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SOME NOTEWORTHY PROPERTIES OF ZERO DIVISORS IN INFINITE RINGS II

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In [3] we have considered four subsets of zero divisors of an infinite ring R which is not a domain: $D = D(R)$ – the set of all zero divisors; $T = T(R)$ – the set of two-sided zero divisors; $S = S(R)$ – the set of zero divisors with nonzero two-sided annihilator; and $N = N(R)$ – the set of nilpotent elements. Our main interest was in the sets $S\backslash N, T\backslash S$ and $D\backslash T$. We have seen that these sets are power closed and root closed, where a subset of a ring is said to be root closed if whenever it contains a positive power of an element, it also contains the element itself. The main results of [3] were: If $S \neq N$ then $S\backslash N$ is infinite; and if N is infinite, then each of the sets $T\backslash S$ and $D\backslash T$ is infinite provided it is nonempty. We have also constructed examples showing that among the eight formal conditions obtainable by choosing sequences of equalities and proper inclusions in $D \supseteq T \supseteq S \supseteq N$, all except perhaps $D \neq T \neq S = N$ can be satisfied. In the present paper we construct a ring satisfying $D \neq T \neq S = N$. The main results of the paper refer to a fifth set of zero divisors which is located between S and N. We first consider the subset of S of elements for which the left and right annihilator coincide, and we denote this set by S_1. For example, all zero divisors belonging to the center are in S_1. The set S_1 need not contain N, so we prefer to consider the set of elements radical over S_1; and we denote it by W. We clearly have $S \supseteq W \supseteq N$. We prove that W is infinite, and if $W \neq N$, then $W\backslash N$ is infinite. If $S \neq W$, then $S\backslash W$ may be finite; but it is infinite when N is infinite or when R has 1. As regards S_1, we prove that it is infinite when N is finite; and if R has 1 and $S_1\backslash N$ is nonempty, then $S_1\backslash N$ is infinite.

We close the paper with the result that S has the same cardinal number as R. This improves the similar result for T which was proved by Lanski [4].

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1. A ring with $D \neq T \neq S = N$.

We first construct a ring with 1 for which $D = T \neq S = N$. We start with the ring of integral polynomials $\mathsf{Z}[t]$ and its subring (ideal) $P = t\mathsf{Z}[t]$. Let V be the zero ring on the additive group of P. Consider V as a P-module (left and right) under the multiplication of elements of P. Let

$$R = \begin{bmatrix} P & V \\ V & 0 \end{bmatrix}$$

with the obvious multiplication. We have $D = T \neq S = N$ as in Example 5 of [3]. In that example, the property $D = T \neq S = N$ is lost when 1 is adjoined, but here this property is preserved. Indeed, if R^1 denotes the ring obtained from R by adjoining 1, then the elements of $R^1 \backslash R$ are easily seen to be regular in R^1 and therefore the sets D, T, S, N remain unchanged. Note that the same idea may be used to construct rings with 1 satisfying the other conditions considered in [3].

The desired example is the ring $Q = R^1[x;\sigma]$ of skew left polynomials in x, where $xa = \sigma(a)x$ for $a \in R^1$ and σ is the endomorphism of R^1 sending 1 to 1 and $\begin{bmatrix} p & v' \\ v & 0 \end{bmatrix}$ to $\begin{bmatrix} p & v' \\ 0 & 0 \end{bmatrix}$. We proceed to show that $D(Q) \neq T(Q) \neq S(Q) = N(Q)$.

We shall use the same notation for annihilators as in [3]. Let $J = \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}$ and $J' = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$, and note that $J = \ker \sigma$. We have $x \in D(Q) \backslash T(Q)$, since $A_\ell(x) = 0$ and $A_r(x) \supset J$. If $a \in T(R) \backslash N(R)$, then we clearly have $A_\ell(a) = J'[x;\sigma]$ and $A_r(a) = J[x;\sigma]$, so $a \in T(Q) \backslash S(Q)$, since $J \cap J' = \{0\}$. It remains to show that $S(Q) = N(Q)$.

Any $f(x) \in Q$ has a unique decomposition $f(x) = f_0(x) + f_1(x)$ with $f_0(x) \in R[x;\sigma]$ and $f_1(x) \in \mathsf{Z}[x]$. If $g(x) \in Q$, then $(fg)_1(x) = f_1(x)g_1(x)$, so $(f^n)_1(x) = (f_1(x))^n$; therefore, if $f(x) \in N(Q)$, then $f_1(x) = 0$. Thus if $f(x) \in N(Q)$, we may identify $f(x) = \Sigma \begin{bmatrix} p_i & v_i' \\ v_i & 0 \end{bmatrix} x^i$ with $\begin{bmatrix} \Sigma p_i x^i & \Sigma v_i' x^i \\ \Sigma v_i x^i & 0 \end{bmatrix}$. The $(1,1)$-entry of $f^n(x)$ is $(\Sigma p_i x^i)^n$; and since $f(x)$ is nilpotent, we have $\Sigma p_i x^i = 0$, so that $p_i = 0$ for all i. Thus $f(x)$ is a polynomial with coefficients in $N(R)$; and since the square of any such polynomial is 0, we can conclude that $N(Q) = N(R)[x;\sigma]$.

To prove $S(Q) = N(Q)$, we take $f(x) \in T(Q) \backslash N(Q)$ and show that $A_\ell(f(x)) = J'[x;\sigma]$ and $A_r(f(x)) = J[x;\sigma]$, so that $f(x) \notin S(Q)$. Our argument will make use of the following lemma:

LEMMA 1. *If* $f(x), g(x) \in Q\backslash\{0\}$ *and* $g(x)f(x) = 0$, *then either* $f_1(x) = 0$ *or* $A_\ell\big(f(x)\big) = 0$.

PROOF. Assume $f_1(x) = \Sigma m_i x^i \neq 0$, and let $g(x)f(x) = 0$ with $g(x) = g_0(x) + g_1(x)$, $g_0(x) \in R[x;\sigma]$, $g_1(x) \in \mathsf{Z}[x]$. Then $g_1(x)f_1(x) = 0$, so $g_1(x) = 0$ and $g(x) = \Sigma \begin{bmatrix} b_j & w_j' \\ w_j & 0 \end{bmatrix} x^j$. We have

$$0 = g(x)f(x) =$$

$$\begin{bmatrix} b_0 & w_0' \\ w_0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} a_i + m_i & v_i' \\ v_i & m_i \end{bmatrix} x^i + \Sigma_{j\geq 1} \begin{bmatrix} b_j & w_j' \\ w_j & 0 \end{bmatrix} x^j \Sigma \begin{bmatrix} a_i + m_i & v_i' \\ v_i & m_i \end{bmatrix} x^i \ ;$$

and since for $j \geq 1$ $x^j \begin{bmatrix} a_i & v_i' \\ v_i & 0 \end{bmatrix} = \begin{bmatrix} a_i & v_i' \\ 0 & 0 \end{bmatrix} x^j = x^j \begin{bmatrix} a_i & v_i' \\ 0 & 0 \end{bmatrix}$, we obtain

$$0 = \begin{bmatrix} b_0 & w_0' \\ w_0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma(a_i + m_i)x^i & \Sigma v_i' x^i \\ \Sigma v_i x^i & \Sigma m_i x^i \end{bmatrix}$$
$$+ \begin{bmatrix} \Sigma_{j\geq 1} b_j x^j & \Sigma_{j\geq 1} w_j' x^j \\ \Sigma_{j\geq 1} w_j x^j & 0 \end{bmatrix} \begin{bmatrix} \Sigma(a_i + m_i)x^i & \Sigma v_i' x^i \\ 0 & \Sigma m_i x^i \end{bmatrix}.$$

The $(1,1)$-entry is $\Sigma b_j x^j \Sigma(a_i + m_i)x^i = 0$; and since $\Sigma(a_i + m_i)x^i \neq 0$, we get $b_j = 0$ for all j. In a similar way, considering the $(2,1)$-entry, we get $w_j = 0$ for all j, hence $w_j' = 0$ for all j since $\Sigma m_i x^i \neq 0$. Thus, $A_\ell\big(f(x)\big) = 0$; and the lemma is established.

Returning to our main argument, let $f(x) \in T(Q)\backslash N(Q)$; and note that by Lemma 1, $f_1(x) = 0$. Let $g(x) \in A_r\big(f(x)\big)$. Since $0 \neq f(x) \in A_\ell\big(g(x)\big)$, Lemma 1 gives $g_1(x) = 0$; therefore,

$$0 = f(x)g(x) = \begin{bmatrix} a_0 & v_0' \\ v_0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma b_j x^j & \Sigma w_j' x^j \\ \Sigma w_j x^j & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} \Sigma_{i\geq 1} a_i x^i & \Sigma_{i\geq 1} v_i' x^i \\ \Sigma_{i\geq 1} v_i x^i & 0 \end{bmatrix} \begin{bmatrix} \Sigma b_j x^j & \Sigma w_j' x^j \\ 0 & 0 \end{bmatrix}.$$

Then $\Sigma a_i x^i \Sigma b_j x^j = 0$; but $\Sigma a_i x^i \neq 0$ since $f(x) \notin N(Q)$, so $b_j = 0$ for all j. Similarly we get $w_j' = 0$ for all j, so $g(x) \in J[x;\sigma]$ and therefore $A_r(f(x)) = J[x;\sigma]$.

Now consider $g(x) \in A_\ell(f(x))$, $g_0(x) = \Sigma \begin{bmatrix} b_j & w_j' \\ w_j & 0 \end{bmatrix} x^j$, $g_1(x) = \Sigma n_j x^j$. Calculating the product $g(x)f(x)$, we obtain $\Sigma(b_j + n_j)x^j \Sigma a_i x^i = 0$, so $\Sigma(b_j + n_j)x^j = 0$ and $b_j = 0$, $n_j = 0$ for all j. Similarly we get $w_j = 0$ for all j, so $g(x) \in J'[x;\sigma]$ and therefore $A_\ell(f(x)) = J'[x;\sigma]$. Thus $S(Q) = N(Q)$, hence for the ring Q we have $D \neq T \neq S = N$.

2. The sets S_1 and W.

Consider the following set of zero divisors:

$$S_1 = \{a \in S \mid A_\ell(a) = A_r(a)\}.$$

For $a \in S_1$, $A(a) = A_\ell(a) = A_r(a)$ is a nonzero ideal, so $S_1 = \{0\}$ if the ring is prime. It is easy to show that S_1 is power closed, however S_1 is not in general root closed. For example, if R is a prime ring which is not a domain, R has nonzero nilpotent elements but $S_1 = \{0\}$.

Given a subset V of a ring R, one may define its root closure by

$$\overline{V} = \{a \in R \mid a^n \in V \quad \text{for some} \quad n \geq 1\}.$$

Clearly $\overline{V}$ is root closed, it contains V, and it is the minimal root closed subset of R containing V. Moreover, if V is power closed, then $\overline{V}$ is power closed. Note that $N = \overline{\{0\}}$.

Now let $W = \overline{S}_1$. Since S is root closed and $S \supseteq S_1 \supseteq \{0\}$, we have $S \supseteq W \supseteq N$; and $W = N$ if and only if $S_1 \subseteq N$. We may have $S \underset{\neq}{\supset} W \underset{\neq}{\supset} N$ – for example in $R = M_2(Q) \oplus \mathsf{Z}_6$, where $(e_{11}, 0) \in S \backslash W$ and $(0,1) \in W \backslash N$. In any ring, W is power closed, since S_1 is power closed.

Recall that we consider only infinite rings with $D \neq \{0\}$. As mentioned in [3], S is infinite, and this result is improved as follows.

THEOREM 1. *If R is any infinite ring with $D \neq \{0\}$, then W is infinite.*

PROOF. If N is infinite, we are done, since $W \supseteq N$. Let N be finite and R semiprime. Then, by [2, Cor. 5], $R = R_1 \oplus R_2$ where R_1 is reduced and R_2 is finite; and clearly R_1 is infinite. If $R_2 \neq \{0\}$, then $S_1 \supseteq R_1$; and if $R_2 = \{0\}$, then $S_1 = D$. Thus S_1 is infinite and so is W.

Now assume N is finite and the prime radical $\mathscr{P}(R)$ is nonzero, and let $\overline{R} = R/\mathscr{P}(R)$. Then, again by [2, Cor. 5], $\overline{R}$ is a direct sum of a reduced ring and a finite ring; and we denote their inverse images in R by R_1 and R_2 respectively. We have $\overline{R} = \overline{R}_1 \oplus \overline{R}_2$, and R_1 has finite index in R, since $\overline{R}_1$ has finite index in $\overline{R}$. By [3, Lemma 1], $A = A(\mathscr{P}(R))$ is an ideal of$R$ of finite index, since $\mathscr{P}(R)$ is a finite ideal. It follows that $A_1 = A \cap R_1$ has finite index, so A_1 is infinite; and we proceed to prove that $A_1 \subseteq W$.

For any two elements u, v of a reduced ring, it is easy to see that if one of the products uv, vu, u^2v, vu^2 is 0, so are all others. Since $\overline{R}_1$ is reduced and $\overline{R} = \overline{R}_1 \oplus \overline{R}_2$, we get for $x \in R_1$, $y \in R$ the result that if one of the products $\overline{x}\,\overline{y}$,$\overline{y}\,\overline{x}$, $\overline{x}^2\overline{y}$, $\overline{y}\,\overline{x}^2$ is $\overline{0}$, so are all others; otherwise put, if one of the products

xy, yx, x^2y, yx^2 is in $\mathscr{P}(R)$, so are all others. It follows for $x \in A_1$, $y \in R$ that $x^2y = 0$ implies $yx \in \mathscr{P}(R)$, hence $yx^2 = 0$; and similarly $yx^2 = 0$ implies $x^2y = 0$. We have shown that $A_\ell(x^2) = A_r(x^2)$,so $x^2 \in S_1$ and $x \in W$. Thus $A_1 \subseteq W$, as we wished to prove.

COROLLARY. *If R is an infinite ring with $D \neq \{0\}$ and N finite, then S_1 is infinite.*

PROOF. The case when R is semiprime is considered at the beginning of the proof of Theorem 1.

If R is not semiprime, we have seen in the proof of Theorem 1 that $x^2 \in S_1$ if $x \in A_1$. Now A_1 is an infinite subring of R_1, so $\overline{A}_1$ is an infinite reduced ring. Assuming S_1 is finite, we have that $\{x^2 \mid x \in A_1\}$ is finite, so $\{\overline{x}^2 \mid \overline{x} \in \overline{A}_1\}$ is finite. But by [1, Th. 4.1] it follows that $\overline{A}_1$ is finite -- a contradiction.

3. The sets $W \backslash N$ and $S_1 \backslash N$.

Since N, W, S are power closed and root closed, so are $W \backslash N$ and $S \backslash W$.

From now on, the results will be stated without saying that it is assumed that R is infinite and $D \neq \{0\}$. The center of R is denoted by Z.

LEMMA 2. *If $e \in W$ is an idempotent, then $e \in Z$.*

PROOF. We have $e = e^2 = \cdots$, so $e \in S_1$ and $A_\ell(e) = A_r(e)$. Since $e(xe - exe) = 0$ and $(ex - exe)e = 0$, we obtain $xe - exe = 0$ and $ex - exe = 0$, so $xe = ex$.

THEOREM 2. *If $W \backslash N$ is nonempty, then it is infinite.*

PROOF. If N is finite, the result follows by Theorem 1.

Let N be infinite and $a \in W \backslash N$. Then $a^m \in W \backslash N$ for any $m \geq 1$, so if a has infinitely many distinct powers, we are done. Otherwise some power of a is a nonzero idempotent e, and $e \in Z$ by Lemma 2.

Now $ne \in Z$ for any integer n, hence $ne \in W$; thus, if e has infinite additive order, we are done. Assume $ke = 0$ for some $k > 1$. Since N is infinite, there are infinitely many elements squaring to 0 [2, Th. 6]; and for each such element u, $(e + u)^k = e + keu = e$. Therefore the infinite set $\{e + u \mid u^2 = 0\}$ is contained in $W \backslash N$.

THEOREM 3. *If R has 1 and $S_1 \backslash N$ is nonempty, then $S_1 \backslash N$ is infinite.*

PROOF. When N is finite, the result follows from the corollary in the previous section. When N is infinite we follow the arguments given in the proof of Theorem 2, starting with $a \in S_1 \backslash N$ and obtaining an idempotent $e \in S_1 \backslash N$

with $ke = 0$ for some $k > 1$. By [2, Th. 6] R has an infinite zero subring U, and either eU or $(1-e)U$ is infinite. Since R has 1, $1-e$ is an idempotent belonging to $S_1 \backslash N$, so we may assume without loss that eU is infinite. For $u \in U$ we have $(e+eu)^k = e$, so $e + eu \notin N$ and $A_\ell(e+eu) = A_\ell(e)$. Similarly $A_r(e+eu) = A_r(e)$, so $e + eu \in S_1$ since $e \in S_1$. Thus $S_1 \backslash N$ contains the infinite set $e + eU$.

4. The set $S\backslash W$.

We start with an example showing that $S\backslash W$ may be finite and nonempty.

Let Z_p be the field of p elements, let C_p be the zero ring on the cyclic group of order p with generator u, and let J be an infinite domain. Let $R = \mathsf{Z}_p \times C_p \times J$ with addition as in $\mathsf{Z}_p \oplus C_p \oplus J$ and multiplication determined by $eu = u$, $ue = 0$, $eJ = Je = 0$ and $uJ = Ju = 0$. This gives a ring structure on R. We have $D = R$ since $uR = 0$; and $T = S = (\mathsf{Z}_p \oplus C_p) \cup (C_p \oplus J)$. Now if $0 \neq a \in J$, then $A_\ell(a) = A_r(a) = \mathsf{Z}_p \oplus C_p$, so $J \subseteq S_1 \subseteq W$; and since $C_p^2 = 0$ we have $C_p \oplus J \subseteq W$, and it follows easily that $C_p \oplus J = W$. Thus $S\backslash W = (\mathsf{Z}_p \backslash \{0\}) \oplus C_p$ is finite and nonempty.

In the previous example, N is finite. We now proceed to consider $S\backslash W$ when N is infinite. We start with a simple result, which holds in arbitrary rings.

LEMMA 3. (1) *Let e be a noncentral idempotent. Then either there is an element $v \neq 0$ satisfying $ev = v$, $ve = v^2 = 0$, or there is an element $u \neq 0$ satisfying $ue = u, eu = u^2 = 0$.*

(2) *If for an element $v \neq 0$ $(u \neq 0)$ there is an element a satisfying $av = v, va = 0$ $(ua = u,\ au = 0)$, then $a \in D\backslash W$.*

PROOF. (1) Since $e \notin Z$, we have $eR(1-e) \neq 0$ or $(1-e)Re \neq 0$. If $eR(1-e) \neq 0$, take $v \neq 0$ in $eR(1-e)$; otherwise take $u \neq 0$ in $(1-e)Re$.

(2) By symmetry it suffices to prove the result for v. We have $a \in D$ since $v \neq 0$; and $a^k v = v$, $va^k = 0$ for any $k \geq 1$, so $a \notin W$.

THEOREM 4. *If N is infinite and $S\backslash W$ is nonempty, then $S\backslash W$ is infinite.*

PROOF. As in the proof of Theorem 2, we may assume there is an idempotent $e \in S\backslash W$. Then $e \notin Z$; and applying Lemma 3, we may assume there is an element $v \neq 0$ satisfying $ev = v$, $ve = v^2 = 0$. As in the proof of Theorem 3, we let U be an infinite zero subring; and we consider separately the two cases: (1) eUe infinite, (2) eUe finite.

In case (1) eU is infinite; and if $eu \in eU\backslash N$, $u \in U$, then for $k \geq 1, (eu)^k u = 0 \neq u(eu)^k$, so $eu \notin W$. Since $e \in S$, $A(e) \neq 0$; and if $0 \neq b \in A(e)$, then $b \in A(eu)$ when $ub = 0$ and $ub \in A(eu)$ when $ub \neq 0$, so $eu \in S\backslash W$. Thus we may assume $eU\backslash N$ is finite and therefore $eU \cap N$ is infinite. It follows that there are infinitely many elements of the form $e + eue$ where $eu \in eU \cap N$. Clearly they are all in S, and we prove that none is in W. We have $v(e + eue)^k = 0$ for all $k \geq 1$. On the other hand, if $(e + eue)^k v = 0$ for some k, then since e commutes with eue and $ev = v$, we have

$$(*) \qquad 0 = v + \Big(\sum_{i=1}^{k} \binom{k}{i} (eu)^i\Big) v \, .$$

Since $eu \in N$, $\sum_{i=1}^{k} \binom{k}{i} (eu)^i \in N$; and it follows from $(*)$ that $v = 0$ -- a contradiction.

In case (2), if eU and Ue are finite, then each of $A_\ell(e) \cap U$ and $A_r(e) \cap U$ has finite index in $(U, +)$ and so does $A(e) \cap U$, which is therefore infinite. It follows that either $v(A(e) \cap U)$ is infinite or $A(e) \cap U \cap A_r(v)$ is infinite. If $v(A(e) \cap U)$ is infinite, we have infinitely many elements of the form $e + vu$ where $u \in A(e) \cap U$. For each such element $v(e + vu) = 0$ and $(e + vu)v = v$, so $e + vu \notin W$; moreover, $e + vu \in S$, since $u \in A(e + vu)$. If $A(e) \cap U \cap A_r(v)$ is infinite, then for any u in this set we have $(e + u)v = v$ and $v(e + u) = 0$, so $e + u \notin W$; and also $u \in A(e + u)$, so $e + u \in S\backslash W$.

It remains to consider case (2) with eU infinite or Ue infinite. If eU is infinite, then $eU \cap A_\ell(e)$ is infinite, since eUe is finite. For any nonzero element $eu \in eU \cap A_\ell(e)$, $u \in U$, we have $eu(e + eu) = 0$ and $(e + eu)eu = eu$, so $e + eu \notin W$. If $0 \neq b \in A(e)$, then $0 \neq b - ub \in A(e + eu)$, hence $e + eu \in S$. In a similar way, when Ue is infinite, the infinite set $e + Ue \cap A_r(e)$ is contained in $S\backslash W$. This completes the proof of Theorem 4.

We have seen that $S\backslash W$ may be nonempty and finite. However, we have

THEOREM 5. *If R has 1 and $S \neq W$, then $S\backslash W$ is infinite.*

PROOF. We may assume N is finite and, as in the proof of Theorem 4, let e be an idempotent in $S\backslash W$ and v a nonzero element satisfying $ev = v$, $ve = v^2 = 0$. Using the notation as in the proof of Theorem 1, we have $\overline{R} = R/\mathscr{P}(R) = \overline{R}_1 \oplus \overline{R}_2$, where $\overline{R}_1$ is infinite and reduced and $\overline{R}_2$ is finite. For $\overline{x} \in \overline{R}$ write $\overline{x} = \overline{x}_1 + \overline{x}_2, \overline{x}_i \in \overline{R}_i$. Letting $\overline{e} = \overline{e}_1 + \overline{e}_2$, we observe that $\overline{e}_1$ is a central idempotent in $\overline{R}_1$ and $[\overline{e}, \overline{x}] = [\overline{e}_2, \overline{x}_2]$; and since $\mathscr{P}(R)$ and $\overline{R}_2$ are finite, we see that there are only finitely many commutators of the form $ex - xe$, $x \in R$. Therefore $C = C_R(e)$ is of finite index in R, hence infinite.

Assume eC is infinite. We have $eCv \subseteq N$, so eCv is finite, and hence $eC \cap A_\ell(v)$ is infinite. For any $u \in eC \cap A_\ell(v)$, $(e + u)v = v$ and $v(e + u) = 0$, so $e + u \notin W$; and since $u \in eC$ and C commutes with e, we see that $A(e) \subseteq A(e + u)$, so $e + u \in S\backslash W$.

If eC is finite, then $(1 - e)C$ is infinite; also, $1 - e$ is an idempotent not in

W and $(1-e)v=0$, $v(1-e)=v$. Thus we may replace e by $1-e$ and proceed as above.

If $D\backslash W$ is nonempty, then at least one of the sets $S\backslash W$, $T\backslash S$, $D\backslash T$ is nonempty; hence by Theorem 4 and [3, Th. 4, Th. 5], if N is infinite, then $D\backslash W$ is infinite. This conclusion may be established directly without assuming that N is infinite.

Theorem 6. *If $D\backslash W$ is nonempty, then it is infinite.*

Proof. As before, we may assume there is an idempotent e in $D\backslash W$ and an element $v \neq 0$ satisfying $ev = v$, $ve = v^2 = 0$.

Let K be the kernel of the map $a \mapsto vav$ from R onto vRv; and note that vRv, Kv, $vK \subseteq A(v)$. Thus, if one of vRv, Kv, vK is infinite, then $A(v)$ is infinite. On the other hand, if all three are finite, then K is infinite and $K \cap A_\ell(v)$ and $K \cap A_r(v)$ have finite index in K, in which case $K \cap A(v)$ has finite index in K. Thus, in any event $A(v)$ is infinite.

Now for $u \in A(v)$ we have $(e+u)v = v$ and $v(e+u) = 0$, so the infinite set $e + A(v)$ is contained in $D\backslash W$.

Note that the example given at the beginning of this section shows that in the above theorem D cannot be replaced by T.

We close the paper by improving a result of Lanski [4, Th. 6], which states that the cardinal number of T equals that of R. Our result is:

Theorem 7. $\mathrm{Card}(S) = \mathrm{Card}(R)$.

Proof. Simply repeat Lanski's proof with T replaced by S. For the convenience of the reader it is suggested to replace S, W appearing in Lanski's proof by N, K respectively.

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