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REMOVABLE SINGULARITIES ON RECTIFIABLE CURVES FOR HARDY SPACES OF ANALYTIC FUNCTIONS

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Abstract.

In this paper we study sets on rectifiable curves removable for Hardy spaces of analytic functions on general domains. With the methods used it seems natural to distinguish between three different classes of rectifiable curves: chord-arc curves, curves of bounded rotation and curves with Dini continuous tangents.

We give results both for sets on rectifiable Jordan curves and for sets on rectifiable curves which intersect. Among the results we prove that if K is a set lying on a rectifiable chord-arc curve, then there exists $p < \infty$ such that K is removable for H^p if and only if the generalized length of K is 0. Furthermore, if the curve is also of bounded rotation, then p can be arbitrarily chosen greater than 1.

1. Introduction and notation.

We let $\mathsf{S} = \mathsf{C} \cup \{\infty\}$ be the Riemann sphere, $\mathsf{D} = \{z \in \mathsf{C} : |z| < 1\}$, $\mathsf{T} = \partial\mathsf{D}$ and $A(\Omega) = \{f : f \text{ is analytic in } \Omega\}$. We also let Λ_d denote the d-dimensional Hausdorff measure and $\dim$ denote the Hausdorff dimension. By a domain we mean a non-empty open connected set.

Definition 1.1. For $0 < p < \infty$ and a domain $\Omega \subset \mathsf{S}$ (or $\Omega \subset \mathsf{C}^n$, $n > 1$) let

$$H^p(\Omega) = \{f \in A(\Omega) : |f|^p \text{ has a harmonic majorant in } \Omega\},$$
$$H^\infty(\Omega) = \{f \in A(\Omega) : \sup_{z\in\Omega} |f(z)| < \infty\}.$$

In this paper we will use the following definition of removability.

Definition 1.2. Let $\Omega \subset \mathsf{S}$ be a domain and $K \subset \Omega$ be compact such that $\Omega\backslash K$ is also a domain. Let $0 < p \leq \infty$. Then the set K is removable for $H^p(\Omega\backslash K)$ if $H^p(\Omega\backslash K) = H^p(\Omega)$ (as sets).

Hejhal [8], [9] showed that the definition is independent of the domain Ω,

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as long as $K \subset \Omega$, and therefore we will normally just say that K is removable for H^p.

It is true that K is removable for $H^p(\Omega \backslash K)$ if and only if $H^p(\Omega \backslash K) \subset A(\Omega)$, i.e. every $f \in H^p(\Omega \backslash K)$ can be extended analytically to the whole of Ω, see Corollary 4.6 in Björn [3].

The inclusion $H^p(\Omega) \supset H^q(\Omega)$ if $0 < p < q \leq \infty$ has as a consequence that if K is a set removable for H^p then K is also removable for H^q for all $q > p$.

It is true that a finite union of disjoint compact sets removable for H^p is removable for H^p. In the plane case removable sets are totally disconnected. Together this implies that removability is a local property in the plane case.

For a more detailed discussion, including the non-compact case and the higher dimensional case, we refer the reader to Björn [3], especially Chapter 4.

In this paper we will be concerned with singularities that lie on rectifiable curves.

The first result of this type was given by Yamashita [21] in 1969. He proved that if Γ is a Jordan curve with continuous tangent angles, $K \subset \Gamma$ is compact with $\Lambda_1(K) = 0$ and $\Omega \supset K$ is a domain, then K is removable for $H^p(\Omega \backslash K)$ for all $p > 1$. If Γ is also analytic he proved that K is removable for $H^1(\Omega \backslash K)$.

At about the same time Heins [7], p. 50, proved that if $K \subset \mathsf{R}$ is compact with $\Lambda_1(K) = 0$ then K is removable for $H^1(\mathsf{S} \backslash K)$.

At that time Hejhal had not yet proved that removability is independent of the surrounding domain (Ω above). Hejhal [8], [9] proved this result and also proved that if Γ is analytic then K is removable for H^1, but he does not seem to have been aware of Yamashita's paper.

In 1987 [15], Theorem 3.1, Øksendal stated the following result.

THEOREM 1.3. *Let E be a relatively closed subset of $\Omega \subset \mathsf{C}^n$. Assume that $E \subset \partial Q$ for a domain Q and that $\Lambda_{2n-1}(E) = 0$.*

(i) *If Q is a $\mathscr{C}^{1+\varepsilon}$ domain for some $\varepsilon > 0$ then $H^1(\Omega \backslash E) \subset A(\Omega)$.*

(ii) *If Q is a $\mathscr{C}^1$ domain then $H^p(\Omega \backslash E) \subset A(\Omega)$ for all $p > 1$.*

(iii) *If Q is a* BMO$_1$ *domain then there exists $p < \infty$ with $H^p(\Omega \backslash E) \subset A(\Omega)$.*

REMARK. A domain is a $\mathscr{C}^{1+\varepsilon}$ (BMO$_1$) domain if the boundary locally can be described as the graph of a function with gradient in the Hölder class $\mathscr{C}^\varepsilon$ (BMO).

As was mentioned above the condition $H^p(\Omega \backslash E) \subset A(\Omega)$ is equivalent to Definition 1.2 for compact E. For non-compact sets, in the plane case, necessary and sufficient conditions for $H^p(\Omega \backslash E) \subset A(\Omega)$ can be obtained from the compact case of removability, in the sense of Definition 1.2, see Theorem 4.10 in Björn [3].

In the higher dimensional case it is known that most sets are removable, e.g. all compact subsets of a domain, see Section 3.3 in Björn [3]. Because of this we will restrict our considerations to the compact case in the plane.

We will generalize Øksendal's theorem in several directions below. When proving his result Øksendal used Brownian motion. We use non-probabilistic methods instead.

We end this section with some remarks about boundary values of analytic functions that will be needed later.

For $f \in A(\mathsf{D})$ we let $f^*(z)$ denote the non-tangential limit at $z \in \mathsf{T}$, if it exists. In the case when $f \in H^p(\mathsf{D})$ well-known results concerning the convergence of f_r and f^* can be found, e.g., in Rudin [17], Chapter 17. We will need one result of this type which we state here for completeness.

Lemma 1.4. *Let* $1 \le p \le \infty, 1/p + 1/p' = 1, f \in H^p(\mathsf{D}), \ g \in H^{p'}(\mathsf{D})$ *and* $h \in \mathscr{C}(\{z \in \mathsf{C} : c \le |z| \le 1\})$, *for some* $c < 1$. *Then* $fg \in H^1(\mathsf{D})$,

$$\|fg\|_{H^1(\mathsf{D})} \le \|f\|_{H^p(\mathsf{D})} \|g\|_{H^{p'}(\mathsf{D})}$$

and

$$\lim_{r \to 1^-} \int_0^{2\pi} |f^*(e^{i\theta})g^*(e^{i\theta})h(e^{i\theta}) - f(re^{i\theta})g(re^{i\theta})h(re^{i\theta})| \, d\theta = 0.$$

2. The main lemma for Jordan curves.

Lemma 2.1. *Let* $1 \le p < \infty$ *and* $1/p + 1/p' = 1$. *Let* $\Gamma \subset \mathsf{C}$ *be a rectifiable Jordan curve. Let* Ω *be the interior of* Γ *and assume that* $0 \in \Omega$. *Let* φ *be a conformal mapping from* D *to* Ω *and* $\widetilde{\varphi}$ *be a conformal mapping from* D *to* $\mathsf{S} \setminus \overline{\Omega}$. *Let* $\sigma(z) = 1/z$. *Assume that* φ' *and* $(\sigma \circ \widetilde{\varphi})'$ *both belong to* $H^{p'}(\mathsf{D})$. *Let* $K \subset \Gamma$ *be compact.*

Then K *is removable for* H^p *if and only if* $\Lambda_1(K) = 0$.

Remark. For any rectifiable Jordan curve Γ, with interior Ω, and a conformal mapping $\varphi : \mathsf{D} \to \Omega$ it is true that $\varphi' \in H^1(\mathsf{D})$, see Koosis [13], p. 69.

Proof. It is a consequence of a theorem by Calderón that if $\Lambda_1(K) > 0$, then K is not removable for H^∞ and hence not for H^p, $p < \infty$, see e.g. Christ [4], Theorem 8, p. 102, for a proof. Thus we can assume that $\Lambda_1(K) = 0$.

By using the conformal invariance of functions in $H^{p'}(\mathsf{D})$ we can assume that $\varphi(0) = 0$ and $\widetilde{\varphi}(0) = \infty$. Since Γ is a Jordan curve we can assume, by a theorem of Carathéodory, see e.g. Rudin [17], Chapter 14.19–20, that φ and

$\widetilde{\varphi}$ are defined on the whole of $\overline{\mathsf{D}}$. Moreover, φ maps $\overline{\mathsf{D}}$ homeomorphically onto $\overline{\Omega}$ and $\widetilde{\varphi}$ maps $\overline{\mathsf{D}}$ homeomorphically onto $\mathsf{S}\backslash\Omega$.

Assume that $f \in H^p(\mathsf{S}\backslash K)$. We have to prove that f can be continued, analytically, to the whole of S. In fact, it is enough to prove that f can be continuously continued to the whole of S, as this shows that f is bounded and we know that K is removable for H^∞ (since all sets with $\Lambda_1(\cdot) = 0$ are removable for H^∞), but we will prove that f can be analytically continued.

We assume that Γ is positively oriented. Let $\Gamma_r = \{\varphi(re^{i\theta}) : 0 \le \theta \le 2\pi\}$ (positively oriented), $0 < r \le 1$, and Ω_r be the interior of Γ_r. Let also $\tilde{\Gamma}_r = \{\widetilde{\varphi}(re^{i\theta}) : 0 \le \theta \le 2\pi\}$ (positively oriented).

Fix $\eta \in \Omega_{1/4}$ and let $\frac{1}{2} \le r \le 1$ for the main part of the derivation below. Substituting $\zeta = \varphi(re^{i\theta})$ we obtain

$$\begin{aligned}\int_{\Gamma_r} \frac{f(\zeta)}{2\pi i(\zeta - \eta)}\, d\zeta &= \int_0^{2\pi} \frac{f\circ\varphi(re^{i\theta})}{2\pi i(\varphi(re^{i\theta}) - \eta)}\, ire^{i\theta}\varphi'(re^{i\theta})\, d\theta \\ &= \int_0^{2\pi} g(re^{i\theta})(f\circ\varphi)(re^{i\theta})\varphi'(re^{i\theta})\, d\theta,\end{aligned}$$

where $g(z) = z/2\pi(\varphi(z) - \eta)$, which is a bounded and continuous function for $\frac{1}{2} \le |z| \le 1$. As $f \in H^p(\mathsf{S}\backslash K)$, conformal invariance shows that $f\circ\varphi \in H^p(\mathsf{D})$. Thus the conditions in Lemma 1.4 are fulfilled and we get, letting $r \to 1^-$,

$$\int_{\Gamma_r} \frac{f(\zeta)}{2\pi i(\zeta - \eta)}\, d\zeta \to \int_0^{2\pi} g(e^{i\theta})(f\circ\varphi)(e^{i\theta})(\varphi')^*(e^{i\theta})\, d\theta = \int_\Gamma \frac{f(\zeta)}{2\pi i(\zeta - \eta)}\, d\zeta.$$

The latter integral is well-defined, as f is defined a.e. on Γ by the assumption $\Lambda_1(K) = 0$.

We now want to perform the same kind of calculation for the outer region. As $\infty \in \mathsf{S}\backslash\overline{\Omega}$ we cannot hope for $\widetilde{\varphi}' \in H^{p'}(\mathsf{D})$, but using the conformal mapping σ we can obtain the desired results. Letting $\zeta = \widetilde{\varphi}(re^{i\theta}) = \sigma\circ\sigma\circ\widetilde{\varphi}(re^{i\theta})$ we obtain

$$\begin{aligned}\int_{\tilde{\Gamma}_r} \frac{f(\zeta)}{2\pi i(\zeta - \eta)}\, d\zeta &= \int_{2\pi}^0 -\frac{f\circ\widetilde{\varphi}(re^{i\theta})}{2\pi i(\widetilde{\varphi}(re^{i\theta}) - \eta)}\, ire^{i\theta}\widetilde{\varphi}^2(re^{i\theta})(\sigma\circ\widetilde{\varphi})'(re^{i\theta})\, d\theta \\ &= \int_0^{2\pi} \tilde{g}(re^{i\theta})(f\circ\widetilde{\varphi})(re^{i\theta})(\sigma\circ\widetilde{\varphi})'(re^{i\theta})\, d\theta,\end{aligned}$$

where $\tilde{g}(z) = z\widetilde{\varphi}^2(z)/2\pi(\widetilde{\varphi}(z) - \eta)$ is bounded and continuous for $\frac{1}{2} \le |z| \le 1$. By conformal invariance $f\circ\widetilde{\varphi} \in H^p(\mathsf{D})$. Applying Lemma 1.4 we get, letting $r \to 1^-$,

$$\int_{\widetilde{\Gamma}_r} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta \to \int_0^{2\pi} \tilde{g}(e^{i\theta})(f\circ\widetilde{\varphi})(e^{i\theta})((\sigma\circ\widetilde{\varphi})')^*(e^{i\theta})\,d\theta = \int_{\Gamma} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta.$$

For $\frac{1}{2} \leq r < 1$ we get, using Cauchy's theorem,

$$\begin{aligned}
f(\eta) &= \int_{\Gamma_r} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta \\
&= \int_{\Gamma_r} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta + \int_{\widetilde{\Gamma}_{1/2}-\widetilde{\Gamma}_r} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta \\
&= \int_{\widetilde{\Gamma}_{1/2}} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta + \int_{\Gamma_r} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta - \int_{\widetilde{\Gamma}_r} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta \\
&\to \int_{\widetilde{\Gamma}_{1/2}} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta + \int_{\Gamma} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta - \int_{\Gamma} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta \\
&= \int_{\widetilde{\Gamma}_{1/2}} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta,
\end{aligned}$$

where the limit is taken as $r \to 1^-$.

Define

$$F(\eta) = \int_{\widetilde{\Gamma}_{1/2}} \frac{f(\zeta)}{2\pi i(\zeta-\eta)}d\zeta,$$

an analytic function inside $\widetilde{\Gamma}_{1/2}$. We see that

$$F(\eta) = f(\eta) \quad \text{for all } \eta \in \Omega_{1/4}.$$

Hence f can be continued analytically across K and $f \in A(\mathsf{S}) = H^p(\mathsf{S})$.

3. Properties of different classes of curves.

In this section we introduce three classes of curves that are suitable when applying Lemma 2.1.

3.1. *Chord-arc curves.*

Definition 3.1. A *chord-arc curve* (*arc*) is a rectifiable Jordan curve (arc) $\Gamma \subset \mathsf{C}$ for which there exists a constant M, such that for any $z_1, z_2 \in \Gamma$ the length of the shorter arc in Γ between z_1 and z_2 is less than $M|z_1 - z_2|$.

A domain bounded by a chord-arc curve is called a *chord-arc domain*.

Remark. A BMO_1 curve is always a chord-arc curve in the plane case.

Theorem 3.2. *Assume that Ω is a bounded chord-arc domain and that φ is a conformal mapping from* D *onto* Ω. *Then there exists $p > 1$, only dependent on the chord-arc constant M of $\Gamma = \partial\Omega$, such that $\varphi' \in H^p(\mathsf{D})$.*

Remark. This is not a new result, however as we have not found a reference with a proof, we here give a proof for completeness.

Proof. Let ω denote the harmonic measure for Ω with respect to some fixed point $z_0 \in \Omega$, and let s denote the arc length on Γ.

By a theorem due to Lavrent′ev, Theorem 7 in [14], $d\omega$ belongs to the Muckenhoupt class $A_\infty(ds)$, and moreover, the A_∞ constants depend only on the chord-arc constant of Γ, see also Jerison and Kenig, Theorem 2.1 in [10] and p. 222 in [11]. By Lemma 5 in Coifman and Fefferman [5] it follows that $ds \in A_\infty(d\omega)$, and moreover,

$$\frac{ds}{d\omega} \in L^p(d\omega)$$

for some $p > 1$. Thus

$$\varphi'(e^{i\theta}) = \frac{1}{ie^{i\theta}} \frac{d\varphi(e^{i\theta})}{d\theta} \in L^p(\mathsf{T}).$$

By examining the proof it is easy to see that p is only dependent on the A_∞ constants and thus only on the chord-arc constant of Γ.

As $\varphi' \in H^1(\mathsf{D})$, see the remark following Lemma 2.1, we can conclude, using a theorem by Smirnov, see e.g. Koosis [13], p. 102, that $\varphi' \in H^p(\mathsf{D})$.

We will be needing the following geometrical lemma.

Lemma 3.3. *Let $\Omega \subset \mathsf{D}$ be a domain with $\Gamma = \partial\Omega \cap \mathsf{D}$ being a chord-arc arc with endpoints on T and chord-arc constant M. Then Ω can be extended to a chord-arc domain $\widetilde{\Omega}$ with $\widetilde{\Omega} \cap \mathsf{D} = \Omega$. Moreover, if $\varepsilon > 0$ then $\widetilde{\Omega}$ can be chosen so that there is a point $z_0 \in \widetilde{\Omega}$, $\partial\widetilde{\Omega} \subset \{z \in \mathsf{C} : (1-\varepsilon)r < |z - z_0| < r\}$ for some $r > 0$, and the chord-arc constant $\widetilde{M}$ of $\widetilde{\Omega}$ only depends on M.*

Sketch of proof. Draw straight radial rays out from the endpoints of Γ, the length depending on M. Close the curve by drawing a circular arc with centre z_0, where $|z_0|$ is large enough, and such that the curve surrounds z_0. That this can be done with control over $\widetilde{M}$, so that $\widetilde{M}$ only depends on M, is elementary, we omit the proof of this fact here.

3.2. *Curves of bounded rotation.*

The following definition was introduced by Radon [16] in 1919.

Definition 3.4. A rectifiable Jordan curve (arc) Γ is of bounded rotation if the forward half-tangent exists at every point and the tangent angle $\tau(s)$, which it makes with a fixed direction, can be defined as a function of bounded variation of the arc length s.

We assume that $\tau(s)$ is so determined that its jumps do not, in modulus, exceed π, and that the arc length parametrization corresponds to the positive orientation of Γ.

The following result is due to Warschawski and Schober, Theorem 2 in [20].

THEOREM 3.5. *Assume that Γ is a chord-arc curve of bounded rotation with interior Ω. Let τ be as above, v_+ be the positive variation of τ and*

$$a_+ = \max_s [v_+(s^+) - v_+(s^-)].$$

Let φ map D conformally onto Ω. Then $\varphi' \in H^p(\mathsf{D})$ for $0 < p < \pi/a_+$.

3.3. *Curves with Dini continuous tangents.*

DEFINITION 3.6. Let $f \in \mathscr{C}(\mathsf{R})$ (or $f \in \mathscr{C}(I)$ for some interval $I \subset \mathsf{R}$) and let

$$c(t) = c_f(t) = \sup_{|x-y|<t} |f(x) - f(y)|$$

be the modulus of continuity. Then the function f is Dini continuous if

$$\int_0^\delta \frac{c(t)}{t}\, dt < \infty,$$

for some $\delta > 0$.

DEFINITION 3.7. Let Γ be a rectifiable Jordan curve (arc) and assume that the tangent function $\tau(s)$ is a Dini continuous function with respect to the arc length s. Then we say that Γ is a curve (arc) with *Dini continuous tangents*.

THEOREM 3.8. *Let Ω be a domain bounded by a closed curve $\Gamma \subset \mathsf{C}$ with Dini continuous tangents. Let φ be a conformal mapping from D onto Ω.*
Then φ' is non-zero and continuous on $\overline{\mathsf{D}}$.

This condition, and hence the conclusion, is true for Γ in the Hölder class $\mathscr{C}^{1+\varepsilon}$.

This result was proved by Warschawski in 1932, p. 443 in [18]. Warschawski gave a simpler proof of this theorem in 1961 [19].

4. Removability on Jordan curves.

As a corollary of Lemma 2.1 and Theorems 3.2, 3.5 and 3.8 we get the following result.

THEOREM 4.1. *Let $\Gamma \subset \mathsf{C}$ be a chord-arc curve with chord-arc constant M. Let $K \subset \Gamma$ be compact. Let τ be the tangent angle of the forward half-tangent, as in Theorem 3.5, whenever it exists. Let v_+ and v_- be the positive and negative variation functions of τ, resp., if they exist, and*

$$a_\pm = \max_s [v_\pm(s^+) - v_\pm(s^-)].$$

Then the following are true:

(a) *there exists $p < \infty$, only dependent on M, such that K is removable for H^p if and only if $\Lambda_1(K) = 0$,*

(b) *if $a_\pm$ exist, $p' < \min(\pi/a_+, \pi/a_-)$ and $1/p + 1/p' = 1$, then K is removable for H^p if and only if $\Lambda_1(K) = 0$,*

(c) *if τ is Dini continuous, then K is removable for H^1 if and only if $\Lambda_1(K) = 0$.*

REMARKS. In Corollary 5.4 we improve upon the results in (b) and (c).

Kobayashi [12], Lemma 2, gave an example of a set $K \subset \mathsf{R}$, or rather a class of such sets, not removable for any $p < 1$. His example can be chosen to have dimension zero. Thus even a lower dimensional Hausdorff condition will not give removability for $p < 1$.

PROOF. We can assume that $0 \in \Omega$. It follows directly from Theorems 3.2, 3.5 and 3.8 that the conditions on φ in Lemma 2.1, necessary for (a), (b) and (c), are fulfilled in the respective cases. We only need to show that $(\sigma \circ \widetilde{\varphi})' \in H^{p'}(\mathsf{D})$ for appropriate p', where $\sigma \circ \widetilde{\varphi}$ is as in Lemma 2.1.

We consider first (b) and (c). Let $\widehat{\Gamma} = \sigma(\Gamma)$ and $\widehat{\Omega} = \sigma(\mathsf{S}\setminus\overline{\Omega})$. Let $\hat{\tau}$ be the tangent angle of the forward half-tangent along $\widehat{\Gamma}$. Let $\hat{v}_+$ and $\hat{v}_-$ be the positive and negative variation functions of $\hat{\tau}$, resp., and

$$\hat{a}_\pm = \max_s [\hat{v}_\pm(s^+) - \hat{v}_\pm(s^-)].$$

Using the conformality of σ, it is easy to see that in (c) $\hat{\tau}$ is also Dini continuous and in (b) $\widehat{\Gamma}$ is also of bounded rotation with $\hat{a}_\pm = a_\mp$. Using Theorems 3.5 and 3.8 we see that the condition on $(\sigma \circ \widetilde{\varphi})'$ is fulfilled in (b) and (c).

In (a) we can, since removability is a local property, assume that $K \subset \Gamma$, where Γ is an arc such that Lemma 3.3 can be applied. Let $\widetilde{\Omega}$ be the domain given by Lemma 3.3. By a translation of the coordinate system we can assume that $\tilde{\Gamma} = \partial\tilde{\Omega} \subset \{z \in \mathsf{C} : (1-\varepsilon)r < |z| < r\}$ and $0 \in \widetilde{\Omega}$. Assume that $K \subset \tilde{\Gamma}$.

Let now $\tilde{\Gamma} = \sigma(\tilde{\Gamma})$ and let $\hat{s}$ and $\tilde{s}$ denote arc length on $\widehat{\Gamma}$ and $\tilde{\Gamma}$, resp. Then for $z \in \tilde{\Gamma}$ we have $r^{-2} < |d\hat{s}(\sigma(z))/d\tilde{s}(z)| < ((1-\varepsilon)r)^{-2}$. This shows that $\widehat{M} \leq \widetilde{M}/(1-\varepsilon)^2$, where $\widehat{M}$ is the chord-arc constant of $\widehat{\Gamma}$.

Let p be suitable for $\widetilde{M}$ and $\widehat{M}$ in Theorem 3.2. Then Theorem 3.2 gives us the condition necessary for applying Lemma 2.1, which shows that K is removable for H^p.

5. Removability on intersecting curves.

5.1. *Intersecting curves of bounded rotation.*

Lemma 5.1. *Let* $1 \leq p < \infty$ *and* $1/p + 1/p' = 1$. *Let* $R \subset \mathsf{C}$ *be a domain whose boundary is a Jordan curve containing* 0. *Let* $\eta > 0$ *(be an angle) and* $\sigma(z) = z^{\pi/\eta}$. *Assume that (a suitable branch of)* σ *is injective on* $\overline{R}$ *and let* $Q = \sigma(R)$. *Let* $\varphi : \mathsf{D} \to Q$ *be a conformal mapping and assume that* $\varphi' \in H^{p'}(\mathsf{D})$. *Let* $f \in H^p(R) \cap A(\mathsf{S}\setminus\{0\})$. *Let* $\gamma \subset R$ *be a rectifiable Jordan arc with* $\xi_1, \xi_2 \in \partial R$ *as endpoints. Then*

$$\left| \int_\gamma \zeta^{k-1} f(\zeta)\, d\zeta \right| \to 0, \quad \text{as } \xi_1, \xi_2 \to 0,$$

whenever $k \geq \pi/\eta$. *Moreover, if* φ' *is continuous and non-zero on* $\overline{\mathsf{D}}$ *and*

$$f(\zeta) = \sum_{k=0}^{N} c_k \zeta^{-k},$$

with $N < \infty$, *then* $c_k = 0$ *whenever* $k \geq \pi/p\eta$.

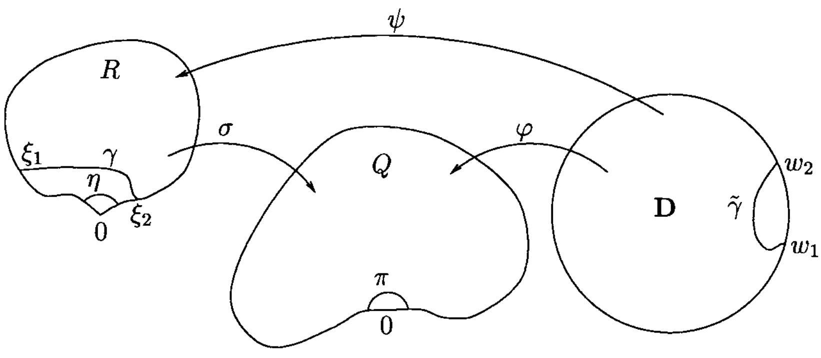

Figure 1. The geometrical situation in Lemma 5.1.

PROOF. We start by proving the first conclusion. As $\varphi' \in H^{p'}(\mathsf{D})$, φ must be bounded and hence both Q and R must be bounded. As ∂Q is a Jordan curve we can assume that φ is defined on $\overline{\mathsf{D}}$ and that $\varphi(1) = 0$.

Let $\psi = \sigma^{-1} \circ \varphi : \overline{\mathsf{D}} \to \bar{R}$, a conformal mapping from D to R with $\psi(1) = 0$. Then $F = f \circ \psi \in H^p(\mathsf{D})$, by conformal invariance. Let $w_l = \psi^{-1}(\xi_l) = e^{i\theta_l}$, $l = 1, 2$. Let $\tilde{\gamma} = \psi^{-1}(\gamma)$ which is a Jordan arc in D from w_1 to w_2. Let $k \geq \pi/\eta$. Using the substitution $\zeta = \psi(z) = \varphi(z)^{\eta/\pi}$ we get

$$\int_\gamma \zeta^{k-1} f(\zeta)\, d\zeta = \int_{\tilde{\gamma}} \varphi(z)^{\eta(k-1)/\pi} \frac{\eta}{\pi} \varphi(z)^{\eta/\pi - 1} \varphi'(z) F(z)\, dz$$

$$= \int_{\tilde{\gamma}} \frac{\eta}{\pi} \varphi(z)^{\eta k/\pi - 1} \varphi'(z) F(z)\, dz.$$

The first factor is bounded and analytic, since $k \geq \pi/\eta$, the second is in $H^{p'}(\mathsf{D})$ and the third in $H^p(\mathsf{D})$. Thus, by Lemma 1.4, the integrand belongs to $H^1(\mathsf{D})$. Let

$$G(z) = \frac{\eta}{\pi} \varphi(z)^{\eta k/\pi - 1} \varphi'(z) F(z).$$

As the integral is independent of the path (in D) we have for $r_0 < 1$,

$$\int_\gamma \zeta^{k-1} f(\zeta)\, d\zeta = \int_{\tilde{\gamma}} G(z)\, dz$$
$$= \int_1^{r_0} G(re^{i\theta_1}) e^{i\theta_1}\, dr + \int_{\theta_1}^{\theta_2} G(r_0 e^{i\theta}) i r_0 e^{i\theta}\, d\theta + \int_{r_0}^1 G(re^{i\theta_2}) e^{i\theta_2}\, dr.$$

Letting $r_0 \to 1^-$, the first and the last integral tend to zero by the Fejér–Riesz inequality, see e.g. Duren [6], p. 46. Thus we see that

$$\left| \int_\gamma \zeta^{k-1} f(\zeta)\, d\zeta \right| \leq \int_{\theta_1}^{\theta_2} |G^*(e^{i\theta})|\, d\theta,$$

which tends to zero as $\theta_1, \theta_2 \to 0$, i.e. as $\xi_1, \xi_2 \to 0$, since $G \in H^1(\mathsf{D})$. This proves the first conclusion.

Assume now that $\varphi' \in \mathscr{C}(\overline{\mathsf{D}})$ is non-zero. Then $|\varphi(e^{i\theta})| \geq A|\theta|$ for some $A > 0$ and θ near 0.

Assume also that $f(\zeta) = \sum_{k=0}^N c_k \zeta^{-k}$. Without loss of generality we can assume that $c_N = 1$. Then for θ near 0

$$|F(e^{i\theta})| = |f(\psi(e^{i\theta}))| \geq \tfrac{1}{2} |\varphi(e^{i\theta})|^{-\eta N/\pi} \geq \tfrac{1}{2} A^{-\eta N/\pi} |\theta|^{-\eta N/\pi}.$$

But $F \in H^p(\mathsf{D})$ so for δ small enough

$$\infty > \int_{-\delta}^{\delta} |F(e^{i\theta})|^p \, d\theta \geq \frac{A^{-\eta Np/\pi}}{2^p} \int_{-\delta}^{\delta} |\theta|^{-\eta Np/\pi} \, d\theta,$$

and thus $\eta Np/\pi < 1$, i.e. $N < \pi/p\eta$.

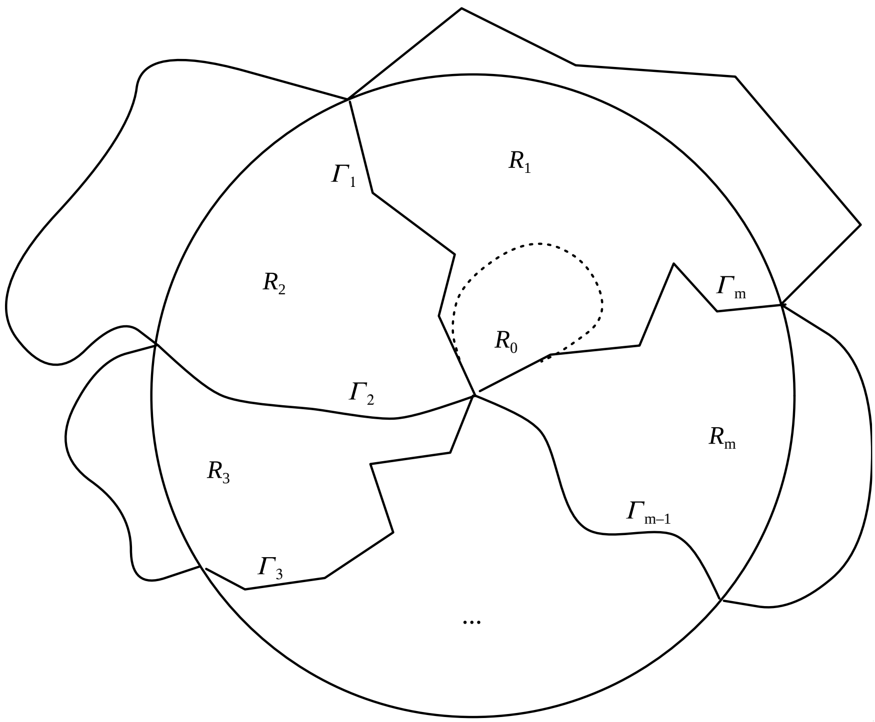

Figure 2. The geometrical situation in Lemma 5.2.

LEMMA 5.2. *Let $1 \leq p < \infty$ and $1/p + 1/p' = 1$. Let for $0 \leq j \leq m$, $R_j \subset \mathsf{C}$ be a domain whose boundary is a Jordan curve containing 0. Assume that $R_0 \subset R_1$, $\overline{\mathsf{D}} \subset \bigcup_{j=1}^m \bar{R}_j$, $R_j \cap \mathsf{T} \neq \varnothing$ if $1 \leq j \leq m$ and $R_j \cap R_k = \varnothing$ if $1 \leq j < k \leq m$. Let further for $0 \leq j \leq m$, $\eta_j > 0$, $\sigma_j(z) = z^{\pi/\eta_j}$, and assume that (a suitable branch of) σ_j is injective on $\bar{R}_j$. Let $\varphi_j : \mathsf{D} \to \sigma_j(R_j)$, $0 \leq j \leq m$, be conformal with $\varphi_j' \in H^{p'}(\mathsf{D})$. Let $f \in A(\mathsf{S}\backslash\{0\}) \cap \bigcap_{j=1}^m H^p(R_j)$ and write*

$$f(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^{-k}.$$

Then $c_k = 0$ if $k \geq \max_{1 \leq j \leq m} \pi/\eta_j$. If, moreover, φ_0' is continuous and non-zero on $\overline{\mathsf{D}}$, then $c_k = 0$ if $k \geq \pi/p\eta_0$.

PROOF. Let $1 \le j \le m$. We can assume that the domains $R_1, \dots, R_m$ are ordered so that $\Gamma_j = (\partial R_j \cap \partial R_{j+1}) \setminus \{0\} \neq \emptyset$ (letting $R_{m+1} = R_1$). Fix $\xi_j \in \Gamma_j$ and consider rectifiable Jordan arcs $\gamma_j \subset R_j$ with endpoints ξ_{j-1} and ξ_j (letting $\xi_0 = \xi_m$). Let γ be the union of these arcs and their endpoints, a rectifiable Jordan curve around 0. Orient γ, and γ_j, positively. Then

$$c_k = \frac{1}{2\pi i} \int_\gamma \zeta^{k-1} f(\zeta)\, d\zeta.$$

Lemma 5.1 shows that if $k \ge \pi/\eta_j$, then

$$\left| \int_{\gamma_j} \zeta^{k-1} f(\zeta)\, d\zeta \right| \to 0, \quad \text{as } \xi_{j-1}, \xi_j \to 0.$$

Hence $c_k = 0$ if $k \ge \max_{1 \le j \le m} \pi/\eta_j$. The function f is thus a polynomial in ζ^{-1} and if φ_0' is continuous and non-zero on $\overline{\mathsf{D}}$, Lemma 5.1 also shows that $c_k = 0$ if $k \ge \pi/p\eta_0$.

THEOREM 5.3. *Assume that we have a finite number of compact chord-arc arcs of bounded rotation and let $\Gamma \subset \mathsf{C}$ be their union. Assume that they only intersect at their endpoints. Let $z_1, \dots, z_m$ be the points of intersection. Let $m_k \ge 2$ be the number of arcs meeting at z_k. Assume that no two arcs have the same tangent at z_k (in the direction towards z_k). Near z_k, $\mathsf{S} \setminus \Gamma$ splits into m_k regions $R_{k,1}, \dots, R_{k,m_k}$. Let $\eta_{k,j}$, $1 \le j \le m_k$, be the angles at z_k for these regions. Let $\sigma_{k,j}(z) = (z - z_k)^{\pi/\eta_{k,j}}$, for some branch containing $R_{k,j}$ near z_k. Assume that $K \subset \Gamma$ is compact. Assume that*

(a) *$p \ge 1$, all arcs have Dini continuous tangents, $\partial\sigma_{k,j}(R_{k,j})$ have Dini continuous tangents near 0 for $1 \le k \le m$, $1 \le j \le m_k$ and*

$$p \ge \max_{1 \le k \le m} \frac{\pi}{\max_{1 \le j \le m_k} \eta_{k,j}},$$

or

(b) *$p > 1$ and for each k, $1 \le k \le m$,*

(i)

$$p > \frac{\pi}{\max_{1 \le j \le m_k} \eta_{k,j}},$$

or

(ii) *there is a domain $R_{k,0} \subset \mathsf{S} \setminus \Gamma$ with angle $\eta_{k,0} = \max_{1 \le j \le m_k} \eta_{k,j}$ at $z_k \in \partial R_{k,0}$ such that $\partial\sigma_{k,0}(R_{k,0})$ has Dini continuous tangents near 0, where $\sigma_{k,0}(z) = (z - z_k)^{\pi/\eta_{k,0}}$ and $p \ge \pi/\eta_{k,0}$.*

Then K is removable for H^p if and only if $\Lambda_1(K) = 0$.

PROOF. As in Lemma 2.1, the theorem of Calderón proves that if $\Lambda_1(K) > 0$ then K is not removable. Therefore we can assume that $\Lambda_1(K) = 0$.

We consider (a) first. Let $f \in H^p(\mathsf{S}\backslash K)$. It follows from Theorem 4.1(a) that f can have singularities only at the points of intersection. As removability is a local property it is enough to assume that the origin is the only point of intersection.

By, if necessary, a scaling, we can assume that the situation near 0 is as in Lemma 1. We only need to verify that the conditions on φ_j' are fulfilled. The domain $R_{k,j}$ in the theorem corresponds to the domain R_j in the lemma.

It is easy to see that a Jordan arc with Dini continuous tangents can be closed to a Jordan curve with Dini continuous tangents. It follows that we can assume that $\partial\sigma_j(R_j)$ have Dini continuous tangents. Using Theorem 3.8 we see that the conditions on φ_j' in Lemma 1 are fulfilled.

In (b) let $1/p + 1/p' = 1$ and thus $p' < \infty$. As the tangents are of bounded variation, there can only be a finite number of corners with (their larger) angles $\geq \pi(1 + 1/p')$. We can split the arcs at these corners, adding only a finite number of points of intersection, and can thus assume that all the interior corners of the arcs have (their larger) angles less than $\pi(1 + 1/p')$.

Let $f \in H^p(\mathsf{S}\backslash K)$. As in (a), by Theorem 4.1(b), the singularities can only be at the points of intersection and we can assume that the origin is the only point of intersection and that the situation near 0 is as in Lemma 1. We only need to verify that the conditions on φ_j' are fulfilled.

A Jordan arc of bounded rotation with all corners having angles less than $\pi(1 + 1/p')$ can be closed to a Jordan curve of bounded rotation with all corners having angles less than $\pi(1 + 1/p')$. It follows that we can assume, using the conformality of σ_j, that $\partial\sigma_j(R_j)$ are of bounded rotation with all corners having (their larger) angles less than $\pi(1 + 1/p')$.

In (b) (i) we notice that we can fit a small sector with angle η at 0, $\pi/p < \eta < \max_{1\leq j\leq m} \eta_j$ into the domain R_j with the largest angle at 0. In (b)(ii) we can assume that $\partial\sigma_0(R_0)$ has Dini continuous tangents. Using Theorem 3.5 we see that the conditions on φ_j' in Lemma 1 are fulfilled.

REMARKS. If ∂R_j near 0 consists of two straight rays for all j (with the notation in the proof of (a) above), the p in the theorem is sharp. This was shown in the proof of the main theorem in Kobayashi [12]. He proved moreover that in this case there exists a zero-dimensional set $K \subset \Gamma$ not removable for H^q for any $q < p$.

Whether the strict inequalities in the conditions on p really are necessary in Theorem 5.3 (b) is not known.

COROLLARY 5.4. *Assume that $\Gamma \subset \mathsf{C}$ is a chord-arc curve of bounded rota-*

tion, that $K \subset \Gamma$ is compact and that $p > 1$. Then K is removable for H^p if and only if $\Lambda_1(K) = 0$.

If, moreover, Γ consists of a finite number of arcs with Dini continuous tangents, and the situations at the endpoints of these arcs are as described in Theorem 5.3 (a) *with $p = 1$, then K is removable for H^1 if and only if $\Lambda_1(K) = 0$.*

Proof. We start with the first part. We can split Γ at two arbitrary points to obtain a situation as in Theorem 5.3 with $k = m_1 = m_2 = 2$. At both intersection points the larger of the (two) angles is $\geq \pi$. Thus Theorem 5.3 (b) (i) gives us the desired result.

For the second part we only need to notice that at every corner (endpoint) always one of the (two) angles is larger than π, to obtain the result from Theorem 5.3 (a).

5.2. *Intersecting chord-arc curves.*

Theorem 5.5. *Assume that we have a finite number of compact Jordan arcs and denote their union by $\Gamma \subset \mathsf{C}$. Assume that there are only a finite number of points of intersection between the arcs. Each component of Γ splits the complex plane into a finite number of domains. Assume that all these domains are chord-arc domains with a common chord-arc constant M. Let $K \subset \Gamma$ be compact. Then there exists $p < \infty$, only dependent on M, such that K is removable for H^p if and only if $\Lambda_1(K) = 0$.*

Proof. The theorem of Calderón proves that if $\Lambda_1(K) > 0$ then K is not removable. As removability is a local property we can consider the components of Γ separately. Let us therefore assume that Γ is connected and $\Lambda_1(K) = 0$.

Let $f \in H^p(\mathsf{S} \backslash K)$. By Theorem 4.1, with p suitable, we see that f can only have singularities at the points of intersection.

Let z_0 be one of the points of intersection. As there are only finitely many points of intersection, we can find a small disc around z_0 without any other point of intersection. By an affine change of coordinates we obtain a situation as described in Lemma 5.2.

Choose all $\eta_j = \pi$, i.e. σ_j is the identity map. For those domains R_j which are bounded Theorem 3.2 shows that $\varphi_j' \in H^{p'}(\mathsf{D})$. If R_j is unbounded apply the first part of Lemma 3.3, with a small enough disc, to obtain a bounded domain $\widetilde{\Omega}$, denote it again by R_j. If we choose p' suitable for $\widetilde{M}$, which still makes it depend only on M, Theorem 3.2 shows that $\varphi_j' \in H^{p'}(\mathsf{D})$.

Thus it follows from Lemma 5.2 that K is removable for H^p.

The results in this paper were part of the author's thesis [1], see also Björn [2]. They were inspired by the works of Hejhal [8], [9] and Øksendal [15].

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