$. Thus $\varphi_2(a)(\xi_1 \otimes \xi_2) = (\varphi(a)\xi_1) \otimes \xi_2$. The correspondence structure on $E^{\otimes n}$ is defined inductively and we write $\varphi_n$ for the left action of $A$ on $E^{\otimes n}$. The direct sum of the $E^{\otimes n}$, $n = 0, 1, 2 \ldots$, denoted $\mathscr{F}(E)$, and endowed with the direct sum structure of (right) Hilbert $C^*$-modules, is a Hilbert $C^*$-module, of course, and it is a correspondence over $A$ determined by the map $\varphi_\infty : A \mapsto \mathscr{L}(\mathscr{F}(E))$, where $\varphi_\infty(a) = \mathrm{diag}(\varphi_0(a), \varphi_1(a), \ldots)$.

For $\xi \in E$, we define the *creation operator* $T_\xi$ on $\mathscr{F}(E)$ by the formula

$$T_\xi \eta = \xi \otimes \eta,$$

$\eta \in \mathscr{F}(E)$. It is not hard to see that $T_\xi \in \mathscr{L}(\mathscr{F}(E))$ with norm dominated by the norm of $\xi$ in $E$. We view $T_\xi$ matricially as

$$T_\xi = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots \\ T_\xi^{(1)} & 0 & 0 & 0 & \ddots & \cdots \\ 0 & T_\xi^{(2)} & 0 & 0 & \ddots & \cdots \\ 0 & 0 & T_\xi^{(3)} & 0 & \ddots & \cdots \\ \vdots & 0 & 0 & T_\xi^{(4)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

where $T_\xi^{(n)} : E^{\otimes n} \mapsto E^{\otimes(n+1)}$ is defined by the same formula as $T_\xi$ except that $\eta$ is constrained to lie in $E^{\otimes n}$. The adjoint of $T_\xi$ is given on vectors of the form $\zeta \otimes \eta$, $\zeta \in E$, $\eta \in \mathscr{F}(E)$, by the formula

$$T_\xi^* \zeta \otimes \eta = \varphi_\infty(\langle \xi, \zeta \rangle)\eta,$$

with a matricial representation

$$T_\xi^* = \begin{bmatrix} 0 & (T_\xi^{(1)})^* & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & (T_\xi^{(2)})^* & 0 & \ddots & \cdots \\ 0 & 0 & 0 & (T_\xi^{(3)})^* & \ddots & \cdots \\ 0 & 0 & 0 & 0 & (T_\xi^{(4)})^* & \cdots \\ \vdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

where again, the $(T_\xi^{(n)})^* : E^{\otimes(n+1)} \mapsto E^{\otimes n}$ are given formally by the same rule as $T_\xi^*$ but with $\varphi_\infty$ replaced by $\varphi_n$.

The $C^*$-algebra generated by $\varphi_\infty(A)$ and all the creation operators $T_\xi$, $\xi \in E$, is called the *Toeplitz algebra* associated to $E$ and is denoted by $\mathscr{T}(E)$. Our $\mathscr{T}(E)$ is what Pimsner would call $\widetilde{\mathscr{T}}(E)$. Ours agrees with his when the span of the inner products $\langle \xi, \eta \rangle$, $\xi, \eta \in E$, is dense in $A$. That is, when this happens, then $\mathscr{T}(E)$ is the $C^*$-algebra generated by all the $T_\xi$. The terminology comes from the fact that when $E = A = C$, with the evident structures, then $\mathscr{T}(E)$ *is* the $C^*$-algebra generated by the unilateral shift, i.e., by all the Toeplitz operators with continuous symbols.

The *Cuntz-Pimsner algebra*, $\mathcal{O}(E)$, *is* defined to be the image of $\mathscr{T}(E)$ in a certain corona algebra. In the setting of this note, where we assume that as a Hilbert $C^*-$module over $A$, $E$ is a direct summand of $C_n(A)$[1], with *n finite*, and where we assume that $\varphi$ is injective, it turns out, thanks to Theorems 3.12 and 3.13 in [16], that $\mathscr{T}(E)$ contains the full algebra of all compact operators on the Fock space, $\mathscr{K}(\mathscr{F}(E))$, as an ideal and the quotient of $\mathscr{T}(E)$ by this ideal is $\mathcal{O}(E)$. (Again, we note that what we are calling $\mathcal{O}(E)$, Pimsner would call $\widetilde{\mathcal{O}}(E)$.) When $A = C$, and $E = C^n$, then $\mathcal{O}(E)$ is the Cuntz algebra $\mathcal{O}_n$. This explains notation and the terminology.

We write $S_\xi$ for the image of the generator $T_\xi$ in $\mathcal{O}(E)$ and we $\varphi_\infty$ for the representation of $A$ in $\mathscr{T}(E)$ or in $\mathcal{O}(E)$. The ambiguity should not cause a problem in context.

Our objective is to give conditions under which $\mathcal{O}(E)$ is simple. We present two theorems giving this conclusion. They are variations of one another and both generalize known theorems in the literature. They are based on the following hypotheses that we will invoke at various stages.

**(H1)** We shall assume that $A$ is unital and strongly amenable in the sense of Johnson [8].

[1] $C_n(A)$ is *column n-space over A* and is defined to be the collection of all $n$-tuples, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, with entries from $A$ endowed with the $A$-vlaued inner product $\langle \xi, \eta \rangle := \sum \xi_i^* \eta_i$.

This hypothesis is made to guarantee that $A$ has a faithful tracial state [4, Corollary 1] and for related purposes. We note also, for emphasis, that since $E$ is assumed to be an essential left module over $A$, we are in fact assuming that $\varphi$ is unital.

**(H2)** We shall assume that $E$ is a summand of $C_n(A)$ for some finite $n$, and we shall write the projection from $C_n(A)$ to $E$ by $P$.

**(H3)** We shall assume that the map $\varphi : A \mapsto \mathscr{L}(E)$ giving the left module structure on $E$ is isometric.

Of course, we have mentioned hypotheses (H2) and (H3) before. Hypothesis (H3) is made, really, to ensure that the algebra $\mathcal{O}(E)$ is nonzero. Hypothesis (H2) is made so that the next definition makes sense and is applicable. First note that since $E = PC_n(A)$, the algebra $\mathscr{L}(E)$ is $PM_n(A)P$, and so we may view $\varphi$ as a homomorphism from $A$ into $M_n(A)$ such that $P\varphi(a) = \varphi(a)P = \varphi(a)$ for all $a \in A$. Let tr denote the formal trace on $M_n(A)$, so that $\mathrm{tr}(a_{ij}) = \sum a_{ii}$. Of course, unless $A$ is abelian, tr is not really a trace on $M_n(A)$. Nor for that matter is *tr* tracial; i.e., in general $\mathrm{tr}(ab) \neq \mathrm{tr}(ba)$. However, the composition of t*r* with any trace on $A$ *is* a trace on $M_n(A)$. We form the map $\Omega$ on $A$ via the formula:

$$\Omega(a) = \mathrm{tr}(P\varphi(a)P) = \mathrm{tr}(\varphi(a)),$$

$a \in A$. Then $\Omega$ is the composition of completely positive maps and therefore is completely positive. Note, however, that $\Omega$ is *not* contractive in general. The next two hypotheses are concerned with the properties of $\Omega$.

**(H4)** We assume that in $A$ there are no nontrivial $\Omega$-invariant ideals.

**(H5)** We assume that $\Omega$ is *non-unital*, i.e., we assume that $\Omega(1_A) \neq 1_A$.

Our first principal result in this note, then, is

THEOREM 1. *If the $C^*$-algebra $A$ and the correspondence $E$ satisfy hypotheses* (H1)–(H5)*, then $\mathcal{O}(E)$ is simple.*

Observe that in the most ellementary, nontrivial case, when $A = \mathbf{C}$ and $E = \mathbf{C}^n$, with $n \geq 2$, it is easy to check that all the hypotheses are satisfied and, indeed, we see that $\Omega(1_A) = n \cdot 1_A \neq 1_A$! Thus, we arrive at what appears to be a curious proof that $\mathcal{O}_n$ is simple when $n \geq 2$.

As a second example, we note that the main theorems of [14] and [2] are easy corollaries of this theorem. (Definitions and details will given in the third section.)

COROLLARY 2 [2, Corollary 2.6]. *If $A$ is a strongly amenable, unital $C^*$-algebra and if $\alpha$ is an endomorphism of $A$ such that $\alpha(1_A) \neq 1_A$ and such that $\alpha$*

*leaves invariant no nontrivial ideals in A, then the Stacey crossed product of order* 1,

$$A \rtimes_{\alpha}^{1} \mathsf{N},$$

*is simple.*

For a third example, we show how certain correspondences arising in index theory give rise to simple $C^*$-algebras of the kind we are considering. Again, a fuller discussion and proofs will be given in the third section.

COROLLARY 3. *Suppose $A$ is a unital, strongly amenable $C^*$-algebra contained properly in a $C^*$-algebra $B$, and suppose that $A$ is the range of an index finite conditional expectation $\Phi : B \mapsto A$ that preserves a faithful trace $\tau$, say, on $B$,* i.e.*, $\tau \circ \Phi = \tau$*[2]*. If $A$ is simple, and if $B$ is viewed as a correspondence over $A$, with the left action of $A$ given by left multiplication and $A$-valued inner product given by the formula $\langle x, y\rangle = \Phi(x^* y)$, $x, y \in B$, then $\mathcal{O}(B)$ is simple.*

The proof of Theorem 1 follows an outline first suggested by Cuntz in [5], and refined by Paschke in [14] and Boyd, Keswani, and Raeburn in [2]. However, the details are not trivial and care must be taken. They will occupy most of the next section.

As already indicated, the fifth hypothesis may seem a bit odd at first glance. It is satisfied, however, in a number of interesting circumstances. A replacement for it that proves to be very effective in those cases when $\Omega$ is unital is:

**Condition F** We say that a correspondence $E$ satisfies *condition F* in case for each $n \geq 1$, the *generalized commutator subspace* $C_n$ of $E^{\otimes n}$,

$$C_n := \overline{\mathrm{span}}\{\varphi_n(a)\xi - \xi a \mid \xi \in E^{\otimes n},\ a \in A\},$$

is all of $E^{\otimes n}$.

The reason for calling this Condition F is that it is a generalization of the notion of *free action* in topological dynamics. Indeed, we have

PROPOSITION 4. *Suppose $X$ is a compact space and that $\tau$ is a homeomorphism of $X$. Let $A$ be $C(X)$, let $E$ be $C(X)$, too, endowed with the usual structure of a right Hilbert $C^*$−module over $A$, and let the left action of $A$ on $E$ be determined by $\tau$ through the formula*

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[2] We do not know if this hypothesis is redundant. That is, if $A$ is strongly amenable and $\Phi : B \mapsto A$ is an index finite conditional expectation, then quite possibly there may be a trace on $B$ of the desired kind. Quite possibly $B$ is automatically strongly amenable, in which case, the existance of $\tau$ is not difficult to show. In any case, the hypothesis seems to be satisfied in numerous interesting examples.

$$\varphi(f)g := (f \circ \tau)g$$

*$f \in A$, $g \in E$. Then the correspondence $E$ satisfies condition F if and only if the homeomorphism $\tau$ determines a free action of the integers in the usual sense; i.e., if and only if, $\tau$ has no periodic points.*

In fact, when the correspondence $E$ is $A$, with the left action of $A$ on $A$ given by an automorphism $\varphi$, then Condition F is equivalent to Kallman's notion of free action [9], considered in the second dual of $A$. We will prove this in Theorem 34 in Section 3.

The second principal result of this note, then, is

THEOREM 5. *If the correspondence $E$ satisfies the hypotheses* (H1)–(H4)*, but in place of* (H5)*, it satisfies Condition F, then $\mathcal{O}(E)$ is simple.*

Of course, appealing to Proposition 2 and the fact that when $A = E = C(X)$ and $\varphi$ is given by $\tau$ then $\mathcal{O}(E)$ is the transformation group $C^*$-algebra $C^*(X, \mathsf{Z})$ determined by $\tau$ [16, Examples in Section 1], we re-arrive at the well known fact that $C^*(X, \mathsf{Z})$ is simple if (and only if) the action of $\mathsf{Z}$ on $X$ is free and minimal.

Notice that in some sense, Condition F and hypothesis (H4) seem rather far apart. In the case of the correspondence $\mathsf{C}^n$ from $\mathsf{C}$ to $M_n(\mathsf{C})$, where we get $\mathcal{O}_n$, the commutator subspaces all reduce to zero. Nevertheless, with a very minor alteration, the proof that we offer for Theorem 1 serves as well for Theorem 5.

We shall prove Theorems 1 and 5 in the next section. In Section 3, we prove Corollaries 2 and 3, Proposition 4, and Theorem 15 relating Condition F to Kallman's notion of free action. Additional results will be found there, too.

## 2. Proofs of Theorems 1 and 5.

We break the proofs into a series of lemmas. We prove Theorem 1 first and then show how to modify the lemmas leading to it in order to prove Theorem 5. Actually, only one part of one lemma needs to be modified. Our correspondence $E$ and $C^*$-algebra $A$ will be fixed throughout the discussion. We shall assume that hypotheses (H1)–(H4) are satisfied. Until the proof of Theorem 1 is complete, we shall assume that hypothesis (H5) is satisfied.

LEMMA 6. *There is a faithful tracial state $\tau_0$ on $A$ such that*

$$\tau_0(a)\tau_0(\Omega(1)) = \tau_0(\Omega(a))* \qquad (*)$$

*for all $a \in A$. In fact, any tracial state satisfying this equation is necessarily faithful under our standing hypotheses.*

Proof. The hypothesis (H1) is that $A$ is unital and strongly amenable. Therefore $A$ has a tracial state by [4]. For any such state $\tau$, the functional $\tau \circ \Omega$ is positive. Therefore, if $\tau \circ \Omega(1) = 0$, then $\tau \circ \Omega = 0$. This implies that $\Omega$ maps $A$ into $N_\tau := \{a \in A | \tau(a^* a) = 0\}$ - a *two-sided* closed ideal, since $\tau$ is a tracial state. In particular, $N_\tau$ is invariant under $\Omega$. Since $\Omega(1) \in N_\tau$, we conclude from hypothesis (H4) that $N_\tau = A$. However, since $A$ is unital, we conclude then that $1 = \tau(1) = 0$, a contradiction. Thus, *for every* tracial state $\tau$, $\tau(\Omega(1)) \neq 0$. Given such a state, $\tau$, we define $\tau'$ by the formula

$$\tau'(a) = \frac{1}{\tau(\Omega(1))} \tau(\Omega(a)),$$

$a \in A$. Then $\tau'$ is also a state. It is, in fact, tracial since $\tau \circ tr$ is a tracial linear functional on $M_n(A)$ and $\varphi$ is a homomorphism. Since the map $\tau \mapsto \tau'$ is affine and continuous in the weak-* topology on the state space of $A$, we conclude from the Schauder fixed point theorem that there is a tracial state $\tau_0$ with $\tau_0 = \tau_0'$. Rewriting this equation yields equation (1). To show that any such state $\tau_0$ is necessarily faithful under our hypotheses, it suffices to show that $N_{\tau_0}$ is $\Omega-$invariant. But if $a \in N_{\tau_0}$, so that $\tau_0(a^* a) = 0$, equation (1) implies that $\tau_0(\Omega(a^* a)) = 0$. Since, however, $\Omega$ is completely positive, the Cauchy-Schwarz-Kadison inequality holds: $\Omega(a)^* \Omega(a) \leq \|\Omega(1)\| \Omega(a^* a)$. Thus, $\tau_0(\Omega(a)^* \Omega(a)) = 0$ and we conclude that $\Omega(a) \in N_{\tau_0}$.

To show $\mathcal{O}(E)$ is simple, it suffices to show that every representation of it is faithful. For this, we fix a (unital) representation $\rho$ of $\mathcal{O}(E)$ on a Hilbert space $H_\rho$ and we let $(V, \sigma)$ be the (necessarily unique) covariant representation of $E$ on $H_\rho$ such that $\rho = V \times \sigma$. That is, $\sigma$ is a unital representation of $A$ on $H_\rho$ and $V$ is a linear map from $E$ to $H_\rho$ such that

1. $V(\xi a) = V(\xi)\sigma(a)$ and $V(\varphi(a)\xi) = \sigma(a)V(\xi)$, $a \in A$, $\xi \in E$;
2. $V(\xi)^* V(\eta) = \sigma(\langle \xi, \eta \rangle)$, $\xi, \eta \in E$; and
3. $\sigma(a) = \sigma^{(1)} \circ \varphi(a)$, where $\sigma^{(1)} : K(E) \mapsto B(H_\rho)$ is defined by the formula $\sigma^{(1)}(\xi \otimes \eta^*) = V(\xi) V(\eta)^*$.

The relation between $\rho$ and $(V, \sigma)$ is: $V(\xi) = \rho(S_\xi)$ and $\sigma(a) = \rho(\varphi_\infty(a))$, where, recall, $S_\xi$ is the image of $T_\xi$ in $\mathcal{O}(E)$. We note that every representation of $\mathcal{T}(E)$ is determined by a covariant representation of $E$ and that such a representation passes to $\mathcal{O}(E)$ precisely when $(V, \sigma)$ satisfies condition (3). (For these things see [16], in particular Theorem 3.12, and [11].)

We write $\alpha$ for the map on $\rho(\mathcal{O}(E))$ defined by the formula

$$\alpha(x) = \sum_i V(P\varepsilon_i)^* x V(P\varepsilon_i),$$

$x \in \rho(\mathcal{O}(E))$, where the $\varepsilon_i$ denote the unit basis vectors in $C_n(A)$. Then $\sigma \circ \Omega = \alpha \circ \sigma$ because

$$\begin{aligned}\sigma(\Omega(a)) &= \sigma(\mathrm{tr}(P\varphi(a)P)) = \sum_i \sigma(\langle P\varepsilon_i, \varphi(a)P\varepsilon_i\rangle) \\ &= \sum_i V(P\varepsilon_i)^* V(\varphi(a)P\varepsilon_i) = \sum_i V(P\varepsilon_i)^* \sigma(a) V(P\varepsilon_i) \\ &= \alpha(\sigma(a)),\end{aligned}$$

$a \in A$.

Lemma 7. *The representation $\sigma$ of $A$ is injective.*

Proof. For this, we need only show that the kernel of $\sigma$ is invariant under $\Omega$. But if $\sigma(a) = 0$, then $\sigma(\Omega(a)) = \alpha(\sigma(a)) = 0$.

Lemma 8. *There is a state $f_0$ on $\rho(\mathcal{O}(E))$ such that*

(a) *The restriction of $f_0$ to $\sigma(A)$ is faithful;*
(b) $f_0(\sigma(a)x) = f_0(x\sigma(a))$, *for all $a \in A$ and $x \in \rho(\mathcal{O}(E))$;*
(c) $f_0(\alpha(x)) = \tau_0(\Omega(1))f_0(x)$, $x \in \rho(\mathcal{O}(E))$; *and*
(d) *for $\xi_i, \eta_i \in E$, $a \in A$,*

$$f_0(V(\xi_1)V(\xi_2)\ldots V(\xi_k)\sigma(a)V(\eta_1)^*V(\eta_2)^*\ldots V(\eta_m)^*) = 0,$$

*whenever $m \neq k$.*

Proof. By Lemma 7, $\sigma$ is injective and so we may set $\tau_1 := \tau_0 \circ \sigma^{-1}$ on $\sigma(A)$, obtaining a faithful tracial state on $\sigma(A)$ satisfying the equation $\tau_1(\sigma(a))\tau_1(\sigma(\Omega(1))) = \tau_1(\sigma(\Omega(a)))$. Observe that we may also write $\sigma(\Omega(a))$ as $\sigma(\mathrm{tr}(P\varphi(a)P) = \mathrm{tr}(\sigma_n(P\varphi(a)P))$ where $\sigma_n$ is the map on $M_n(A)$ to $M_n(B(H))$ obtained by applying $\sigma$ to the entries of the elements in $M_n(A)$ and where the second use of tr, on $M_n(B(H))$ refers to the process of adding up the diagonal terms of matrices in $M_n(B(H))$.

Pick a state $g$ on $\rho(\mathcal{O}(E))$ that extends $\tau_1$ from $\sigma(A)$ and note that for every unitary $u$ in $\sigma(A)$ the state $u \cdot g \cdot u^*$ on $\rho(\mathcal{O}(E))$, defined by the formula $u \cdot g \cdot u^*(x) = g(u^*xu)$ also extends $\tau_1$. Since $A$ is strongly amenable, so is $\sigma(A)$, and there is a state $f$ in the weak-* closed convex hull of $\{u \cdot g \cdot u^* | u \in \mathcal{U}(\sigma(A))\}$ which *centralizes* $\sigma(A)$; i.e., which satisfies the equation $f(\sigma(a)x) = f(x\sigma(a))$, $x \in \rho(\mathcal{O}(E))$, $a \in A$. Write $K$ for the (necessarily weak-* compact) set of all such states on $\rho(\mathcal{O}(E))$. For $f \in K$, define $f'$ by the formula

$$f'(x) = \tau_0(\Omega(1))^{-1} f(\alpha(x)),$$

$x \in \rho(\mathcal{O}(E))$. We show that $f'$ is also in $K$.

First of all, $f'$ is clearly a positive functional since $\alpha$ is completely positive. Furthermore, $f'(1) = \tau_0(\Omega(1))^{-1} f(\alpha(1)) = \tau_0(\Omega(1))^{-1} f(\sigma(\Omega(1))) = 1$, so $f'$ is a state. So to show that $f'$ is in $K$, we need only show that $f \circ \alpha$ centralizes $\sigma(A)$. Now for $a \in A$ and $x \in \rho(\mathcal{O}(E))$, we have

$$\begin{aligned} f \circ \alpha(\sigma(a)x) &= f(\alpha(\sigma(a)x)) = \sum_i f(V(P\varepsilon_i)^* \sigma(a) x V(P\varepsilon_i)) \\ &= \sum_i f(V(\varphi(a^*)P\varepsilon_i)^* x V(P\varepsilon_i)). \end{aligned}$$

However, $\varphi(a^*)P\varepsilon_i = \sum \varepsilon_j \langle \varepsilon_j, \varphi(a^*)P\varepsilon_i \rangle = \sum P\varepsilon_j \langle \varepsilon_j, \varphi(a^*)P\varepsilon_i \rangle$, and so

$$V(\varphi(a^*)P\varepsilon_i)^* = \sum_j \sigma(\langle \varphi(a^*)P\varepsilon_i, P\varepsilon_j \rangle) V(P\varepsilon_j)^*.$$

Inserting this into the last sum for $f \circ \alpha(\sigma(a)x)$, we conclude that

$$f \circ \alpha(\sigma(a)x) = \sum_{i,j} f(\sigma(\langle \varphi(a^*)P\varepsilon_i, P\varepsilon_j \rangle) V(P\varepsilon_j)^* x V(P\varepsilon_i)).$$

Since $f$ centralizes $\sigma(A)$, we may write

$$\begin{aligned} f \circ \alpha(\sigma(a)x) &= \sum_{i,j} f(V(P\varepsilon_j)^* x V(P\varepsilon_i) \sigma(\langle \varphi(a^*)P\varepsilon_i, P\varepsilon_j \rangle)) \\ &= \sum_{i,j} f(V(P\varepsilon_j)^* x V(P\varepsilon_i) \sigma(\langle P\varepsilon_i, \varphi(a)P\varepsilon_j \rangle)) \\ &= \sum_j f(V(P\varepsilon_j)^* x V(\sum_i P\varepsilon_i \sigma(\langle P\varepsilon_i, \varphi(a)P\varepsilon_j \rangle))) \\ &= \sum_j f(V(P\varepsilon_j)^* x V(\varphi(a)P\varepsilon_j)) \\ &= \sum_j f(V(P\varepsilon_j)^* x \sigma(a) V(P\varepsilon_j)) = f \circ \alpha(x\sigma(a)), \end{aligned}$$

showing that $f \circ \alpha$ centralizes $\sigma(A)$.

Since the map $f \mapsto f'$ is continuous and affine on the weak-* compact, convex set $K$, we may apply Schauder's fixed point theorem again to conclude that there is a state $f_0 \in K$ such that $f_0 = f_0'$; i.e., such that $f_0(x)\tau_0(\Omega(1)) = f_0(\alpha(x))$. Thus $f_0$ satisfies (b) and (c) in the statement of the lemma. It also satisfies (a), since $f_0$ extends $\tau_1$. Thus it remains to prove (d).

For this, first calculate $\alpha^m$ to get

$$\alpha^m(x) = \sum_{1 \le i_1, \ldots, i_m \le n} V(P\varepsilon_{i_1})^* \ldots V(P\varepsilon_{i_m})^* x V(P\varepsilon_{i_m}) \ldots V(P\varepsilon_{i_1}),$$

$x \in \rho(\mathcal{O}(E))$. We will show next that for $y \in \rho(\mathcal{O}(E))$, and $\xi_1, \ldots, \xi_m \in E$,

$$f_0(\alpha^m(yV(\xi_1)^* \cdots V(\xi_m)^*)) = f_0(V(\xi_1)^* \cdots V(\xi_m)^* y). \tag{1}$$

Write $x = yV(\xi_1)^* \cdots V(\xi_m)^*$ and note that

$$V(\xi_1)^* \cdots V(\xi_m)^* V(P\varepsilon_{i_m}) \ldots V(P\varepsilon_{i_1})$$

lies in $\sigma(A)$. Then, since $f_0$ centralizes $\sigma(A)$, we may write

$$\begin{aligned} &f_0(V(P\varepsilon_{i_1})^* \ldots V(P\varepsilon_{i_m})^* x V(P\varepsilon_{i_m}) \ldots V(P\varepsilon_{i_1})) \\ &= f_0(V(\xi_1)^* \cdots V(\xi_m)^* V(P\varepsilon_{i_m}) \ldots V(P\varepsilon_{i_1}) V(P\varepsilon_{i_1})^* \ldots V(P\varepsilon_{i_m})^* y). \end{aligned}$$

Now note that our hypotheses on $E$ imply that $\mathcal{L}(E) = \mathcal{K}(E)$. Recall also from [16, Lemma 3.2] that $\mathcal{K}(E)$ is embedded in $\mathcal{O}(E)$ as the span of the elements $S_\xi S_\eta^*$, $\xi, \eta \in E$ and these are mapped by $\rho$ to $V(\xi)V(\eta)^*$; this is the map that Pimsner denotes by $\sigma^{(1)}$. Since $\rho$ is a representation of $\mathcal{O}(E)$, and $\varphi(A) \subseteq \mathcal{K}(E)$, we have $\sigma = \sigma^{(1)} \circ \varphi$, as we noted above. Finally, since $\sum_i P\varepsilon_i \otimes (P\varepsilon_i)^*$ is the identity operator on $E$, we deduce that $\sum_i V(P\varepsilon_i)V(P\varepsilon_i)^* = \sigma^{(1)}(\sum_i P\varepsilon_i \otimes (P\varepsilon_i)^*) = \sigma^{(1)} \circ \varphi(1_A) = \sigma(1_A) = I_{H_\rho}$. Thus we have

$$\begin{aligned} f_0(\alpha^m(x)) &= \sum_{1 \le i_1, \ldots, i_m \le n} f_0(V(P\varepsilon_{i_1})^* \ldots V(P\varepsilon_{i_m})^* x V(P\varepsilon_{i_m}) \ldots V(P\varepsilon_{i_1})) \\ &= \sum_{1 \le i_1, \ldots, i_m \le n} f_0(V(\xi_1)^* \cdots V(\xi_m)^* \times \\ &\qquad V(P\varepsilon_{i_m}) \ldots V(P\varepsilon_{i_1}) V(P\varepsilon_{i_1})^* \ldots V(P\varepsilon_{i_m})^* y) \\ &= f_0(V(\xi_1)^* \cdots V(\xi_m)^* y), \end{aligned}$$

proving equation (1).

Also, since $f_0$ is self-adjoint ($f_0(x^*) = \overline{f_0(x)}$), we find that for vectors $\eta_1, \ldots, \eta_k \in E$, and $z \in \rho(\mathcal{O}(E))$, we have

$$f_0(\alpha^k(V(\eta_1) \cdots V(\eta_k)z)) = f_0(zV(\eta_1) \cdots V(\eta_k)).$$

Since $\tau_0$ is faithful, so $\tau_0(\Omega(1)) \neq 0$, we conclude, that whenever $\xi_i, \eta_i \in E$, $a \in A$, and $m \neq k$, then,

$$
\begin{aligned}
& f_0(V(\xi_1)V(\xi_2)\dots V(\xi_k)\sigma(a)V(\eta_1)^*V(\eta_2)^*\dots V(\eta_m)^*) \\
&= \tau_0(\Omega(1))^{-m} f_0(\alpha^m(V(\xi_1)V(\xi_2)\dots V(\xi_k)\sigma(a)V(\eta_1)^*V(\eta_2)^*\dots V(\eta_m)^*)) \\
&= \tau_0(\Omega(1))^{-m} f_0(V(\eta_1)^*V(\eta_2)^*\dots V(\eta_m)^*V(\xi_1)V(\xi_2)\dots V(\xi_k a)) \\
&= \tau_0(\Omega(1))^{-m} f_0(\alpha^k(V(\xi_1)V(\xi_2)\dots V(\xi_k)\sigma(a)V(\eta_1)^*V(\eta_2)^*\dots V(\eta_m)^*)) \\
&= \tau_0(\Omega(1))^{k-m} f_0(V(\xi_1)V(\xi_2)\dots V(\xi_k)\sigma(a)V(\eta_1)^*V(\eta_2)^*\dots V(\eta_m)^*).
\end{aligned}
$$

Since $\tau_0(\Omega(1)) \neq 1$, either, we see that all the terms in this equation must be zero; i.e., we have proved (d).

Our next goal is to show that a state $f_0$ on $\rho(\mathcal{O}(E))$ satisfying the conclusions of Lemma 8 is faithful on all of $\rho(\mathcal{O}(E))$. This is accomplished in steps.

Lemma 9. *A state $f_0$ on $\rho(\mathcal{O}(E))$ satisfying the conclusions of Lemma 8 is faithful*[3] *on the subalgebra $D$ consisting of all sums of the form*

$$V(\eta_1)V(\eta_2)\cdots V(\eta_k)\sigma(a)V(\xi_1)^*V(\xi_2)^*\cdots V(\xi_k)^*,$$

*where $k \geq 0$, $a \in A$, and the $\eta_i$ and $\xi_i$ run over $E$.*

Proof. First note that for every element $x \in D$, there is a $k$, depending on $x$, such that $\alpha^k(x)$ lies in $\sigma(A)$. Indeed, it suffices to assume that $x$ has the form $x = V(\eta_1)V(\eta_2)\cdots V(\eta_k)\sigma(a)V(\xi_1)^*V(\xi_2)^*\cdots V(\xi_k)^*$. Then $V(P\varepsilon_{i_1})^*\dots V(P\varepsilon_{i_k})^* x V(P\varepsilon_{i_k})\dots V(P\varepsilon_{i_1})$ lies in $\sigma(A)$ for all choices of $i_1,\dots,i_k$. Since $\alpha^k(x)$ is the sum of all such expressions, we conclude that $\alpha^k(x)$ lies in $\sigma(A)$. Suppose $b \in D$ satisfies the equation $f_0(b^*b) = 0$ and choose $k$ so that $\alpha^k(b^*b) \in \sigma(A)$. Then, from Lemma 8 we see that $f_0(\alpha^k(b^*b)) = \tau_0(\Omega(1))^k f_0(b^*b) = 0$. Since $f_0|\sigma(A)$ is faithful, again by Lemma 8, we conclude that $\alpha^k(b^*b) = 0$. But $\alpha^k(b^*b) =$

$$\sum_{1\leq i_1,\dots,i_k\leq n} V(P\varepsilon_{i_1})^*\dots V(P\varepsilon_{i_k})^* b^*b V(P\varepsilon_{i_k})\dots V(P\varepsilon_{i_1}),$$

so this gives $bV(P\varepsilon_{i_k})\dots V(P\varepsilon_{i_1}) = 0$, for all choices of $i_1,\dots,i_k$. On the other hand, as we have seen in the proof of Lemma 8,

$$\sum_{1\leq i_1,\dots,i_k\leq n} V(P\varepsilon_{i_k})\dots V(P\varepsilon_{i_1})V(P\varepsilon_{i_1})^*\dots V(P\varepsilon_{i_k})^* = I_{H_\rho}.$$

Hence we conclude that

$$b = \sum_{1\leq i_1,\dots,i_k\leq n} bV(P\varepsilon_{i_k})\dots V(P\varepsilon_{i_1})V(P\varepsilon_{i_1})^*\dots V(P\varepsilon_{i_k})^* = 0.$$

[3] To say that a state $\omega$ on a $C^*$-algebra $\mathcal{A}$ is *faithful* on a not-necessarily closed *-subalgebra [$\mathcal{D}$ of $\mathcal{A}$ is to say that the equation $\omega(d^*d) = 0$, $d \in \mathcal{D}$, implies that $d = 0$.

We call the *degree* of an expression of the form

$$V(\eta_1)V(\eta_2)\cdots V(\eta_m)\sigma(a)V(\xi_1)^*V(\xi_2)^*\cdots V(\xi_n)*$$

the integer $m-n$. We extend this to linear combinations of such elements in the obvious way. When this is done, then, $D$ is precisely the collection of degree zero elements in the *-algebra generated by $\sigma(A)$ and the $V(\xi)$, $\xi\in E$. We write $D_k$ for the linear span of the set of all monomials of the form

$$V(\eta_1)V(\eta_2)\cdots V(\eta_m)\sigma(a)V(\xi_1)^*V(\xi_2)^*\cdots V(\xi_m)^*$$

with $m\leq k$, where the $\eta_i$ and $\xi_i$ range over $E$, and $a\in A$.

Lemma 10. *The space $D_k$ is a $C^*$-subalgebra of $D$.*

Proof. The only issue is whether $D_k$ is closed. However, as Pimsner notes in the discussion following Definition 3.8 in [16], $D_k$ is the image under $\sigma^{(k)}$ of the **-subalgebra* generated by all the $\xi\otimes\eta^*$, where $\xi$ and $\eta$ range over $E^{\otimes k}$. (Recall that $\sigma^{(k)}((\xi_1\otimes\ldots\otimes\xi_k)\otimes(\eta_1\otimes\ldots\eta_k)^*)=V(\xi_1)\cdots V(\xi_k)V(\eta_1)^*\cdots V(\eta_k)^*$, by definition.) Since $E$ is a summand of $C_n(A)$, and so finitely generated, the *-subalgebra coincides with all of $K(E^{\otimes k})$. Thus $D_k=\sigma^{(k)}(K(E^{\otimes k}))$ - a $C^*$-algebra.

Of course $D$ is the algebraic inductive limit of the $D_k$.

Following Pimsner, we write $\lambda$ for the one parameter group of automorphisms of $\mathcal{O}(E)$ defined by the formulae: $\lambda_t(\varphi_\infty(a))=\varphi_\infty(a)$, for all $a\in A$, and $\lambda_t(S_\xi)=tS_\xi$, for all $\xi\in E$, $t\in\mathsf{T}$. We also write $\Lambda$ for the conditional expectation $\int_{\mathsf{T}}\lambda_t\,dt$. Its range is the fixed point algebra of $\mathcal{O}(E)$, denoted $\mathcal{O}(E)^\lambda$.

Lemma 11. *The representation $\rho$ is faithful on $\mathcal{O}(E)^\lambda$.*

Proof. Write $C_k$ for the *-algebra in $\mathcal{O}(E)$ generated by the elements of the form

$$S_{\eta_1}\cdots S_{\eta_m}\varphi_\infty(a)S^*_{\xi_1}\cdots S^*_{\xi_m}$$

with $m\leq k$. As we noted in the proof of Lemma 10, $C_k$ is a $C^*$-subalgebra of $\mathcal{O}(E)^\lambda$ (with $\rho(C_k)=D_k$, of course.) Pimsner proves in the paragraph preceding Lemma 3.2 of [16] that, in fact $\mathcal{O}(E)^\lambda$ is the closure of $\cup C_k$. (Thus $D$ is dense in $\rho(\mathcal{O}(E)^\lambda)$.) Also, we define on $\mathcal{O}(E)$ the map $\tilde\alpha$ by the formula

$$\tilde\alpha(x)=\sum_i S^*_{P\varepsilon_i}xS_{P\varepsilon_i},$$

$x\in\mathcal{O}(E)$. Of course $\alpha\circ\rho=\rho\circ\tilde\alpha$. The argument in Lemma 4 shows that $\tilde\alpha^k(C_k)\subseteq\varphi_\infty(A)$ and so if $x$ is a non-negative element in $C_k$ with $\rho(x)=0$, then $\sigma(\tilde{\alpha}^k(x)) = \rho(\tilde{\alpha}^k(x)) = \alpha^k(\rho(x)) = 0$. Since $\sigma$ is faithful, $\tilde{\alpha}^k(x) = 0$. The argument used in Lemma 24, then, shows that $x = 0$. Thus $\rho$ is faithful on $C_k$. Since $\mathcal{O}(E)^\lambda$ is the closure of $\cup C_k$, we see that if $J = \ker \rho|\mathcal{O}(E)^\lambda$, then $J = \cup_k(J \cap C_k) = 0$ by [1, Lemma 1.3].

*Completion of the Proof of Theorem 1.*

With a state $f_0$ on $\rho(\mathcal{O}(E))$ of the form guaranteed by Lemma 8 fixed, let $(\pi, H, \zeta)$ be the usual GNS data associated with $f_0$ and $\rho(\mathcal{O}(E))$. Fix $k \geq 0$ and let $F$ be the projection of $H$ onto $\overline{\pi(D_k)\zeta}$. Then for $x \in \rho(\mathcal{O}(E))$, we have

$$\begin{aligned}\|F\pi(x)F\| &= \sup\{|\langle \pi(x)h, k\rangle \mid \|h\| = \|k\| = 1,\ h, k \in FH\}\\ &= \sup\{|f_0(b_2^* x b_1)|\, b_i \in D_k,\ f_0(b_i^* b_i) = 1,\ i = 1, 2\}.\end{aligned}$$

Consider, now, an $x = V(\eta_1)V(\eta_2)\cdots V(\eta_m)\sigma(a)V(\xi_1)^*V(\xi_2)^*\cdots V(\xi_l)^*$ in $\rho(\mathcal{O}(E))$ with $m \neq l$. Then for elements $b_i \in D_k$, $i = 1, 2$, $b_2^* x b_1$ is a finite sum of elements of this form with $m \neq l$. Hence, by Lemma 8, $f_0(b_2^* x b_1) = 0$, and we see that $F\pi(x)F = 0$. It follows that if we next let $x$ range over the *-algebra $\mathcal{B}$ generated by $\sigma(A)$ and $\{V(\xi) | \xi \in E\}$ and if we let $x_0$ denote its component in $D$,i.e., if $x_0$ is the sum of elements with zero degree, then

$$\|F\pi(x)F\| = \|F\pi(x_0)F\|.$$

If, however, $x_0$ lies in $D_k$ then $F\pi(x_0)F$ is the restriction of $\pi(x_0)$ to the range of $F$. Since this space contains $\xi$ and since the state it determines on $\pi(D_k)$ is $f_0$ restricted to $D_k$, which is faithful there by Lemma 24, we conclude that the GNS representation of $D_k$ associated to $f_0|D_k$ is the restriction of $\pi$, restricted to $D_k$, acting on $FH$, and that it is faithful. Thus we conclude that $\|x_0\| = \|\pi(x_0)|FH\| = \|F\pi(x)F\| \leq \|x\|$. Since $k \geq 0$ was chosen arbitrarily, we conclude that the inequality $\|x_0\| \leq \|x\|$ holds for all $x$ in $\mathcal{B}$.

This, in turn, shows that for $z$ in the dense *-subalgebra of $\mathcal{O}(E)$ generated by $\varphi_\infty(A)$ and $\{S_\xi | \xi \in E\}$, we have the inequality

$$\|\rho(\Lambda(z))\| \leq \|\rho(z)\|.$$

This inequality, together with the fact that $\rho$ is faithful on $\mathcal{O}(E)^\lambda$, by Lemma 3, allows us to invoke Lemma 2.2 of [2] to conclude that $\rho$ is faithful on all of $\mathcal{O}(E)$.

*Proof of Theorem 5.*

We want to replace Hypothesis (H5) by Condition F. The only place Hypothesis (H5) is used in our proof of Theorem 1 is to establish part (d) in Lemma 4. We shall show how Condition F also yields part (d). Thus, we want to show $f_0(V(\xi_1)V(\xi_2)\ldots V(\xi_k)\sigma(a)V(\eta_1)^*V(\eta_2)^*\ldots V(\eta_m)^*) = 0$,

$\xi_i, \eta_i \in E$, $a \in A$, whenever $m \neq k$. For this we simplify notation and write $\rho(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n)$ for $V(\xi_1)V(\xi_2)\ldots V(\xi_n)$ and extend by linearity. We then have $\rho(\varphi_n(b)\xi c) = \sigma(b)\rho(\xi)\sigma(c)$, for all $b, c \in A$ and $\xi \in E^{\otimes n}$. Part (d) of Lemma 4, then, amounts to the assertion that

$$f_0(\rho(\xi)\rho(\eta)^*) = 0$$

whenever $\xi \in E^{\otimes k}$ and $\eta \in E^{\otimes m}$, with $k \neq m$. We shall assume that $k > m$; the alternative follows from this case by taking adjoints. The proof of part (d) shows that

$$\begin{aligned} f_0(\rho(\xi)\rho(\eta)^*) &= \tau_0(\Omega(1))^{-m} f_0(\alpha^m(\rho(\xi)\rho(\eta)^*)) \\ &= \tau_0(\Omega(1))^{-m} f_0(\rho(\eta)^*\rho(\xi)), \end{aligned}$$

for all $\eta \in E^{\otimes m}$ and all $\xi \in E^{\otimes k}$. (This uses only the fact that $\tau_0(\Omega(1)) \neq 0$, which in turn uses only (H4), as we saw in the proof of Lemma 2.) Since $\rho(\zeta_1)^*\rho(\zeta_2) = V(\zeta_1)^*V(\zeta_2) = \sigma(\langle\zeta_1, \zeta_2\rangle)$ for all $\zeta_1, \zeta_2 \in E$, and since $k > m$, $\rho(\eta)^*\rho(\xi)$ is of the form $\rho(\zeta)$ for a $\zeta \in E^{\otimes(k-m)}$. So we need to show that $f_0(\rho(\zeta)) = 0$ for all $\zeta \in E^{\otimes n}$, for every $n > 0$. Fix such an $n$ and $\zeta \in E^{\otimes n}$. By Condition F, we may assume that $\zeta = \varphi_n(a)\xi - \xi a$ for some $\xi \in E^{\otimes n}$ and $a \in A$. Then

$$\begin{aligned} f_0(\rho(\zeta)) &= f_0(\rho(\varphi_n(a)\xi - \xi a)) \\ &= f_0(\sigma(a)\rho(\xi) - \rho(\xi)\sigma(a)) \\ &= 0, \end{aligned}$$

by part (b) of Lemma 8.

## 3. Applications.

We attend first to Corollary 1. Suppose $\alpha$ is an endomorphism of a unital $C^*$-algebra $A$. A *covariant representation of* $(A, \alpha)$ *of multiplicity* $n$ is defined to be a pair $(\pi, S)$, where $\pi : A \mapsto B(H_\pi)$ is a unital representation of $A$ and where $S = \{S_i\}$ is a set of $n$ isometries on $H_\pi$ with orthogonal ranges such that

$$\pi(\alpha(a)) = \sum_{i=1}^{n} S_i \pi(a) S_i^*,$$

for all $a \in A$. In general, there need be no such representations. However, if $\alpha$ is *injective*, there are. (In fact they exist under weaker hypotheses that we do not need here.) In [18], Stacey proves that given $(A, \alpha)$, with $\alpha$ injective, there is a triple $(B, i_A, t)$ consisting of a unital $C^*$-algebra $B$, a unital homomorphism $i_A : A \mapsto B$, and a family $t = \{t_i\}$ of $n$ isometries in $B$ such that

$$i_A(\alpha(a)) = \sum_{i=1}^{n} t_i i_A(a) t_i^*,$$

for all $a \in A$, and such the following two conditions are satisfied:

1. $B$ is generated by $i_A(A)$ and the $t_i$, $i = 1, 2, \ldots, n$; and
2. for every covariant representation $(\pi, S)$ of $(A, \alpha)$, there is a unital representation, denoted $\pi \times S$, of $B$ such that $(\pi \times S) \circ i_A = \pi$ and such that $(\pi \times S)(t_i) = S_i$, $i = 1, 2, \ldots, n$.

Furthermore, $(B, i_A, t)$ is uniquely determined up to isomorphism, in an obvious sense, by these properties. We therefore write $A \rtimes_\alpha^n \mathsf{N}$ for $B$ and call $A \rtimes_\alpha^n \mathsf{N}$, or the triple $(A \rtimes_\alpha^n \mathsf{N}, i_A, t)$, the *Stacey crossed product of order n determined by* $(A, \alpha)$. In essentially the same sense, the algebras $\mathcal{O}(E)$ are universal objects associated to covariant representations $(V, \sigma)$ of $E$. So, we are led to seek a realization of $A \rtimes_\alpha^n \mathsf{N}$ as an $\mathcal{O}(E)$. We are able to do this only when $n = 1$. For $n > 1$, the relation between $A \rtimes_\alpha^n \mathsf{N}$ and any $\mathcal{O}(E)$ seems somewhat tenuous, as we shall indicate shortly.

When $n = 1$, we take $E = pA$, with $p = \alpha(1)$. We let $A$ act on the right by multiplication and we give $E$ the $A$-valued inner product that it inherits from $A$. The left action $\varphi$, is given by the formula $\varphi(a)\xi = \alpha(a)\xi$. Then $E$ certainly is a correspondence of the type we have been discussing. We define a covariant representation $(V, \sigma)$ of $E$ into $A \rtimes_\alpha^1 \mathsf{N}$ by setting $\sigma = i_A$ and defining $V(\xi) = t^* i_A(\xi)$. Then, it is routine to check the defining properties of a covariant representation which comes from a representation of $\mathcal{O}(E)$, i.e., which satisfies the equation $\sigma^{(1)} \circ \varphi = \sigma$. We conclude, then, that $V \times \sigma$ is a $C^*$-representation of $\mathcal{O}(E)$ in $A \rtimes_\alpha^1 \mathsf{N}$. On the other hand, if we define $\pi$ to be $\varphi_\infty$ and set $S = S_p^*$, we obtain a covariant representation of $(A, \alpha)$ of multiplicity one. Moreover, it is immediate that $(\pi \times S) \circ (V \times \sigma) = \iota_{\mathcal{O}(E)}$, while the composition in the other order yields the identity on $A \rtimes_\alpha^1 \mathsf{N}$. We thus have proved the following lemma, which we record for reference.

Lemma 12. *If $\alpha$ is an endomorphism of a unital $C^*$-algebra $A$, then the Stacey crossed product of order* 1, $A \rtimes_\alpha^1 \mathsf{N}$ *is isomorphic to* $\mathcal{O}(E)$, *where $E$ is the correspondence $pA$ just defined.*

*Proof of Corollary* 2. Simply note that in the case when $E$ comes from $A$ and $\alpha$ in this way, then $\mathscr{L}(E) = pEp$ and the map $\Omega$ is simply $\alpha$. Thus the hypotheses of the Corollary are (H4) and (H5) in this case. Since the other hypotheses, (H1)–(H3), are also satisfied, we conclude from Theorem 1 and Lemma 12 that $A \rtimes_\alpha^1 \mathsf{N}$ is simple.

To see what the problem seems to be when $n > 1$, consider a Stacey crossed product, $(A \rtimes_\alpha^n \mathsf{N}\ ,\ i_A, t)$, and *suppose we knew that the projections* $p_i := t_i t_i^*$ *lay in* $A$. (Of course there is no reason *a priori* for this to happen.) Then we would let $E = \{\xi = (\xi_1, \ldots, \xi_n)^t \in C_n(A) \,|\, \xi_i = p_i \xi_i,\ i = 1, \ldots, n\}$ - a summand of $C_n(A)$, and we would give $E$ the structure of a correspondence simply by defining $\varphi(a)(\xi_1, \ldots, \xi_n)^t = (\alpha(a)\xi_1, \ldots, \alpha(a)\xi_n)^t$. (Note that the projections $p_i$ all commute with $\alpha(A)$.) If we then map $i_A(a)$ to $\varphi_\infty(a)$, $a \in A$, and $t_k$ to $V(p_k \varepsilon_k)^*$, where $\varepsilon_k$ is the unit basis vector in $C_n(A)$ whose only nonzero entry occurs in the $k$th slot, where it is 1, we obtain an isomorphism between $A \rtimes_\alpha^n \mathsf{N}$ and $\mathcal{O}(E)$ as we did in Lemma 2. If we think of $A \rtimes_\alpha^n \mathsf{N}$ and $\mathcal{O}(E)$ as universal objects, given by generators and relations, then for this $E$, at any rate, there appear to be a few more relations in $\mathcal{O}(E)$ than in $A \rtimes_\alpha^n \mathsf{N}$.. Whether another choice of $E$ can be made so that $\mathcal{O}(E)$ is isomorphic to $A \rtimes_\alpha^n \mathsf{N}$, we do not know. We do, however, have the following proposition and corollary that seem to be worth recording.

PROPOSITION 13. *Suppose $\alpha$ is an automorphism of the unital $C^*$-algebra $A$. Then for each $n$, the Stacey crossed product $A \rtimes_\alpha^n \mathsf{N}$ is isomorphic to $\mathcal{O}(E)$ where $E$ is the correspondence $C_n(A)$ with $\varphi$ given by the formula $\varphi(a) = \alpha^{-1}(a) \otimes I_n$.*

PROOF. The proof is the obvious one. Simply form the covariant representation $(V, \sigma)$ where $\sigma = i_A$ and $V$ sends the coordinate basis vectors to the corresponding $t_i$'s. A moment's reflection reveals that it implements an isomorphism between $\mathcal{O}(E)$ and $A \rtimes_\alpha^n \mathsf{N}$.

As Stacey notes in developing Proposition 3.4 of [18], the algebra $A \rtimes_\alpha^n \mathsf{N}$, when $\alpha$ is an automorphism was considered by Cuntz in [6] and called there a *twisted* tensor product.

COROLLARY 14. *Suppose $\alpha$ is an automorphism of the unital, amenable, $C^*$-algebra $A$ and suppose that $A$ has no $\alpha$-invariant ideals. Then, for each $n > 1$, the Stacey crossed product $A \rtimes_\alpha^n \mathsf{N}$ is simple.*

PROOF. Use the preceding proposition to identify $A \rtimes_\alpha^n \mathsf{N}$ with $\mathcal{O}(E)$, where $E = C_n(A)$, and $\varphi$ is given by $\alpha^{-1}$. Then the hypotheses guarentee that $E$ and $A$ satisfy hypotheses (H1)–(H4). As for hypothesis (H5), simply note that the map $\Omega$ is $n \cdot \alpha^{-1}$. Consequently, $\Omega(1) = n1$. So, if $n > 1$, as we are assuming, hypothesis (H5) is satisfied, too. Thus, the result is an immediate consequence of Theorem 1.

Turning to Corollary 1, suppose that $B$ is a $C^*$-algebra containing $A$ and that $A$ is the image of $B$ under a conditional expectation $\Phi$. Recall that this means that $\Phi$ is a (completely) positive, contractive projection of $B$ onto $A$

such that $\Phi(a_1ba_2) = a_1\Phi(b)a_2,\ a_i \in A,\ b \in B$. Suppose, too, that $\Phi$ is index finite in the sense of [19]. This means that there is a finite set of elements $\{u_i\}_{i=1}^n$ in $B$ (called a *quasi-basis*), such that for all $x \in B$,

$$x = \sum u_i\Phi(u_i^*x) = \sum \Phi(xu_i)u_i^*$$

(see [19, Lemma 2.1.6]). With respect to the inner product

$$\langle x, y\rangle = \Phi(x^*y)$$

$x, y \in B$, $B$ becomes a Hilbert $C^*$-module over $A$ which is isomorphic to a summand of $C_n(A)$, where $n$ is the number of elements in a quasi-basis $\{u_i\}_{i=1}^n$. (Note, $n$ is not uniquely determined, but that does not matter.) The isomorphism sends $x \in B$ to $(\Phi(u_1^*x), \Phi(u_2^*x), \ldots, \Phi(u_n^*x))^t$. Of course left multiplication converts $B$ into a correspondence over $A$ of the kind we have been discussing, i.e., $\varphi(a)x = ax,\ a \in A,\ x \in B$.

*Proof of Corollary* 3. We need to verify hypotheses (H1)–(H5) and apply Theorem 1. Hypothesis (H1) is explicit in the hypotheses of the corollary. We already have remarked that (H2) is a consequence of the assumption that $\Phi$ is index finite. Further, since $\varphi$ is given by left multiplication, which certainly is faithful, (H3) is satisfied. Hypothesis (H4) is automatically satisfied, since $A$ is assumed to be simple. To verify hypothesis (H5), we need to identify $\Omega$. If $W$ is the isomorphism from $B$ to $PC_n(A)$ just described, then $Wu_i = P\varepsilon_i$. This gives, for $a \in A$, $\Omega(a) = \sum\langle P\varepsilon_i, W\varphi(a)W^{-1}P\varepsilon_i\rangle = \sum\langle u_i, au_i\rangle = \sum\Phi(u_i^*au_i)$. Consequently, $\Omega(1_A) = \sum\Phi(u_i^*u_i)$. Recall that the *index* of $\Phi$, $ind(\Phi)$, is defined to be $\sum u_iu_i^*$, an element of the center of $B$. In fact, $ind(\Phi) \geq 1_B$, with equality holding only when $A = B$ (See Propositions 1.2.8, 2.3.1, and 2.3.7 of [19].) Since we are assuming $A$ is proper, $ind(\Phi)1_B = 1_A$. Since $\tau$ is a faithful trace on $B$ preserving $\Phi$, we find that

$$\begin{aligned}\tau(\Omega(1_A)) &= \sum \tau(\Phi(u_i^*u_i)) = \sum \tau(u_i^*u_i)\\ &= \sum \tau(u_iu_i^*) = \tau(ind(\Phi))\\ &\geq \tau(1_B) = \tau(1_A).\end{aligned}$$

This shows that $\Omega(1_A) \neq 1_A$, verifying hypothesis (H5).

*Proof of Proposition* 4. With the notation of the proposition, observe that if a point $x_0$ in $X$ is fixed by a power of $\tau$, $\tau^n$, then the point mass at $x_0$ is a measure that annihilates $C_n$ but doesn't annihilate $E^{\otimes n} = C(X)$. Thus if Condition F is satisfied, then the $\mathsf{Z}$-action determined by $\tau$ is free. For the other direction, we can replace $\tau$ with $\tau^n$, if necessary, and worry only about the case when $n = 1$. Suppose, then, that $\tau$ has no fixed points, but that $C_1$ is smaller than $E = C(X)$, and let $\mu$ be a measure on $X$ that annihilates $C_1$. Since $\tau$ has no fixed points and $X$ is assumed to be compact, there is a finite cover $\{U_i\}$ of $X$ such that $\tau^{-1}(U_i) \cap U_i = \emptyset$, for all $i$. The condition that $\mu$ annihilates $C_1$ means that $\int \varphi \circ \tau \cdot \xi d\mu = \int \varphi \cdot \xi d\mu$ for all $\xi \in C(X)$. Choosing a sequence of functions in $C(X)$ converging to the characteristic function of $U_i$, we see that $\int_{\tau^{-1}(U_i)} \xi d\mu = \int_{U_i} \xi d\mu$ for all $\xi$. Since $\tau^{-1}(U_i) \cap U_i = \emptyset$, this, in turn, implies that $\mu$ annihilates every bounded Borel function that is supported on $U_i$. Since $i$ is arbitrary and their number is finite, we conclude that $\mu$ is the zero measure.

The proof of Proposition 4 suggests that when studying Condition F, one should look at the Banach space dual of $E$, $E^\dagger$, and, in fact, one should think also about the Banach space double dual, $E^{\dagger\dagger}$. Perhaps the easiest and best way to do this, is first to imbed $E$ into its *linking algebra* $\mathscr{L}$. This algebra is a $C^*$-algebra, expressed as an algebra of $2 \times 2$ matrices with entries coming from $A$, $E$, $E^*$, and $\mathscr{K}(E)$, by the formula

$$\mathscr{L} = \begin{bmatrix} A & E^* \\ E & \mathscr{K}(E) \end{bmatrix}$$

(see [3]). If $p$ denotes the projection $\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ in the multiplier algebra of $\mathscr{L}$, $M(\mathscr{L})$, and if $q$ denotes its complement, then $E$ may be identified with $q\mathscr{L}p$ and $E^{\dagger\dagger}$ may be identified with $q\mathscr{L}^{\dagger\dagger}p$, isometrically and homeomorphically with respect to the weak-* topologies (see [7, Proposition 2.1]). Of course, $\mathscr{L}^{\dagger\dagger}$ is a von Neumann algebra containing $p$ and $q$, and it contains the double dual of $A$, $A^{\dagger\dagger}$, as $p\mathscr{L}^{\dagger\dagger}p$. Furthermore, from this perspective, it is not hard to conclude Paschke's observation [13, Corollary 4.3] that $E^{\dagger\dagger}$ is what he called a *self-dual* Hilbert $C^*$-module over $A^{\dagger\dagger}$. To keep the notation from becoming ponderous, we shall use $\langle \cdot, \cdot \rangle$ to denote the $A^{\dagger\dagger}$-valued inner product on $E^{\dagger\dagger}$ as well as the $A$-valued inner product on $E$. Of course, looking in $\mathscr{L}$ or in $\mathscr{L}^{\dagger\dagger}$, the inner products are given by products via the formula: $\langle \xi, \eta \rangle = \xi^* \eta$. If, as we are assuming, $E$ is a correspondence over $A$, via a homomorphism $\varphi : A \mapsto \mathscr{L}(E)$, then the double transpose of $\varphi$ turns $E^{\dagger\dagger}$ into a "self-dual correspondence" over $A^{\dagger\dagger}$. Functionals $f \in E^\dagger$ may be viewed as functionals in $\mathscr{L}^\dagger$ that satisfy the equation $f = pfq$, where, in general, for any algebra $\mathscr{A}$ and any functional $\phi$ on any $\mathscr{A}$-bimodule $\mathscr{X}$, we write $a\phi b(x) = \phi(bxa)$, $x \in \mathscr{X}$, $a, b \in \mathscr{X}$. When this is done, we conclude that given $f \in E^\dagger$, we may form its polar decomposition $|f| \cdot v$ as an element of $\mathscr{L}^\dagger$, where $|f| \in \mathscr{L}^\dagger$ is a positive linear functional and $v$ is a partial isometry in $\mathscr{L}^{\dagger\dagger}$. The functional $|f|$ is uniquely determined by $f$ and $v$ is uniquely determined by the condition that its final projection is dominated by the support projection of $|f|$. The condition that $pfq = f$ and the uniqueness of the polar decomposition imply that $p|f|p = |f|$ and that $v = pvq$. This means that $v \in (E^*)^{\dagger\dagger} = p\mathscr{L}^{\dagger\dagger}q$, so that $v^*$ lies in $E^{\dagger\dagger}$ and we may write $f(\xi) = |f|(\langle v^*, \xi\rangle)$, $\xi \in E$. All these statements are fairly easy to verify directly. Alternatively, one can appeal to the full treatment found in [7, Theorem 1].

The (double dual) *fixed point subspace* $F_n$ of $E^{\otimes n\dagger\dagger}$ is defined to be the collection of all $\xi \in E^{\otimes n\dagger\dagger}$ such that $\varphi_n^{\dagger\dagger}(a)\xi = \xi a$ for all $a \in A^{\dagger\dagger}$.

Theorem 15. *If $f$ is functional in $(E^{\otimes n})^{\dagger}$ that annihilates $C_n$ and if $|f| \cdot v$ denotes its polar decomposition, then $|f|$ is a (positive) tracial functional on $A$ and $v^*$ lies in $F_n$. Conversely, if $f = |f| \cdot v$ is a functional in $\mathscr{L}^{\dagger}$ such that $|f|$ is a trace on $A = p\mathscr{L}p$ and $v$ is a partial isometry with $v^* \in F_n$, then $f$ annihilates $C_n$. Thus, Condition F is equivalent to the assertion that all the spaces $F_n$ vanish.*

Proof. Since the analysis we wish to make works the same in all correspondences, we assume without loss of generality that $n = 1$. Suppose, then, that $f \in E^{\dagger}$ annihilates $C_1$. This is the same as saying that $f(\xi a) = f(\varphi(a)\xi)$ for all $\xi \in E$ and $a \in A$, or, equivalently, that $f(\varphi(u)\xi u^*) = f(\xi)$, $\xi \in E$, and $u$ in the unitary group of $A$. Furthermore, this equation persists when $E$ is replaced by $E^{\dagger\dagger}$ and $A$ by $A^{\dagger\dagger}$. Now in the proof of the polar decomposition of $f = |f| \cdot v$ found in [7], which is a modification of a (the?) standard proof in von Neumann algebras [15, Proposition 3.6.7] (which applies here, too), $v^*$ is taken to be *any* extreme point in the weak-* compact, convex set $K := \{\xi \in E^{\dagger\dagger} | f(\xi) = 1 = \|\xi\|\}$. Since $f(\varphi(u)\xi u^*) = f(\xi)$ for all unitaries $u \in A^{\dagger\dagger}$ and all $\xi \in E^{\dagger\dagger}$, $K$ is preserved under the maps $\xi \mapsto \varphi(u)\xi u^*$, $u$ unitary in $A^{\dagger\dagger}$. Further, these maps clearly carry extreme points of $K$ to extreme points of $K$. Thus $\varphi(u)v^*u^*$ is an extreme point of $K$ for every unitary in $A^{\dagger\dagger}$. But then, the uniqueness part of the polar decomposition implies that there is only *one* extreme point in $K$. Whence, $\varphi(u)v^*u^* = v^*$ for all unitaries $u \in A^{\dagger\dagger}$, and so $v^* \in F_1$. This, in turn, implies that for all $a \in A^{\dagger\dagger}$ and all unitaries $u \in A^{\dagger\dagger}$, we have $|f|(au) = f(v^*au) = f(\varphi(u)v^*a) = f(v^*ua) = |f|(ua)$, which means that $|f|$ is tracial. For the converse, simply note that if $|f|$ is tracial on $A^{\dagger\dagger}$ while $v^* \in F_1$, then for all $\xi \in E^{\dagger\dagger}$ and all unitaries $u \in A^{\dagger\dagger}$, $f(\varphi(u)\xi u^*) = |f|(v\varphi(u)\xi u^*) = |f|(u^*v\varphi(u)\xi) = |f|(v\xi) = f(\xi)$.

Suppose $M$ is a von Neumann algebra and that $\alpha$ is an automorphism of $M$. In [9], Kallman termed $\alpha$ to be *free*, or *freely acting*, in case the only solution $b$ to the equation $\alpha(a)b = ba$, for all $a \in M$, is $b = 0$. If one views $M$ as a correspondence $E$ over itself, with $\varphi = \alpha$, then $\alpha$ is free precisely when $F_1$ vanishes. (We don't make any distinction, at this point, between analysis in $E$ and analysis in $E^{\dagger\dagger}$.) Thus, Theorem 15 says that our Condition F is a correspondence-theoretic way of saying that all the powers of an automorphism are freely acting. In this case, one sometimes says that the group generated by $\alpha$, $\{\alpha^n\}_{n \in \mathbf{Z}}$, is *properly outer*. We could adopt this terminology for correspondences here, too, saying that $E$ is *properly outer* in case all the spaces $F_n$ vanish. Then our Theorem 2 would become: If *$E$ is a properly out $C^*$ -correspondence over a $C^*$-algebra $A$ and if hypotheses* (H1)–(H4) *are satisfied, then the $C^*$ -algebra $\mathcal{O}(E)$ is simple*. Of course what we would really like is a notion of Connes spectrum for a correspondence and a concomitant necessary and sufficient condition for the simplicity of $\mathcal{O}(E)$ that parallels the theorem of Olesen and Pedersen [12] (see also [15, Theorem 8.11.12]). The extreme difference that seems to exist between the simplest examples, $\mathcal{O}_n$ and transformation group $C^*$-algebras, makes the existence of such a notion – one yielding such a simplicity result – seem rather unlikely.

Added In Proof. After this paper was submitted, we received the preprint, *Ideal structure and simplicity of the $C^*$-algebras generated by Hilbert bimodules*, by T. Kajiwara, C. Pinzari, and Y. Watatani. These authors also obtain conditions for $\mathcal{O}(E)$ to be simple. They avoid our assumption that $A$ is strongly amenable. However, they assume a condition they call *($I$)-free*, which seems to be in the spirit of our Condition F, but the precise relation remains to be determined. They also assume a condition they call *$X$-simple* (or *$E$-simple*, with our notation) that is essentially our hypothesis (H4). Their proofs are rather different from ours. Nevertheless, in their Theorem 14, the authors are able to prove our Corollary 3 without the hypotheses that $A$ is strongly amenable and the conditional expectation $\Phi$ preserves a faithful trace. Their analysis also identifies the ideals in $\mathcal{O}(E)$ under suitable hypotheses on $E$, when $\mathcal{O}(E)$ is not simple.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF IOWA
IOWA CITY, IA 52242
U.S.A.

DEPARTMENT OF MATHEMATICS
TECHNION
32000 HAIFA
ISRAEL