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ALGEBRAS WITH VANISHING $\mathrm{Ext}^2(X, X)$ FOR INDECOMPOSABLE MODULES

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Let k be an algebraically closed field and A be a finite dimensional k-algebra. We denote by mod_A the category of finitely generated left A-modules. Recall that A is said to be representation-finite if there are only finitely many indecomposable A-modules up to isomorphism. The algebra A is tame if the indecomposables occur, in each dimension, in a finite number of discrete and a finite number of one-parameter families. If the number of discrete families growths polynomially with the dimension, then A is said to be of polynomial growth. See [11,15,17] and section 1 for these concepts.

In this work we shall say that A *satisfies* the condition (E^s) for some $s \in \mathsf{N}$ if $\mathrm{Ext}^s_A(X, X) = 0$ holds for every indecomposable A-module X. Important classes of algebras satisfying (E^s) for some s have been studied. If (E^1) is satisfied, then A is representation-finite [9,10]. Tilted, and more generaly, quasi-tilted algebras satisfy (E^2) [7,15]. Strongly simply connected algebras of (tame) polynomial growth satisfy (E^2) [12]. In this paper we study tame algebras satisfying (E^2).

Let A be a basis connected finite dimensional k-algebra. Then A has a presentation $A = kQ/I$, where $Q = (Q_0, Q_1)$ is the ordinary quiver of A with set of vertices (resp. arrows) Q_0 (resp. Q_1). By $\mathrm{mod}_A(v)$ we denote the variety of A-modules with dimension vector v. We recall from [6] that the condition $\mathrm{Ext}^s_A(X, X) = 0$ for some module $X \in \mathrm{mod}_A(v)$ implies the existence of an open neighborhood $\mathscr{U}$ of X such that $\mathrm{Ext}^s_A(Y, Y) = 0$ for any $Y \in \mathscr{U}$.

The main results of the paper are the following

Theorem 1. *Let A be an algebra satisfying (E^2). Then the following are equivalent:*

(a) *A is tame;*

(b) *for every $v \in \mathsf{N}^{Q_0}$, there is an open and dense subset $\mathscr{U}$ of $\mathrm{mod}_A(v)$ such that for any $X \in \mathscr{U}$, $\dim_k \mathrm{Ext}^1_A(X, X) \le \dim_k X$ holds.*

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Moreover, in this case the following property is satisfied:

(c) $\dim_k \operatorname{End}_A(X) - \dim_k \operatorname{Ext}^1_A(X,X) \geq 0$, *for every indecomposable* $X \in \operatorname{mod}_A$.

An algebra A is *strongly simply connected* if every convex subcategory B of A satisfies that the first Hochschild cohomology group $H^1(B,B)$ vanishes, [16]. Strongly simply connected algebras of polynomial growth have been extensively studied, see [12,17].

THEOREM 2. *Let A be a strongly simply connected algebra satisfying (E^2). The following are equivalent:*

(a) *A is of polynomial growth.*

(b) *A is tame.*

(c) *For every indecomposable $X \in \operatorname{mod}_A$, we have* $\dim_k \operatorname{Ext}^1_A(X,X) \leq \dim_k X$.

(d) *For every indecomposable $X \in \operatorname{mod}_A$, we have* $\dim_k \operatorname{End}_A(X) \leq \dim_k X$.

We shall prove the theorem and some consequences in section 2, after some general remarks in section 1. In section 3 we shall consider some properties of the structure of the Auslander-Reiten quiver Γ_A of tame algebras A satisfying (E^2). In section 4 we give some examples.

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1. Module varieties.

1.1. Let $A = kQ/I$ be a finite dimensional k-algebra. A module $X \in \operatorname{mod}_A$ will be considered as a representation of Q satisfying the ideal I, see [4]. The dimension vector $X = (\dim_k X(i))_{i \in Q_0}$ is the class of X in the Grothendieck group $K_0(A) \cong \mathbb{Z}^{Q_0}$.

We denote by $\operatorname{mod}_A(v)$ the closed subset of $k^v := \prod_{(i \xrightarrow{v} j) \in Q_0} k^{v(j) \times v(i)}$ of those tuples $(X(\alpha))_{\alpha \in Q_1}$ satisfying the relations imposed by I. The set $\operatorname{mod}_A(v)$ is called the *variety of modules* of dimension v. The affine group $G(v) = \prod_{i \in Q_0} GL_{v(i)}(k)$ acts on $\operatorname{mod}_A(v)$ in such a way that the orbits form the isoclasses of A-modules. The indecomposable modules in $\operatorname{mod}_A(v)$ form the constructible set $\operatorname{ind}_A(v)$.

The following lemma is well-known, see for example [6].

LEMMA. a) *For given $v \in \mathbb{N}^{Q_0}$ and $s \in \mathbb{N}$, the function*

$$e^s : \operatorname{mod}_A(v) \to \mathbb{N},\ X \mapsto \dim_k \operatorname{Ext}^s_A(X,X)$$

is upper semicontinuous.

b) *If* $\mathrm{Ext}^s_A(X,X)=0$ *for some* $X\in \mathrm{mod}_A(v)$, *there exists an open neighborghood of* X *in* $\mathrm{mod}_A(v)$ *and an integer* c_X *such that for all* $Y\in\mathcal{U}$ *we have,*

(i) $\mathrm{Ext}^s_A(Y,Y)=0$;

(ii) $\sum_{i=0}^{s-1}(-1)^i\dim_k \mathrm{Ext}^i_A(Y,Y)=c_X$

1.2. We recall that as examples of algebras satisfying (E^1) we have the representation-finite algebras A whose Auslander-Reiten quiver Γ_A has no oriented cycles.

Lemma. [9] *If* A *satisfies* (E^1), *then* A *is representation-finite.*

We recall the argument of the *proof* as an opportunity to introduce some concepts.

Let $X\in\mathrm{mod}_A(v)$. By T_X we denote the tangent space to $\mathrm{mod}_A(v)$ at the point X and by T^0_X the tangent space to the orbit $G(v)X$ at X. By Voigt's theorem [18] (see also [11]), there is a vector space embedding, $T_X/T^0_X\hookrightarrow \mathrm{Ext}^1_A(X,X)$. In case $\mathrm{Ext}^1_A(X,X)=0$, then $\dim T_X=\dim G(v)X$ which implies that $G(v)X$ is open in $\mathrm{mod}_A(v)$. Obviously this may happen only for finitely many $G(v)$-orbits in $\mathrm{mod}_A(v)$. The result follows.

1.3. Using the scheme of modules $\underline{\underline{\mathrm{mod}}}_{A(v)}$, the following is shown.

Proposition. [6] *Let* $X\in\mathrm{mod}_A(v)$ *be a module satisfying* $\mathrm{Ext}^2_A(X,X)=0$, *then the following happens:*

(i) $\mathrm{mod}_A(v)$ *is smooth at* X;

(ii) *the inclusion* $T_X/T^0_X\hookrightarrow \mathrm{Ext}^1_A(X,X)$ *is an isomorphism.*

Corollary. *For* $X\in\mathrm{mod}_A(v)$ *satisfying* $\mathrm{Ext}^2_A(X,X)=0$, *the following equality holds:*

$$\dim G(v)-\dim_X \mathrm{mod}_A(v)=\dim_k \mathrm{End}_A(X)-\dim \mathrm{Ext}^1_A(X,X)$$

Proof. Since $\mathrm{mod}_A(v)$ is smooth at X, then $\dim_X\mathrm{mod}_A(v)=\dim T_X$. Since the orbits are homogeneous spaces, then $\dim G(v)-\dim_k\mathrm{End}_A(X)=\dim G(v)X=\dim T^0_X$. Then the result follows from (ii) above.

1.4. An algebra A is *tame* if for every $d\in\mathbb{N}$ there is a finite number of $A-k[T]$-bimodules $M_1,\dots,M_{S(d)}$ which are free as right $k[T]$-modules and such that for almost every indecomposable A-module X with dimension d, X is isomorphic to $M_i\otimes_{k[T]}k[T]/(T-\lambda)$ for some $1\le i\le s(d)$ and some $\lambda\in k$. In this case we denote $\mu(d)$ the minimal number $s(d)$ in the definition. We say that A is *domestic* (resp. *of polynomial growth*) if there is a constant $m\in\mathbb{N}$ such that $\mu(d)\le m$ (resp. $\mu(d)\le d^m$) for all $d\in\mathbb{N}$.

For a tame algebra A the following is known:

(i) [3] for every $v \in \mathbb{N}^{Q_0}$, almost all $X \in \mathrm{ind}_A(v)$ lie in homogeneous tubes of Γ_A;

(ii) [9] for every $v \in \mathbb{N}^{Q_0}$, the inequality $\dim G(v) - \dim \mathrm{mod}_A(v) \geq 0$ holds.

For $v \in \mathbb{N}^{Q_0}$ and $t \in \mathbb{N}$, let $\mathrm{mod}_A(v,t) = \{X \in \mathrm{mod}_A(v) \colon \dim G(v)X = t\}$ which by (1.1) is a closed subset of $\mathrm{mod}_A(v)$. By [5], A is tame if and only if $\dim \mathrm{mod}_A(v,t) \leq |v| + t$, for every $v \in \mathbb{N}^{Q_0}$ (here $|v| = \sum_{i \in Q_0} v(i)$).

1.5. For a module $X \in \mathrm{mod}_A$, let $\cdots \to P_1(X) \xrightarrow{p_1} P_0(X) \xrightarrow{p_0} X \to 0$ be a minimal projective resolution and let $\Omega^{i+1}(X) = \ker p_i$ be the corresponding syzygies.

For any $Y \in \mathrm{mod}_A$, Auslander and Reiten [2] showed the following formula:

$$\begin{aligned}\dim_k \mathrm{Hom}_A(X,Y) - \dim_k \mathrm{Hom}_A(Y, \tau_A X) = \\ = \dim_k \mathrm{Hom}_A(P_0(X), Y) - \dim \mathrm{Hom}_A(P_1(X), Y).\end{aligned}$$

2. On algebras satisfying (E^2).

2.1. We recall some *examples* of algebras satisfying (E^2).

(a) Obviously, hereditary algebras $A = k\Delta$ (which satisfy $\mathrm{g}\ell \dim A \leq 1$) have property (E^2). More generaly, tilted algebras A satisfy that for every indecomposable A-module X, either $p\dim_A X \leq 1$ or $i\dim_A X \leq 1$, hence (E^2) holds.

(b) An algebra A is said to be *quasi-tilted* if $\mathrm{g}\ell \dim A \leq 2$ and for every indecomposable A-module X, either $p\dim_A X \leq 1$ or $i\dim_A X \leq 1$, see [7]. Thus these algebras satisfy (E^2).

(c) For strongly simply connected algebras the main result in our context is the following.

THEOREM. [12] *Let A be a strongly simply connected algebra. Then the following are equivalent:*

(a) *A is of polynomial growth.*

(b) *For every* $v \in \mathbb{N}^{Q_0}$ *and every indecomposable* $X \in \mathrm{mod}_A(v)$, $\dim_k \mathrm{Ext}^1_A(X,X) \leq \dim_k \mathrm{End}_A(X)$ *and* $\mathrm{Ext}^2_A(X,X) = 0$.

Moreover, if this holds, then $\mathrm{Ext}^s_A(X,X) = 0$ *for every* $v \in \mathbb{N}^{Q_0}$, $X \in \mathrm{ind}_A(v)$ *and every* $s \geq 2$.

COROLLARY. *Let A be a strongly simply connected algebra satisfying* (E^2). *Then the following are equivalent:*

(a) *A is of polynomial growth.*
(b) *For every* $X \in \mathrm{ind}_A$, *we have* $\dim_k \mathrm{Ext}^1_A(X,X) \leq \dim \mathrm{End}_A(X)$.
(c) *A is tame.*

PROOF. Obviously, it is enough to show (c) $\Rightarrow$ (a). By [17], A is of polynomial growth if it does not contain a convex subcategory B which is either hypercritical or pg-critical. A hypercritical algebra B is not tame. Moreover, in [14] it was shown that pg-critical algebras do not satisfy (E^2). Therefore, A is of polynomial growth.

In section 4 we will show more examples.

2.2. We shall prove our characterization of algebras satisfying (E^2).
Proof of Theorem 1: Implication (b) $\Rightarrow$ (a) was shown in [10]. Nevertheless it follows as part of the following argument. Assume first that A is tame.
Let $v \in \mathbb{N}^{Q_0}$ and C be an irreducible component of $\mathrm{mod}_A(v)$. Let $t \in \mathbb{N}$ be such that $C \cap \mathrm{mod}_A(v,t)$ is dense in $\mathrm{mod}_A(v)$.
Assume first that $C \cap \mathrm{ind}_A(v)$ is dense in C. Then there is an open and dense subset $\mathcal{U}$ of C such that every $Y \in \mathcal{U}$ satisfies $\mathrm{Ext}^2_A(Y,Y) = 0$ and $\dim G(v) Y = t$. By (1.3) and (1.4), the following holds for any $Y \in \mathcal{U}$

$$\begin{aligned}\dim_k \mathrm{Ext}^1_A(Y,Y) &= \dim_k \mathrm{End}_A(Y) - \dim G(v) + \dim C \\ &\leq -t + (|v| + t) = |v| = \dim_k Y.\end{aligned}$$

In the general case, consider the *generic decomposition* $v = \sum_{i=1}^{s} w_i$ of v in C, [9]. That is, $w_1, \ldots, w_s \in \mathbb{N}^{Q_0}$ and the following conditions hold:

(i) $\mathcal{V} = \{Y \in C\colon Y = \bigoplus_{i=1}^{s} Y_i$ with $Y_i \in \mathcal{U}_i\}$ is open and dense in C, where $\mathcal{U}_i$ is an open subset of $\mathrm{mod}_A(w_i)$ formed by indecomposable modules;

(ii) if $Y = \bigoplus_{i=1}^{s} Y_i \in \mathcal{V}$, with $Y_i \in \mathcal{U}_i$, then $\mathrm{Ext}^1_A(Y_i, Y_j) = 0$ for $i \neq j$.

For $Y = \bigoplus_{i=1}^{s} Y_i \in \mathcal{V}$, we get by the first case, $\dim_k \mathrm{Ext}^1_A(Y,Y) \leq \sum_{i=1}^{s} \dim_k \mathrm{Ext}^1_A(Y_i, Y_i) \leq \sum_{i=1}^{s} \dim_k Y_i = \dim_k Y$. We are done.
By [9] and (1.3), for every $X \in \mathrm{ind}_A(v)$ the following holds:

$$0 \leq \dim G(v) - \dim_X \mathrm{mod}_A(v) = \dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Ext}^1_A(X,X).$$

2.3. PROPOSITION. *Let A be a tame algebra satisfying* (E^2). *Then for every* $X \in \mathrm{ind}_A(v)$ *with* $\tau_A X \cong X$, *there is an open neighborhood* $\mathcal{U}$ *of X such that for all* $Y \in \mathcal{U}$, *the following equality holds:*

$$\dim_k \mathrm{End}_A(Y) - \dim_k \mathrm{Ext}^1_A(Y, Y) = \dim_k \mathrm{Hom}_A(\Omega^2(X), X).$$

PROOF. Let $X \in \mathrm{ind}_A(v)$ with $\tau_A X \cong X$ and let $\mathscr{U}$ be as in (2.2). We get the two exact sequences

$$0 \to \mathrm{End}_A(X) \to \mathrm{Hom}_A(P_0(X), X) \to \mathrm{Hom}_A(\Omega^1(X), X) \to \mathrm{Ext}^1_A(X, X) \to 0$$
$$0 \to \mathrm{Hom}_A(\Omega^1(X), X) \to \mathrm{Hom}_A(P_1(X), X) \to \mathrm{Hom}_A(\Omega^2(X), X)$$
$$\to \mathrm{Ext}^1_A(\Omega^1(X), X) \cong \mathrm{Ext}^2_A(X, X) = 0.$$

Hence by (1.5), we get

$$\dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Ext}^1_A(X, X) =$$
$$= \dim_k \mathrm{Hom}_A(P_0(X), X) - \dim_k \mathrm{Hom}_A(P_1(X), X) +$$
$$+ \dim_k \mathrm{Hom}_A(\Omega^2(X), X) = \dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Hom}_A(X, \tau_A X) +$$
$$+ \dim_k \mathrm{Hom}_A(\Omega^2(X), X)$$

and the result follows.

COROLLARY. *Assume that A is a tame algebra satisfying (E^2). Then for any $X \in \mathrm{ind}_A(v)$ with $\tau_A X \cong X$, there is an open subset $\mathscr{U}$ of $\mathrm{mod}_A(v)$ such that for any $Y \in \mathscr{U}$, the following inequality holds:*

$$\dim_k \mathrm{End}_A(Y) \leq \dim_k Y + \dim_k \mathrm{Hom}_A(\Omega_2(Y), Y).$$

2.4. In our proof of Theorem 2 we shall use results on the structure of module categories of polynomial growth algebras proved in [17]. Namely, any indecomposable module X over a polynomial growth strongly simply connected algebra A is either directing or it lies in the coil of a multicoil component of the Auslander-Reiten quiver Γ_A. Any such coil $\mathscr{C}$ is obtained from a tube by a sequence of admissible operations as defined in [1]. This component $\mathscr{C}$ is standard and the *rank* of $\mathscr{C}$ is defined as the number of modules in the mouth of $\mathscr{C}$.

LEMMA. *Let $\mathscr{C}$ be a coil in the Auslander-Reiten quiver Γ_B of a polynomial growth strongly simply connected algebra B. Let $X \in \mathscr{C}$ be a module with a maximal sectional path*

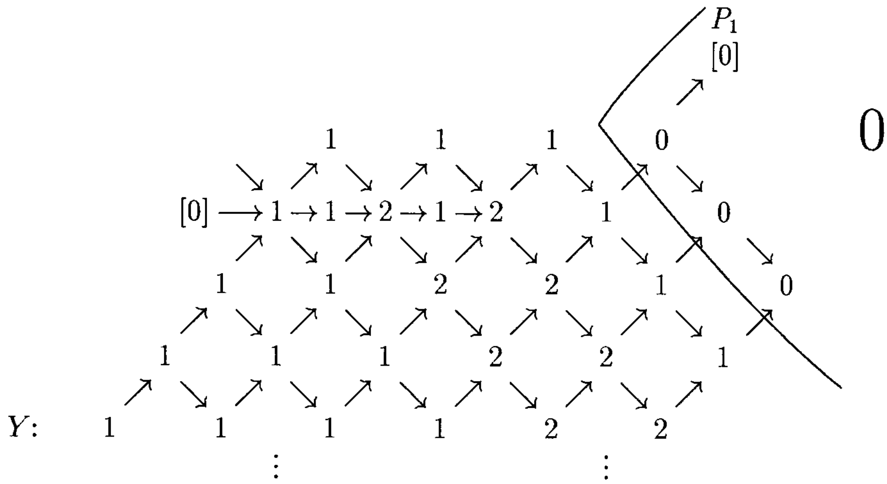
$$X = X_1 \to X_2 \to \cdots \to X_m$$

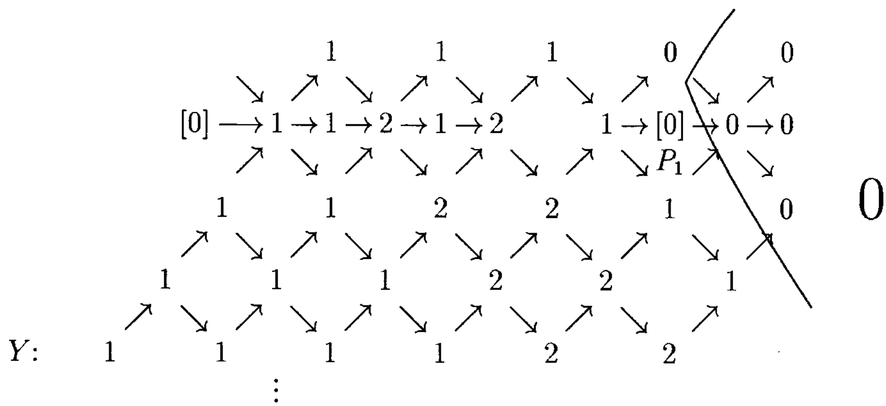
in $\mathscr{C}$ (then X_m is in the mouth of $\mathscr{C}$). Let $m = sp + t$ with $0 \leq t < p$, where p is the rank of $\mathscr{C}$. Then

$$\dim_k \mathrm{Ext}^1_B(X, X) \leq s + 2.$$

PROOF. Recall that $\mathrm{Ext}_B^1(X,X) \cong D\,\underline{\mathrm{Hom}}_B(\tau^-{}_B X, X)$ and let $Y = \tau_B^- X$. We shall consider the values of $\dim_k \underline{\mathrm{Hom}}_B(Y,-)$ in $\mathscr{C}$. For this purpose we shall use that $\mathscr{C}$ is a standard component of Γ_B, [16].

Let $P_1, \ldots, P_t$ be all projective modules in the mouth of $\mathscr{C}$. By [17, 4.5], we may assume that $\mathrm{Hom}_B(P_i, P_j) \neq 0$ implies $i < j$. Moreover, any projective P in $\mathscr{C}$ which is not in the mouth of $\mathscr{C}$, is injective. Consider a Galois covering of translation quivers $\pi\colon \Delta \to \mathscr{C}$ defined by the action of an infinite cyclic group. Fix $\pi(Y_0) = Y$ and $X_0 = \tau_\Delta^- Y_0$. Examples of the values of $\dim_k \underline{\mathrm{Hom}}(Y_0, -)$ in Δ are the following:





In general, we observe that for the mesh category $k(\Delta)$ we have:

(a) $\dim_k \underline{\mathrm{Hom}}_{k(\Delta)}(Y_0, \tau_\Delta^{-i} X_0) = 1$ for $1 \leq i \leq m$;

(b) $\dim_k \underline{\mathrm{Hom}}_{k(\Delta)}(Y_0, \tau_\Delta^{-i} X_0) \leq 2$ for $m+1 \leq i \leq m+p$;

(c) $\dim_k \underline{\mathrm{Hom}}_{k(\Delta)}(Y_0, \tau_\Delta^{-i} X_0) = 0$ for $i > m+p$.

Since $\tau_B^{-i}X = X$ for $1 \leq i \leq m$ at most s times and $\tau_B^{-i}X = X$ for $m+1 \leq i \leq m+p$ at most once, then

$$\dim_k \mathrm{Ext}_B^1(X,X) \leq \sum \dim_k \underline{\mathrm{Hom}}_{k(\Delta)}(Y_0, \tau_\Delta^{-i}X_0) \leq s+2.$$

2.5. *Proof of Theorem 2*: In (2.1), we proved the equivalence of (a) and (b). By Theorem 1, we have (c) $\Rightarrow$ (b). Moreover, by [10] we have (d) $\Rightarrow$ (b) and then by Theorem 1 we get (d) $\Rightarrow$ (c).

(a) $\Rightarrow$ (d): Let X be an indecomposable A-module. If X is directing, then $\dim_k \mathrm{End}_A(X) = 1$. Assume that X is not directing, then there exists a convex coil subcategory B of A and a coil $\mathscr{C}$ of Γ_B such that X lies on $\mathscr{C}$.

Let C be the tame concealed algebra which is a convex subcategory of B such that $\mathscr{C}$ is obtained from a tube $\mathscr{T}$ of Γ_C by a sequence of admissible operations. Let p be the rank of $\mathscr{C}$. Consider a sectional path

$$X = X_1 \to X_2 \to \cdots \to X_m$$

in $\mathscr{C}$ such that X_m is at the mouth of $\mathscr{C}$. Write $m = sp + t$ with $0 \leq t < p$, then by (2.4) we have $\dim_k \mathrm{Ext}_A^1(X,X) \leq s+2$.

On the other hand, let $e(X)$ be the number of projective-injective indecomposable A-modules P such that $P \in \mathscr{C}$ and $\mathrm{Hom}_A(P,X) \neq 0$. By [13], we have $\dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Ext}_A^1(X,X) \leq e(X) + 2$. Altogether we get, $\dim_k \mathrm{End}_A(X) \leq s + e(X) + 4$.

Since C is a tame concealed simply connected algebra, the sum of the dimensions of the modules in the mouth of $\mathscr{T}$ is at least 5. Hence, $\dim_k X \geq 5s + e(X)$. If $s \geq 1$, we get $\dim_k \mathrm{End}_A(X) \leq \dim_k X$ as desired. If $s = 0$, then the arguments given in [13, (3.3)] show that $\dim_k \mathrm{End}_A(X) \leq 1$.

2.6. We recall that the *Tits form* of A is the quadratic form $q_A: \mathbb{Z}^{Q_0} \to \mathbb{Z}$ such that

$$q_A(v) = \sum_{i \in Q_0} v(i)^2 - \sum_{(i \to j) \in Q_1} v(i)v(j) + \sum_{i,j \in Q_0} \dim_k \mathrm{Ext}_A^2(S_i, S_j)v(i)v(j),$$

where S_i denotes the simple module corresponding to the vertex i of Q, see [11].

For A a tame algebra, q_A is weakly non-negative, [9]. From [12] we get that for A strongly simply connected satisfying (E^2), A is tame if and only if q_A is weakly non-negative. We obtain also the more general result.

Proposition. *Let $F: \tilde{A} = k\tilde{Q}/\tilde{I} \to A = kQ/I$ be a Galois covering defined by the action of the group G. Assume that* (i) *A satisfies* (E^2) *and* (ii) *$\tilde{A}$ is strongly simply connected. Then the following are equivalent:*

(a) *$\tilde{A}$ is tame.*

(b) *$\tilde{A}$ is of polynomial growth.*

(c) *A is tame.*
(d) *The Tits form $q_{\tilde{A}}: \mathbb{Z}^{(\tilde{Q}_0)} \to \mathbb{Z}$ is weakly non-negative.*
(e) *$\tilde{A}$ does not contain any hypercritical convex subcategory.*

For concepts not defined before, see [11,17].

Proof. Let $F_\lambda: \mathrm{mod}_{\tilde{A}} \to \mathrm{mod}_A$ be the push-down functor. By [17], G is torsion-free and F_λ preserves indecomposable modules. For any $X \in \mathrm{ind}_{\tilde{A}}$ we get

$$0 = \mathrm{Ext}_A^2(F_\lambda X, F_\lambda X) \cong \bigoplus_{g \in G} \mathrm{Ext}_{\tilde{A}}^2(X, X^g),$$

where X^g is the shift of X defined by the action of G on $\mathrm{mod}_{\tilde{A}}$. Therefore, $\mathrm{Ext}_{\tilde{A}}^2(X, X) = 0$ and $\tilde{A}$ satisfies (E^2). The equivalence of (a), (b), (d) and (e) follows from [12], see (2.1). The equivalence of (a) and (c) follows from [17].

3. On the structure of the Auslander-Reiten quiver of a tame algebra satisfying (E^2).

3.1. Recall that the number of arrows from X to Y in Γ_A is the dimension of the space $\mathrm{rad}_A(X, Y)/\mathrm{rad}_A^2(X, Y)$. The powers rad_A^n of the rad_A are ideals of the category mod_A, as well as rad_A^∞ defined by $\mathrm{rad}_A^\infty(X, Y) = \bigcap_{n \in \mathbb{N}} \mathrm{rad}_A^n(X, Y)$. If X, Y belong to two different components of Γ_A and $\mathrm{Hom}_A(X, Y) \neq 0$, then $\mathrm{rad}_A^\infty(X, Y) \neq 0$.

We recall that a component $\mathscr{C}$ of Γ_A is *standard* if the full subcategory of mod_A induced by the modules in $\mathscr{C}$ is equivalent to the mesh-category $k(\mathscr{C})$, see [4].

Lemma. *Let $A = kQ/I$ be an algebra satisfying (E^2). The following conditions are equivalent:*

(a) *for every $v \in \mathbb{N}^{Q_0}$, almost every $X \in \mathrm{ind}_A(v)$ lies in a homogeneous standard tube in Γ_A;*

(b) *for every $v \in \mathbb{N}^{Q_0}$, almost every $X \in \mathrm{ind}_A(v)$ has $\mathrm{rad}_A^\infty(X, X) = 0$.*

Moreover, if these conditions are satisfied, then A is tame.

Proof. First observe that in [10] it was shown that condition (a) implies that A is tame. That (b) implies tameness is left as an easy exercise.

(a) $\Rightarrow$ (b): If X is an indecomposable module in a homogeneous standard tube of Γ_A, clearly $\mathrm{rad}_A^\infty(X, X) = 0$.

If (b) is satisfied, then A is tame and by [3], almost every $X \in \mathrm{ind}_A(v)$ lies in a homogeneous tube of Γ_A. If X is in the mouth of a homogeneous tube T and $\mathrm{rad}_A^\infty(X, X) = 0$, then $\mathrm{End}_A(X) = k$. Moreover, $\mathrm{Ext}_A^2(X, X) = 0$ implies that T is standard by [15].

3.2. A well-known conjecture says that a homogeneous tube in a tame algebra always belongs to a tubular family. For algebras satisfying (E^2) we may prove the following.

PROPOSITION. *Let A be a tame algebra satisfying (E^2). Let T be an homogeneous tube in Γ_A. Assume that the indecomposable module Y in the mouth of T satisfies* $\mathrm{End}_A(Y) = k$. *Then there exists an infinite family $(T_\lambda)_\lambda$ of homogeneous tubes in Γ_A such that the module X_λ in the mouth of T_λ has* $\mathbf{dim}\ X_\lambda = \mathbf{dim}\ Y$.

PROOF. Consider the point $Y \in \mathrm{mod}_A(v)$ satisfying $\mathrm{Ext}^2_A(Y, Y) = 0$. By (1.2) and (1.3), there is an open neighbourhood $\mathscr{U}$ of Y in $\mathrm{mod}_A(v)$ such that for any $X \in \mathscr{U}$ the following are satisfied:

(i) $\dim_k \mathrm{End}_A(X) = 1$, hence X is indecomposable.

(ii) $\dim G(v) - \dim_X \mathrm{mod}_A(v) = \dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Ext}^1_A(X, X)$ is a constant $c \geq 0$.

(iii) $\mathrm{Ext}^2_A(X, X) = 0$; hence $\mathrm{mod}_A(v)$ is smooth at X.

Evaluating the difference (ii) at Y we get $c = 0$. Let C be the unique irreducible component of $\mathrm{mod}_A(v)$ containing Y. Consider any $X \in C \cap \mathscr{U}$. Then

$$\begin{aligned}\dim_X \mathrm{mod}_A(v) &= \dim T_X = \dim_k \mathrm{Ext}^1_A(X, X) + \dim G(v) - \dim_k \mathrm{End}_A(X)\\ &= \dim_k \mathrm{Ext}^1_A(Y, Y) + \dim G(v) - \dim_k \mathrm{End}_A(Y) > \dim G(v)X,\end{aligned}$$

where we have use (iii), (1.3), (ii) and (i) for the succesive steps. Therefore there is an infinite family $(X_\lambda)_\lambda$ of pairwise non-isomorphic modules in $C \cap \mathscr{U}$. Most of these modules lie on homogeneous tubes of Γ_A.

3.3. By [12], coil algebras and strongly simply connected polynomial growth algebras are examples of algebras A satisfying (E^2) and such that for every $v \in \mathbb{N}^{Q_0}$, almost every $X \in \mathrm{ind}_A(v)$ lies on a homogeneous standard tube. These algebras are also *cycle-finite*.

Recall that a *cycle* in ind_A is a chain $X = X_0 \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_s} X_s = X$ of non-zero non-isomorphisms between indecomposable A-modules. Such a cycle is infinite if $f_i \in \mathrm{rad}^\infty_A(X_{i-1}, X_i)$ for some $1 \leq i \leq s$. The algebra A is *cycle-finite* if it does not accept any infinite cycle in ind_A. By [1], a cycle-finite algebra is tame.

PROPOSITION. *Let $A = kQ/I$ be a cycle-finite algebra. Then*

(a) *for $v \in \mathbb{N}^{Q_0}$ and almost every indecomposable $X \in \mathrm{mod}_A(v)$, we have* $\mathrm{Ext}^2_A(X, X) = 0$ *and X lies in a homogeneous standard tube.*

(b) *Assume that there are infinitely many $G(v)$-orbits of indecomposable A-modules in* $\mathrm{mod}_A(v)$. *Then* $\mathrm{supp}\ v = \{i \in Q_0 : v(i) \neq 0\}$ *is convex in Q and the* *induced convex subcategory B is a tame quasi-tilted algebra (in particular, satisfying (E^2)).*

Proof. (a): Since A is tame, almost every $X \in \operatorname{ind}_A(v)$ lies in a homogeneous tube of Γ_A. If X is in such a tube T, then $\operatorname{rad}_A^\infty(X, X) = 0$ because A is cycle-finite. Then T is standard as in (3.1). Assume that $\operatorname{Ext}_A^2(X, X) \neq 0$. Consider the exact sequence $0 \to \Omega^1(X) \to P_0(X) \to X \to 0$ as in (1.5), then $0 \neq \operatorname{Ext}_A^2(X, X) \cong \operatorname{Ext}_A^1(\Omega^1(X), X) \cong D\,\overline{\operatorname{Hom}}_A(X, \tau_A \Omega^1(X))$. We get indecomposable direct summands Z of $\Omega^1(X)$, P of $P_0(X)$ a non-zero maps

$$\begin{array}{ccccccccc} X & \to & \tau_A X & & & & Z & \to & P \to X. \\ & & & \searrow & & \nearrow & & & \\ & & & & \cdot & & & & \end{array}$$

Hence all these modules belong to T. In particular, $P \in T$, which contradicts that T is a stable tube.

(b) Assume $(X_\lambda)_\lambda$ is an infinite family of pairwise non-isomorphic modules in $\operatorname{ind}_A(v)$. Since A is tame, we may assume $X_0 = X_{\lambda_0}$ belongs to a homogeneous standard tube T_0. By a standard argument, $\operatorname{supp} v = \operatorname{supp} X_0$ is convex in Q.

Let B the convex subcategory of A induced by $\operatorname{supp} v$. We show first that $\operatorname{g\ell\,dim} B \leq 2$. Otherwise, there is an indecomposable summand R of P for an indecomposable projective B-module P, such that $p\dim_B R > 1$, that is, $\operatorname{Hom}_B(I, \tau_B R) \neq 0$ for some indecomposable injective B-module I. Since X_0 is sincere as B-module, we get a chain of non-zero maps

$$\begin{array}{ccccccccccc} X_0 & \to & I & \to & \tau_B R & & & & R & \to & P \to X_0. \\ & & & & & \searrow & & \nearrow & & & \\ & & & & & & \cdot & & & & \end{array}$$

Again, $P \in T_0$. Moreover, for any $Y \in T_0$, $\operatorname{supp} Y = \operatorname{supp} v$, hence T_0 is a component of Γ_B, therefore T_0 cannot contain projective B-modules, a contradiction showing that $\operatorname{g\ell\,dim} B \leq 2$.

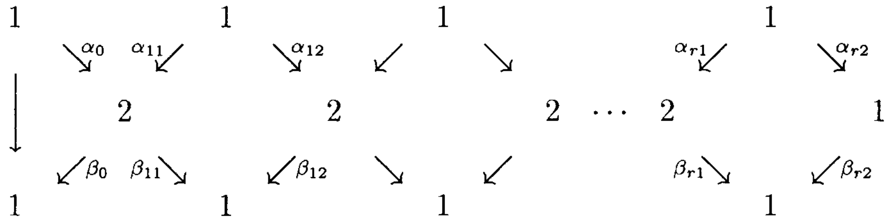
Finally, if $Y \in \operatorname{ind}_B(w)$ and $p\dim_B Y > 1$, $i\dim_B Y > 1$, we get a cycle

$$\begin{array}{ccccccccccccc} X_0 & \to & I & \to & \tau_B Y & & & & Y & & & & \tau_B^- Y \to P \to X_0. \\ & & & & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & & & & & \cdot & & & & \cdot & & \end{array}$$

for some indecomposable projective (resp. injective) B-module P (resp. I). As above, $P \in T_0$ and we get a contradiction.

4. Some examples.

4.1. Consider the algebra B_r given by the following quiver



with r squares and relations: $\beta_{11}\alpha_0$, $\beta_0\alpha_{11}$, $\beta_{i,2}\alpha_{i+1,1}$, $\beta_{i+1,1}\alpha_{i,2}$ $(1 \le i \le r-1)$ and $\beta_{i1}\alpha_{i1} - \beta_{i2}\alpha_{i2}$ $(1 \le i \le r)$. The algebra B_r is a coil algebra (obtained from an algebra of type $\tilde{\mathbf{A}}_2$ by a sequence of admissible operations of type (ad 1) and (ad 2*) as defined in [1]) and therefore B_r is cycle-finite. As observed in (3.2), B_r satisfies (E^2).

Let X be the indecomposable B_r-module whose dimension vector is as indicated in the drawing. Since $g\ell \dim B_r = 2$, then

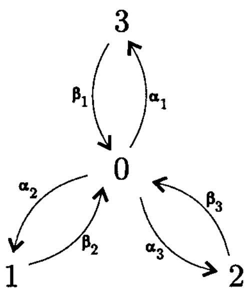
$$r = q_{B_r}(\mathbf{dim}X) = \dim_k \operatorname{End}_A(X) - \dim_k \operatorname{Ext}^1_A(X, X).$$

Moreover, we may check that $\operatorname{Ext}^1_A(X, X) = 0$ and hence $\dim_k \operatorname{End}_A(X) = \frac{1}{6}(\dim_k X + 1)$.

Finally, observe that B_r is not simply connected and there is a Galois covering $\tilde{B}_r \to B_r$ defined by the action of Z.

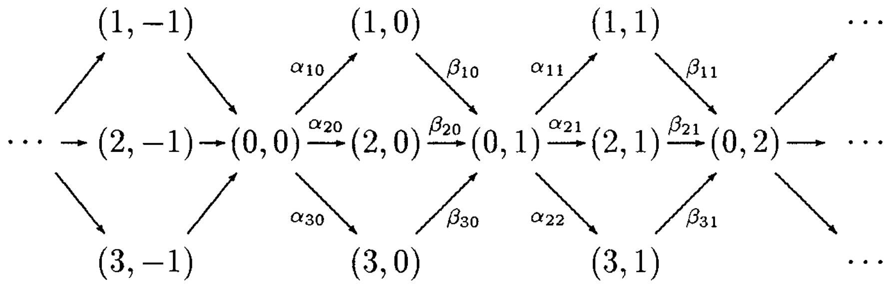
4.2. We shall show an example of a polynomial growth algebra satisfying (E^2) but not (E^3).

Consider the algebra A_1 given by the quiver



with relations: $\alpha_i\beta_j = 0$, for $i \neq j$ and $\beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3$. We have numbered the vertices $Q_0 = \{0, 1, 2, 3\}$.

There is a Galois covering $F_1 : \tilde{A}_1 \to A_1$ where $\tilde{A}_1$ is given by the quiver



with the relations: $\alpha_{i,s+1}\beta_{j,s}$, for $i \neq j$, $s \in \mathbb{Z}$ and $\sum_{i=1}^{3} \beta_{is}\alpha_{is}$ for $s \in \mathbb{Z}$. The group defining F_1 is $\mathbb{Z}$ acting as horizontal shifts.

It is an easy exercise to construct $\Gamma_{\tilde{A}_1}$. If X is an indecomposable $\tilde{A}_1$-module, then either $\operatorname{supp} X \subset \{(0,s),(1,s),(2,s),(3,s),(0,s+1)\}$ for some $s \in \mathbb{Z}$ or X is a projective-injective module of the form $P_{(i,s)} = I_{(i,s+1)}$ for some $i \in \{1,2,3\}$ and some $s \in \mathbb{Z}$. It is easy to check that $\operatorname{Ext}^2_{\tilde{A}_1}(X, X^g) \xrightarrow{\sim} \operatorname{Ext}^1_{\tilde{A}_1}(\Omega^1(X), X^g) = 0$ for any horizontal shift $g \in \mathbb{Z}$. It follows that A_1 satisfies (E^2).

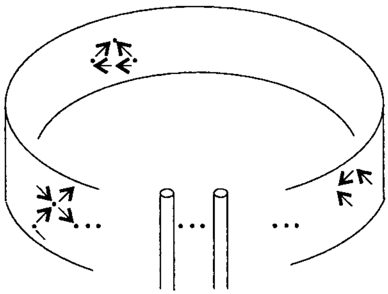
The projective resolution of the simple $\tilde{A}_1$-module $S_{(0,0)}$ is:

$$0 \to S_{(0,2)} \oplus S_{(0,2)} \to P_{(0,1)} \to \bigoplus_{i=1}^{3} P_{(i,0)} \to P_{(0,0)} \to S_{(0,0)} \to 0.$$

Hence $\operatorname{Ext}^3_{A_1}(S_0, S_0) \neq 0$ and A_1 does not satisfy (E^3).

We may apply (2.5) to obtain that A_1 is tame. In fact, the Auslander-Reiten quiver Γ_{A_1} is of the form $\Gamma_{\tilde{A}_1}/\mathbb{Z}$ and has the following shape,

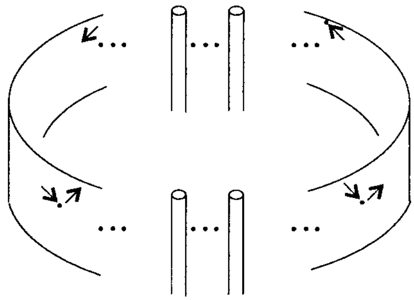
Γ_{A_1}:



There is a unique family $\mathscr{T} = (T_\lambda)_{\lambda \in \mathbf{P}_1 k}$ of (homogeneous) tubes in Γ_{A_1}. The module X_λ at the mouth of T_λ has dimension $\mathbf{dim}\, X_\lambda = \begin{smallmatrix} & 1 & \\ & 2 & \\ 1 & & 1 \end{smallmatrix}$. Moreover $\dim_k \operatorname{End}_{A_1}(X_\lambda, X_\lambda) = 2$ and hence $\operatorname{rad}^\infty_{A_1}(X_\lambda, X_\lambda) \neq 0$. By (3.1), the tubes T_λ are not standard.

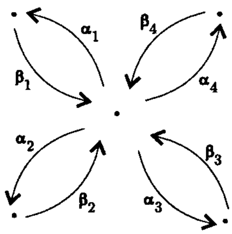
We may consider the algebra $A_2 = \tilde{A}_1/2\mathbb{Z}$ which is a double covering of

A_1. As above, A_2 is tame and satisfies (E^2). Moreover Γ_{A_2} has the following shape:



The tubes in Γ_{A_2} are all homogeneous and standard. Clearly, A_2 is not cycle-finite.

4.3. Consider the algebra A given by the quiver



with relations: $\alpha_i\beta_j = 0$ for $i \neq j$; $\beta_3\alpha_3 - \beta_1\alpha_1 - \lambda_3\beta_2\alpha_2$, $\beta_4\alpha_4 - \beta_1\alpha_1 - \lambda_4\beta_2\alpha_2$ for some $\lambda_3 \neq \lambda_4$ elements of $k \setminus \{0\}$. Using covering techniques the following is easy to verify:

(i) A is a tame algebra of polynomial growth but not domestic.

(ii) A does not satisfy (E^2). Indeed, there is an infinite family $(X_\lambda)_{\lambda \in \mathbf{P}_1 k}$ of pairwise non-isomorphic indecomposable modules with $X_\lambda = {}^{1}_{1}2^{1}_{1}$ such that $\mathrm{Ext}^2_A(X_\lambda, X_\lambda) \neq 0$.

(iii) The modules X_λ belong to homogeneous non-standard tubes of Γ_A.

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