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MEASURE HOMOLOGY

SØREN KOLD HANSEN

Abstract.

Let X be a topological space, $\mathrm{Sin}_k(X)$ the space of singular k-simplices with the compact-open topology, and let $\mathscr{C}_k(X)$be the real vector space of all compactly supported signed Borel Measures of bounded total variation on $\mathrm{Sin}_k(X)$. There are linear operators $\partial : \mathscr{C}_k(X) \to \mathscr{C}_{k-1}(X)$, so that $\{\mathscr{C}_*(X), \partial\}$ is a chain complex. The homology $H_*^\mu(X)$ is the measure homology of X of Thurston and Gromov. The main results in this paper are that $H_*^\mu(-)$ satisfies the Eilenberg-Steenrod axioms for a wide class of topological spaces including all metric spaces, and is ordinary homology with real coefficients for CW-complexes.

1. Introduction.

Measure homology was introduced by Gromov and Thurston in [T] §6 in connection with Gromov's theorem that the Gromov norm of a closed oriented hyperbolic n-manifold M equals the volume of M divided by the supremum of the volumes of the geodesic n-simplices in the hyperbolic n-space.

For a measurable space $(X, \wp)$, let $\mathscr{V}(X, \wp)$ be the vector space of all signed measures of bounded total varation. The total variation of a signed measure μ on $(X, \wp)$ is $\|\mu\| = \mu^+(X) - \mu^-(X)$ where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ into its positive and negative variation. A measure μ on $(X, \wp)$ has support in $A \in \wp$, $\mathrm{Supp}(\mu) \subseteq A$, if $\mu(A \cap B) = \mu(B)$ for all $B \in \wp$. We write $\mathscr{B}(X)$ for the Borel σ-algebra on the space X, and define a linear subspace of $\mathscr{V}(X, \mathscr{B}(X))$ by

$$\mathscr{M}_c(X) = \{\mu \in \mathscr{V}(X, \mathscr{B}(X)) \mid \mu \,\text{has compact support}\}.$$

A continuous map $f : X \to Y$ induces a linear map $f_* : \mathscr{M}_c(X) \to \mathscr{M}_c(Y)$, namely the image measure of μ under f.

Let $\mathrm{Sin}_k(X)$ be the set of continuous maps from the standard k-simplex Δ^k to the space X with the compact-open topology, and set

$$\mathscr{C}_k(X) = \mathscr{M}_c(\mathrm{Sin}_k(X)).$$

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The ith face map $\delta^i : \Delta^{k-1} \to \Delta^k$ induces a continuous map $\partial_i : \mathrm{Sin}_k(X) \to \mathrm{Sin}_{k-1}(X)$, $\partial_i(\sigma) = \sigma \circ \delta_i$, and hence a linear map $(\partial_i)_* : \mathscr{C}_k(X) \to \mathscr{C}_{k-1}(X)$. The measure chain complex is the spaces $\mathscr{C}_k(X)$ together with the boundary operators $\partial = \sum_{i=0}^{k}(-1)^i(\partial_i)_*$. The homology of $\mathscr{C}_*(X)$ is denoted $H_*^\mu(X)$ and is the measure homology of X, cf. [T]. Actually, in [T], the authors only defined $\mathscr{C}_*(X)$ when X is a smooth manifold and used the sets $\mathrm{Sin}_k^1(X)$ of singular k-simplices of class C^1 with the C^1 topology instead of $\mathrm{Sin}_k(X)$. We shall see that this makes no difference. The main theorem of this paper is the following result, listed without proof in the case of smooth manifolds in [T] §6 p. (6.7):

THEOREM 1.1. *The measure homology functor satisfies the Eilenberg-Steenrod axioms on the category of metric spaces.*

REMARKS. 1) Actually we prove that $H_*^\mu(X)$ satisfies the Eilenberg-Steenrod axioms for all Hausdorff spaces X such that $\mathrm{Sin}_k(X)$ and $\mathrm{Sin}_k(A)$ are normal for all $k \geq 0$ and all $A \subseteq X$. This is indeed satisfied if X is a metric space. Note that normality of X does not imply normality of $\mathrm{Sin}_k(X)$. Actually A. H. Stone showed in [S] that if $I = [0, 1]$ and Y is the product of uncountably many copies of I then Y^I is not normal, where Y^I is the space of maps of I into Y with the compact-open topology.

2) If X is a smooth manifold theorem 1.1 and the proof we give for it is still valid if one uses the sets $\mathrm{Sin}_k^r(X)$ of singular k-simplices of class C^r with the C^r topology instead of $\mathrm{Sin}_k(X) = \mathrm{Sin}_k^0(X)$ to define measure homology, $1 \leq r \leq \infty$.

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2. The measure homology functor.

In the preceding section we introduced the measure chain complex $\mathscr{C}_*(X)$ for an arbitrary topological space X. A map $f : X \to Y$ induces linear maps $\bar{f} : \mathscr{C}_k(X) \to \mathscr{C}_k(Y)$ by $\bar{f} = (f_\#)_*$ where $f_\# : \mathrm{Sin}_k(X) \to \mathrm{Sin}_k(Y)$ is as usual. Instead of $\bar{f}$ we usually write $f : \mathscr{C}_k(X) \to \mathscr{C}_k(Y)$. This makes $\mathscr{C}_*(-)$ a covariant functor and turns $H_*^\mu(-)$ into a covariant functor in a standard way.

One can generalize the above to pairs of Hausdorff spaces (X, A). We have $\mathscr{B}(A) = \{Z \cap A \,|\, Z \in \mathscr{B}(X)\}$, so that $\mathscr{B}(A) = \{Z \in \mathscr{B}(X) \,|\, Z \subseteq A\} \subseteq \mathscr{B}(X)$ if $A \in \mathscr{B}(X)$. For an arbitrary set E, $\mathscr{P}(E) = \{A \mid A \subseteq E\}$ denotes the power set of E. Taking direct images, $f : E \to F$ induces $\mathscr{P}(f) : \mathscr{P}(E) \to \mathscr{P}(F)$ and makes $\mathscr{P}(-)$ a covariant functor. For a homeomorphism $f : X \to Y$, $\mathscr{P}(f)|_{\mathscr{B}(X)} : \mathscr{B}(X) \to \mathscr{B}(Y)$ is a bijection. We use below that $\mathrm{Sin}_k(X)$ is a Hausdorff space if and only if X is.

LEMMA 2.1. *For (X, A) a pair of Hausdorff spaces, the inclusion $i : A \to X$ induces a monomorphism $i : \mathscr{C}_*(A) \to \mathscr{C}_*(X)$.*

PROOF. If K is a compact subset of $\mathrm{Sin}_k(A)$ then $L = i_\#(K)$ is a compact subset of $\mathrm{Sin}_k(X)$ and $i_\#|_K : K \to L$ is a homeomorphism. Let $\mu_1, \mu_2 \in \mathscr{C}_k(A)$ with compact supports K_1, K_2, and $i(\mu_1) = i(\mu_2)$. If $B \in \mathscr{B}(\mathrm{Sin}_k(A))$ then $B \cap (K_2 \setminus K_1) = K_2 \cap (B \setminus K_1) \in \mathscr{B}(K_2)$, so

$$\begin{aligned}\mu_2(B \cap (K_2 \setminus K_1)) &= i(\mu_2)(i_\#(B \cap (K_2 \setminus K_1))) = i(\mu_1)(i_\#(B \cap (K_2 \setminus K_1))) \\ &= \mu_1(B \cap (K_2 \setminus K_1)) = 0.\end{aligned}$$

Thus $K_1 \cap K_2$ is a support for μ_2, and, symmetrically for μ_1. If $B \in \mathscr{B}(\mathrm{Sin}_k(X))$,

$$\mu_\nu(i_\#^{-1}(B)) = \mu_\nu(i_\#^{-1}(B) \cap K_1 \cap K_2) = \mu_\nu(i_\#^{-1}(B \cap L_1 \cap L_2))$$

where $L_\nu = i_\#(K_\nu)$, $\nu = 1, 2$. But

$$\begin{aligned}\mathscr{B}(K_1 \cap K_2) &= \left\{ i_\#^{-1}(D) \mid D \in \mathscr{B}(L_1 \cap L_2) \right\} \\ &= \left\{ i_\#^{-1}(B \cap L_1 \cap L_2) \mid B \in \mathscr{B}(\mathrm{Sin}_k(X)) \right\}\end{aligned}$$

so $\mu_1 = \mu_2$ on $\mathscr{B}(K_1 \cap K_2)$ hence on all of $\mathscr{B}(\mathrm{Sin}_k(A)$.

We let $\mathscr{C}_*(X, A)$ be the cokernel of $i : \mathscr{C}_*(A) \to \mathscr{C}_*(X)$, so that we have an exact sequence

$$0 \longrightarrow \mathscr{C}_*(A) \xrightarrow{i} \mathscr{C}_*(X) \xrightarrow{\pi} \mathscr{C}_*(X, A) \longrightarrow 0$$

of chain complexes. The homology groups of $\mathscr{C}_*(X, A)$ are the relative measure homology groups of (X, A) and are denoted $H_*^\mu(X, A)$. A map $f : (X, A) \to (Y, B)$ of pairs of Hausdorff spaces induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{C}_*(A) & \xrightarrow{i} & \mathscr{C}_*(X) & \xrightarrow{\pi} & \mathscr{C}_*(X, A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{C}_*(B) & \underset{i}{\longrightarrow} & \mathscr{C}_*(Y) & \underset{\pi}{\longrightarrow} & \mathscr{C}_*(Y, B) & \longrightarrow & 0 \end{array}$$

of chain maps. Thus we get as usual a long exact homology sequence, natural in (X, A):

$$\cdots \xrightarrow{\partial_*} H_k^\mu(A) \xrightarrow{i_*} H_k^\mu(X) \xrightarrow{j_*} H_k^\mu(X, A) \xrightarrow{\partial_*} H_{k-1}^\mu(A) \xrightarrow{i_*} \cdots.$$

3. Proof of theorem 1.1.

In this section we verify the homotopy axiom, excision and the dimension axiom, i.e.

i) *If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic as maps of pairs of Hausdorff spaces, then $H_*^\mu(f_0) = H_*^\mu(f_1) : H_*^\mu(X, A) \to H_*^\mu(Y, B)$.*

ii) *If (X,A) is a pair of metric spaces and $U \subseteq A$ has $\bar{U} \subseteq \mathrm{Int}(A)$, then the inclusion map $i : (X - U, A - U) \to (X, A)$ induces an isomorphism on homology.*

iii) *If X is a one-point space, then $H_k^\mu(X) = 0$ for $k \neq 0$ and $H_0^\mu(X) = \mathsf{R}$.*

If one works with complex measures instead of real measures the only difference is that $H_0^\mu(X) = \mathsf{C}$ for a one-point space X. We start by showing the easy i) and iii).

Proof of iii). Since $\mathrm{Sin}_k(X)$ has only one element φ_k, $\mathscr{B}(\mathrm{Sin}_k(X)) = \{\emptyset, \{\varphi_k\}\}$ and $\mu \in \mathscr{C}_k(X)$ is completely determined by the value $\mu(\{\varphi_k\})$. If $r \in \mathsf{R}$ we get an element $\mu_r^k \in \mathscr{C}_k(X)$ defined by $\mu_r^k(\emptyset) = 0$, $\mu_r^k(\{\varphi_k\}) = r$. This shows that $\mathscr{C}_k(X) \cong \mathsf{R}$, and a simple calculation shows that

$$\partial(\mu_r^k) = \begin{cases} 0 & ,k \text{ odd} \\ \mu_r^{k-1} & ,k \text{ even and } k > 0. \end{cases}$$

Since $\partial\mu = 0$ for all $\mu \in \mathscr{C}_0(X)$ by definition the result follows.

Proof of i). We just do the absolute case, $A = B = \emptyset$. Let $\lambda_t : X \to X \times I$ be given by $\lambda_t(x) = (x, t)$, $I = [0, 1]$ and let $F : X \times I \to Y$ be a homotopy between f_0 and f_1. Then $F\lambda_0 = f_0$ and $F\lambda_1 = f_1$ and it suffices to show that $H_*^\mu(\lambda_0) = H_*^\mu(\lambda_1) : H_*^\mu(X) \to H_*^\mu(X \times I)$. To show this we construct a chain homotopy $P : \mathscr{C}_*(X) \to \mathscr{C}_*(X \times I)$ between λ_0 and λ_1. For $i = 0, 1, \ldots, k$ we define maps $Q_i : \mathrm{Sin}_k(X) \to \mathrm{Sin}_{k+1}(X \times I)$ by

$$Q_i(\sigma)(t_0, \ldots, t_{k+1}) = \sigma(t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{k+1}) \times \left(1 - \sum_{l=0}^{i} t_l\right)$$

for $\sigma \in \mathrm{Sin}_k(X)$ and $(t_0, \ldots, t_{k+1}) \in \Delta^{k+1}$. The Q_i are continuous and induce linear maps $(Q_i)_* : \mathscr{C}_k(X) \to \mathscr{C}_{k+1}(X \times I)$. Define $P_k : \mathscr{C}_k(X) \to \mathscr{C}_{k+1}(X \times I)$ by $P_k = \sum_{i=0}^{k}(-1)^i (Q_i)_*$. A tedious calculation shows that the P_k form a natural chain homotopy between λ_0 and λ_1. The general case now follows in a standard way, by using naturality of P.

We now begin the proof of ii). Let U be an open subset of X and $i : U \to X$ the inclusion map. Then $V = i_\#(\mathrm{Sin}_k(U))$ is open in $\mathrm{Sin}_k(X)$ and $i_\# : \mathrm{Sin}_k(U) \to V$ is a homeomorphism. It follows that $\mathscr{P}(i_\#) : \mathscr{B}(\mathrm{Sin}_k(U)) \to \mathscr{B}(V)$ is a bijection so that

$$\mathscr{B}(\mathrm{Sin}_k(U)) = \left\{ i_{\#}^{-1}(B) \mid B \in \mathscr{B}(V) \right\} = \left\{ i_{\#}^{-1}(B) \mid B \in \mathscr{B}(\mathrm{Sin}_k(X)) \right\}.$$

For a family $\mathscr{U} = \{U_\alpha \mid \alpha \in I\}$ of (not necessarily open) subsets of X we consider the subchain complex of $\mathscr{C}_*(X)$ of "$\mathscr{U}$-small" measures

$$\mathscr{C}_k^{\mathscr{U}}(X) = \sum_{\alpha \in I} \mathscr{C}_k^{U_\alpha}(X), \qquad \mathscr{C}_k^{U_\alpha}(X) = i_\alpha(\mathscr{C}_k(U_\alpha)).$$

For the family $\mathscr{U}$ we let $\mathrm{Int}(\mathscr{U})$ be the collection of interiors of elements of $\mathscr{U}$. We then have

Theorem 3.1. *Let X be a metric space and let $\mathscr{U}$ be a family of subsets of X such that $Int(\mathscr{U})$ is a covering of X. Then the inclusion chain map $I : \mathscr{C}_*^{\mathscr{U}}(X) \to \mathscr{C}_*(X)$ induces an isomorphism on homology.*

The proof for this theorem is deferred to §4. As in the case of singular homolog the excision axiom follows at once. Let $\mathscr{U} = \{U_\alpha \mid \alpha \in I\}$ and X be as in theorem 3.1 and let $\mathscr{U} \cap A = \{U_\alpha \cap A \mid \alpha \in I\}$. The commutative diagram of inclusion maps

$$\begin{array}{ccc} U_\alpha \cap A & \xrightarrow{m_\alpha} & U_\alpha \\ {\scriptstyle j_\alpha}\downarrow & & \downarrow{\scriptstyle i_\alpha} \\ A & \xrightarrow[i]{} & X \end{array}$$

shows that $i(j_\alpha(\mathscr{C}_k(U_\alpha \cap A))) \subseteq i_\alpha(\mathscr{C}_k(U_\alpha))$ so $i(\mathscr{C}_*^{\mathscr{U}\cap A}(X))$ is a subcomplex of $\mathscr{C}_*^{\mathscr{U}}(X)$. Setting $\mathscr{C}_*(\mathscr{U}, \mathscr{U} \cap A) = \mathscr{C}_*^{\mathscr{U}}(X)/i(\mathscr{C}_*^{\mathscr{U}\cap A}(A))$, we have a commutative diagram of chain maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{C}_*^{\mathscr{U}\cap A}(A) & \xrightarrow{i} & \mathscr{C}_*^{\mathscr{U}}(X) & \xrightarrow{\pi} & \mathscr{C}_*(\mathscr{U}, \mathscr{U}\cap A) & \longrightarrow & 0 \\ & & {\scriptstyle I}\downarrow & & {\scriptstyle I}\downarrow & & {\scriptstyle \Pi}\downarrow & & \\ 0 & \longrightarrow & \mathscr{C}_*(A) & \xrightarrow[i]{} & \mathscr{C}_*(X) & \xrightarrow[\pi]{} & \mathscr{C}_*(X, A) & \longrightarrow & 0. \end{array}$$

By the preceding theorem the inclusions I induce isomorphisms on homology, and the five-lemma yields that Π induces an isomorphism on homology.

Proof of ii). Let $\mathscr{U} = \{X - U, \mathrm{Int}(A)\}$. Then $\mathscr{U} \cap A = \{A - U, \mathrm{Int}(A)\}$ and we have that $\mathrm{Int}(\mathscr{U})$ and $\mathrm{Int}(\mathscr{U} \cap A)$ cover respectively X and A. Now let

$$\begin{array}{ccc} \mathrm{Int}(A) & \xrightarrow{j_X} & X \\ {\scriptstyle j_A}\downarrow & & \downarrow{\scriptstyle id} \\ A & \xrightarrow[i]{} & X \end{array} \qquad \begin{array}{ccc} A - U & \xrightarrow{m} & X - U \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_X} \\ A & \xrightarrow[i]{} & X \end{array}$$

be commutative diagrams of inclusion maps. Then we have

$$\mathscr{C}_k^{\mathscr{U}\cap A}(A) = i_A(\mathscr{C}_k(A-U)) + j_A(\mathscr{C}_k(\mathrm{Int}(A)))$$
$$\mathscr{C}_k^{\mathscr{U}}(X) = i_X(\mathscr{C}_k(X-U)) + j_X(\mathscr{C}_k(\mathrm{Int}(A)))$$

and therefore $i(\mathscr{C}_k^{\mathscr{U}\cap A}(A)) = i_X \circ m(\mathscr{C}_k(A-U)) + j_X(\mathscr{C}_k(\mathrm{Int}(A)))$.This implies the isomorphisms

$$\mathscr{C}_*(\mathscr{U}, \mathscr{U}\cap A) = \mathscr{C}_*^{\mathscr{U}}(X)/i(\mathscr{C}_*^{\mathscr{U}\cap A}(A)) \cong i_X(\mathscr{C}_*(X-U))/i_X(m(\mathscr{C}_*(A-U)))$$
$$\cong \mathscr{C}_*(X-U)/m(\mathscr{C}_*(A-U)) = \mathscr{C}_*(X-U, A-U)$$

The first isomorphism follows by the fact that

$$[i_X \circ m(\mathscr{C}_k(A-U)) + j_X(\mathscr{C}_k(\mathrm{Int}(A))] \cap i_X(\mathscr{C}_k(X-U)) = i_X \circ m(\mathscr{C}_k(A-U))$$

Since $\Pi : \mathscr{C}_*(\mathscr{U}, \mathscr{U}\cap A) \to \mathscr{C}_*(X, A)$ induces an isomorphism on homology the result follows.

4. Proof of theorem 3.1.

The proof of theorem 3.1 uses the standard ideas from barycentric subdivision in singular theory, which we begin by recalling, cf. [D]. The subdivision homomorphisms $\beta_q : S_q(X) \to S_q(X)$, $q \in \mathsf{Z}$, are inductively defined in the following way:

Let $\imath_q \in S_q(\Delta^q)$ denotes the identity map of the standard q-simplex Δ^q with vertices the standard basis $\{e_i\}$ in R^{q+1} and let $B_q = \sum_{i=0}^{q} \frac{1}{q+1} e_i$ be the barycenter of Δ^q. Write $B_q\cdot$ for the cone construction (see [D] chap. III (4.7) p. 34), and set

$$(1) \qquad \beta_0 = \mathrm{id} \qquad \beta_q(\imath_q) = B_q \cdot \beta_{q-1}(\partial \imath_q), \quad q > 0.$$

One defines $\beta_q : \mathrm{Sin}_q(X) \to S_q(X)$ by $\beta_q(\sigma) = \sigma_\#(\beta_q(\imath_q))$. Then

$$(2) \qquad \beta : S_*(X) \to S_*(X)$$

is a natural chain map, [D] p. 41. For later use we need to explicate the natural chain homotopy $s : \beta \simeq \mathrm{id}_{S_*(X)}$, [D] p. 42. It is 0 for $q = 0$ and is given by

$$s_q(\imath_q) = B_q \cdot (\beta_q(\imath_q) - \imath_q - s_{q-1}(\partial \imath_q)) \in S_{q+1}(\Delta^q)$$

on $\imath_q \in S_q(\Delta^q)$ for $q > 0$. For a general $\sigma \in \mathrm{Sin}_q(X)$, $s_q(\sigma) = \sigma_\#(s_q(\imath_q))$.

We now want to define a "subdivision" homomorphism $\beta : \mathscr{C}_*(X) \to \mathscr{C}_*(X)$ for the measure theoretical chain complex and a chain homotopy $s : \beta \simeq \mathrm{id}_{\mathscr{C}_*(X)}$. To this end we write out the construction in (1) in a form $\beta_q = \sum_{\nu \in A_q} r_\nu \beta_q^\nu$ where $\beta_q^\nu : \mathrm{Sin}_q(X) \to \mathrm{Sin}_q(X)$ are continuous and induce linear maps $(\beta_q^\nu)_* : \mathscr{C}_q(X) \to \mathscr{C}_q(X)$. Thus we can define our measure theoretical "subdivision" homomorphism by $\beta_q = \sum_{\nu \in A_q} r_\nu (\beta_q^\nu)_*$. This procedure will also be used to define the chain homotopy $s : \beta \simeq id_{\mathscr{C}_*(X)}$. For $q \geq 1$ the explicit formula is

$$\beta_q(\imath_q) = \sum_{k_q=0}^{q} \sum_{k_{q-1}=0}^{q-1} \cdots \sum_{k_1=0}^{1} (-1)^{k_1+\cdots+k_q} \sigma_q^{k_1\ldots k_q}$$

where

$$\sigma_q^{k_1\ldots k_q} = B_q \cdot (\delta^{k_q} \circ (B_{q-1} \cdot (\delta^{k_{q-1}} \circ \cdots B_2 \cdot (\delta^{k_2} \circ (B_1 \cdot \delta^{k_1}))\cdots))) \in \mathrm{Sin}_q(\Delta^q).$$

Thus we get maps $\beta_q : \mathrm{Sin}_q(X) \to S_q(X)$,

$$\beta_q = \sum_{k_q=0}^{q} \sum_{k_{q-1}=0}^{q-1} \cdots \sum_{k_1=0}^{1} (-1)^{k_1+\cdots+k_q} \beta_q^{k_1\ldots k_q} \tag{3}$$

where $\beta_q^{k_1\ldots k_q}(\sigma) = \sigma \circ \sigma_q^{k_1\ldots k_q}$. Similarly when $q \geq 1$ we have

$$\begin{aligned} s_q(\imath_q) = &\sum_{\nu=1}^{q} (-1)^{q-\nu} \sum_{k_q=0}^{q} \sum_{k_{q-1}=0}^{q-1} \cdots \sum_{k_1=0}^{1} (-1)^{k_1+\cdots+k_q} f_{q,\nu}(k_1,\ldots,k_q) \\ &- \sum_{\nu=1}^{q-1} (-1)^{q-\nu} \sum_{k_q=0}^{q} \sum_{k_{q-1}=0}^{q-1} \cdots \sum_{k_{\nu+1}=0}^{\nu+1} (-1)^{k_{\nu+1}+\cdots+k_q} g_{q,\nu}(k_{\nu+1},\ldots,k_q) - g_{q,q} \end{aligned}$$

where

$$\begin{aligned} f_{q,\nu}(k_1,\ldots,k_q) &= B_q \cdot (\delta^{k_q} \circ (B_{q-1} \cdot (\delta^{k_{q-1}} \circ \cdots (\delta^{k_{\nu+1}} \circ (B_\nu \cdot \sigma_\nu^{k_1\cdots k_\nu}))\cdots))) \\ g_{q,\nu}(k_{\nu+1},\ldots,k_q) &= B_q \cdot (\delta^{k_q} \circ (B_{q-1} \cdot (\delta^{k_{q-1}} \circ \cdots (\delta^{k_{\nu+1}} \circ (B_\nu \cdot \imath_\nu))\cdots))), \end{aligned}$$

$\nu = 1, \ldots, q-1$, and

$$\begin{aligned} f_{q,q}(k_1,\ldots,k_q) &= B_q \cdot \sigma_q^{k_1\cdots k_q} \\ g_{q,q} &= B_q \cdot \imath_q \end{aligned}$$

are elements of $\mathrm{Sin}_{q+1}(\Delta^q)$, so $s_q : \mathrm{Sin}_q(X) \to S_{q+1}(X)$ is given by

$$\begin{aligned} s_q = &\sum_{\nu=1}^{q} (-1)^{q-\nu} \sum_{k_q=0}^{q} \sum_{k_{q-1}=0}^{q-1} \cdots \sum_{k_1=0}^{1} (-1)^{k_1+\cdots+k_q} f_{q,\nu}^{k_1\ldots k_q} \\ &- \sum_{\nu=1}^{q-1} (-1)^{q-\nu} \sum_{k_q=0}^{q} \sum_{k_{q-1}=0}^{q-1} \cdots \sum_{k_{\nu+1}=0}^{\nu+1} (-1)^{k_{\nu+1}+\cdots+k_q} g_{q,\nu}^{k_{\nu+1}\ldots k_q} - h_{q,q} \end{aligned} \tag{4}$$

where $f_{q,\nu}^{k_1\ldots k_q}, g_{q,\nu}^{k_{\nu+1}\ldots k_q}, h_{q,q} : \mathrm{Sin}_q(X) \to \mathrm{Sin}_{q+1}(X)$ are defined by

$$f_{q,\nu}^{k_1\ldots k_q}(\sigma) = \sigma \circ f_{q,\nu}(k_1,\ldots,k_q)$$
$$g_{q,\nu}^{k_{\nu+1}\ldots k_q}(\sigma) = \sigma \circ g_{q,\nu}(k_{\nu+1},\ldots,k_q)$$
$$h_{q,q}(\sigma) = \sigma \circ q_{q,q}.$$

In the following we shorten notation and write (3) as $\beta_q = \sum_{\nu\in A_q} r_\nu \beta_q^\nu$ where A_q is the set of q-tuples $(a_1,\ldots,a_q)$, $a_i = 0,1,\ldots,i$, and $r_{(k_1,\ldots,k_q)} = (-1)^{k_1+\cdots+k_q}$. For $q \geq 2$ we then have

$$\partial\beta_q = \sum_{j=0}^{q}(-1)^j\partial_j\left(\sum_{\nu\in A_q} r_\nu\beta_q^\nu\right) = \sum_{j=0}^{q}\sum_{\nu\in A_q}(-1)^j r_\nu \partial_j \circ \beta_q^\nu$$

$$\beta_{q-1}\partial = \sum_{\alpha\in A_{q-1}} r_\alpha\beta_{q-1}^\alpha\left(\sum_{j=0}^{q}(-1)^j\partial_j\right) = \sum_{j=0}^{q}\sum_{\alpha\in A_{q-1}}(-1)^j r_\alpha \beta_{q-1}^\alpha \circ \partial_j.$$

Now $\partial\beta_q = \beta_{q-1}\partial$ by (2) so in particular $\partial\beta_q(\imath_q) = \beta_{q-1}\partial(\imath_q)$, i.e.

$$\sum_{j=0}^{q}\sum_{\nu\in A_q}(-1)^j r_\nu \sigma_q^\nu \circ \delta^j = \sum_{j=0}^{q}\sum_{\alpha\in A_{q-1}}(-1)^j r_\alpha \delta^j \circ \sigma_{q-1}^\alpha.$$

Since $\mathrm{Sin}_{q-1}(\Delta^q)$ is a basis for $S_{q-1}(\Delta^q)$ we can write

$$\sum_{j=0}^{q}\sum_{\nu\in A_q}(-1)^j r_\nu \sigma_q^\nu \circ \delta^j = \sum_{\lambda\in M_q} t_\lambda \tau_q^\lambda$$
$$\sum_{j=0}^{q}\sum_{\alpha\in A_{q-1}}(-1)^j r_\alpha \delta^j \circ \sigma_{q-1}^\alpha = \sum_{\mu\in N_q} s_\mu \omega_q^\mu,$$

where

$$\left\{\tau_q^\lambda | \lambda \in M_q\right\} \subseteq \left\{\sigma_q^\nu \circ \delta^j | (\nu,j) \in A_q \times \{0,1,\ldots,q\}\right\}$$
$$\left\{\omega_q^\mu | \mu \in N_q\right\} \subseteq \left\{\delta^j \circ \sigma_{q-1}^\alpha | (j,\alpha) \in \{0,1,\ldots,q\} \times A_{q-1}\right\},$$

and $\lambda_1 \neq \lambda_2 \Rightarrow \tau_q^{\lambda_1} \neq \tau_q^{\lambda_2}$ and $\mu_1 \neq \mu_2 \Rightarrow \omega_q^{\mu_1} \neq \omega_q^{\mu_2}$ and $t_\lambda \neq 0$ for all $\lambda \in M_q$ and $s_\mu \neq 0$ for all $\mu \in N_q$. We observe that M_q and N_q contain the same number of elements and that for all $\lambda \in M_q$ there exists a $\mu \in N_q$ such that $s_\mu = t_\lambda$ and $\omega_q^\mu = \tau_q^\lambda$. Now let $T_q^\lambda, \Omega_q^\mu : \mathrm{Sin}_q(X) \to \mathrm{Sin}_{q-1}(X)$ be given by

$$T_q^\lambda(\sigma) = \sigma \circ \tau_q^\lambda, \quad \Omega_q^\mu(\sigma) = \sigma \circ \omega_q^\mu.$$

Then we have that

$$\left\{T_q^\lambda | \lambda \in M_q\right\} \subseteq \left\{\partial_j \circ \beta_q^\nu | (\nu, j) \in A_q \times \{0, 1, \ldots, q\}\right\}$$
$$\left\{\Omega_q^\mu | \mu \in N_q\right\} \subseteq \left\{\beta_{q-1}^\alpha \circ \partial_j | (j, \alpha) \in \{0, 1, \ldots, q\} \times A_{q-1}\right\}$$

and $\partial\beta_q = \sum_{\lambda \in M_q} t_\lambda T_q^\lambda$, $\beta_{q-1}\partial = \sum_{\mu \in N_q} s_\mu \Omega_q^\mu$. These results are also true for $q = 1$ with some small, obvious changes in the notation (put $A_0 = \{0\}$,$r_0 = 1$,$\beta_0^0 = \text{id}$ and $\sigma_0^0 = \text{id}$).

Define $\beta_q : \mathscr{C}_q(X) \to \mathscr{C}_q(X)$ by $\beta_q = \sum_{\nu \in A_q} r_\nu (\beta_q^\nu)_*$ for $q \geq 1$ and $\beta_0 = \text{id}$.

LEMMA 4.1. *$\beta : \mathscr{C}_*(X) \to \mathscr{C}_*(X)$ is a natural chain map.*

PROOF. The map is natural by definition. For $q \geq 1$ we have

$$\partial\beta_q = \sum_{j=0}^{q} (-1)^j (\partial_j)_* \left(\sum_{\nu \in A_q} r_\nu (\beta_q^\nu)_*\right) = \sum_{j=0}^{q} \sum_{\nu \in A_q} (-1)^j r_\nu (\partial_j \circ \beta_q^\nu)_*$$

$$\beta_{q-1}\partial = \sum_{j=0}^{q} \sum_{\alpha \in A_{q-1}} (-1)^j r_\alpha (\beta_{q-1}^\alpha \circ \partial_j)_* \quad \left(\beta_0 \partial = \sum_{j=0}^{1} (-1)^j (\partial_j)_*\right).$$

From the remarks before the lemma we conclude that

$$\partial\beta_q = \sum_{\lambda \in M_q} t_\lambda (T_q^\lambda)_*, \quad \beta_{q-1}\partial = \sum_{\mu \in N_q} s_\mu (\Omega_q^\mu)_*$$

which implies $\partial\beta_q = \beta_{q-1}\partial$.

In the following we write (4) as $s_q = \sum_{\alpha \in B_q} r_\alpha s_q^\alpha$ where $r_\alpha \in \{-1, 1\}$ and $\{s_q^\alpha \,|\, \alpha \in B_q\} = M_f^q \cup M_g^q$. Here

$$M_f^q = \left\{f_{q,\nu}^{k_1 \ldots k_q} \,|\, \nu \in \{1, \ldots, q\}, (k_1, \ldots, k_q) \in A_q\right\}$$
$$M_g^q = \left\{g_{q,\nu}^{k_{\nu+1} \ldots k_q} \,|\, \nu \in \{1, \ldots, q-1\}, (k_{\nu+1}, \ldots, k_q) \in A_q^{\nu+1}\right\} \cup \left\{h_{q,q}\right\}$$

where A_q^p is the set of tuples $(a_p, \ldots, a_q)$, $a_i = 0, 1, \ldots, i$, $p = 1, 2, \ldots, q$. We then have that

$$\partial s_q = \sum_{j=0}^{q+1} \sum_{\alpha \in B_q} (-1)^j r_\alpha \partial_j \circ s_q^\alpha, \quad s_{q-1}\partial = \sum_{j=0}^{q} \sum_{\gamma \in B_{q-1}} (-1)^j r_\gamma s_{q-1}^\gamma \circ \partial_j$$

so $\partial s_q + s_{q-1}\partial = \beta_q - \text{id}$ is equivalent to

$$\sum_{j=0}^{q+1} \sum_{\alpha \in B_q} (-1)^j r_\alpha \partial_j \circ s_q^\alpha + \sum_{j=0}^{q} \sum_{\gamma \in B_{q-1}} (-1)^j r_\gamma s_{q-1}^\gamma \circ \partial_j = \sum_{\nu \in A_q} r_\nu \beta_q^\nu - \text{id}. \tag{5}$$

Define $s_q : \mathscr{C}_q(X) \to \mathscr{C}_{q+1}(X)$ by $s_q = \sum_{\alpha\in B_q} r_\alpha (s_q^\alpha)_*$ for $q \geq 1$. For $q \leq 0$, $s_q = 0$.

LEMMA 4.2. *$s : \beta \simeq \mathrm{id}_{\mathscr{C}_*(X)}$ is a natural chain homotopy.*

PROOF. Naturality follows from the definition. We have $\beta_q - \mathrm{id} = \sum_{\nu\in A_q} r_\nu (\beta_q^\nu)_* - \mathrm{id}$ and

$$\partial s_q = \sum_{j=0}^{q+1} \sum_{\alpha\in B_q} (-1)^j r_\alpha (\partial_j \circ s_q^\alpha)_*, \quad s_{q-1}\partial = \sum_{j=0}^{q} \sum_{\gamma\in B_{q-1}} (-1)^j r_\gamma (s_{q-1}^\gamma \circ \partial_j)_*.$$

Now evaluate (5) on $\imath_q$ and use the same procedure as in the remarks before lemma 4.1 to deduce that s defines a chain homotopy $\partial s_q + s_{q-1}\partial = \beta_q - \mathrm{id}$.

Let $\mathscr{U} = \{U_\alpha | \alpha \in I\}$ be a family of subsets of X. We put $W_\alpha^k = (i_\alpha)_\#(\mathrm{Sin}_k(U_\alpha))$ where $i_\alpha : U_\alpha \to X$ is the inclusion map. If U_α is open in X then W_α^k is open in $\mathrm{Sin}_k(X)$ and $(i_\alpha)_\# : \mathrm{Sin}_k(U_\alpha) \to W_\alpha^k$ is a homeomorphism.

LEMMA 4.3. *Let $n \in \mathsf{N}$. Then we have a natural chain homotopy $c : \beta^n \simeq \mathrm{id}_{\mathscr{C}_*(X)}$. If $\mu \in \mathscr{C}_q(X)$ and $\partial\mu \in \mathscr{C}_{q-1}^{\mathscr{U}}(X)$ then $c_{q-1}(\partial\mu) \in \mathscr{C}_q^{\mathscr{U}}(X)$.*

PROOF. Let $c_q = s_q(\mathrm{id} + \beta_q + \ldots + \beta_q^{n-1}) : \mathscr{C}_q(X) \to \mathscr{C}_{q+1}(X)$. Then c is a natural chain homotopy between β^n and $\mathrm{id}_{\mathscr{C}_*(X)}$. Now let $q \geq 1$ and $\mu \in \mathscr{C}_q(X)$ and assume that $\partial\mu \in \mathscr{C}_{q-1}^{\mathscr{U}}(X)$. Write $\partial\mu = \sum_{j=1}^n r_j\mu_j, r_j \in \mathsf{R}$, $\mu_j \in \mathscr{C}_{q-1}^{U_{\alpha_j}}(X)$ and choose $\nu_j \in \mathscr{C}_{q-1}(U_{\alpha_j})$ such that $\mu_j = i_{\alpha_j}(\nu_j)$. By naturality of c

$$c_{q-1}(\partial\mu) = \sum_{j=1}^n r_j c_{q-1}(i_{\alpha_j}(\nu_j)) = \sum_{j=1}^n r_j i_{\alpha_j}(c_{q-1}(\nu_j)) \in \mathscr{C}_q^{\mathscr{U}}(X).$$

The main lemma is

LEMMA 4.4. *Let $\mathscr{U} = \{U_\alpha \,|\, \alpha \in I\}$ be a family of subsets of X such that $\mathrm{Int}(\mathscr{U})$ is a covering of X. Then for all $\mu \in \mathscr{C}_q(X)$ there exists a natural number n so that the n'th iterate $(\beta_q)^n(\mu) \in \mathscr{C}_q^{\mathscr{U}}(X)$.*

PROOF. We may assume that U_α is open since $\mathscr{C}_q^{\mathrm{Int}(\mathscr{U})} \subseteq \mathscr{C}_q^{\mathscr{U}}(X)$. Now let $q \geq 1$. Since $\beta_q = \sum_{\nu\in A_q} r_\nu (\beta_q^\nu)_*$ we have

$$(\beta_q)^k = \sum_{\nu_1\in A_q} \cdots \sum_{\nu_k\in A_q} r_{\nu_1\ldots\nu_k} (\beta_q^{\nu_k})_* \circ \cdots \circ (\beta_q^{\nu_1})_*$$
$$= \sum_{\nu_1\in A_q} \cdots \sum_{\nu_k\in A_q} r_{\nu_1\ldots\nu_k} (\beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1})_*$$

for every $k \in \mathsf{N}$ where $r_{\nu_1 \dots \nu_k} = \prod_{j=1}^{k} r_{\nu_j}$. From standard singular theory (cf. [D] (6.3) p. 41) we know that for given $\epsilon > 0$ $(\beta_q)^k(\imath_q)$ is a formal linear combination of simplices of diameter less than ϵ if k is sufficiently large, say $k \geq n_0$. Now

$$(\beta_q)^k(\imath_q) = \sum_{\nu_1 \in A_q} \cdots \sum_{\nu_k \in A_q} r_{\nu_1 \dots \nu_k} (\beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1})(\imath_q)$$

for every $k \in \mathsf{N}$ so $\operatorname{diam}(\beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1}(\imath_q)(\Delta^q)) < \epsilon$ for all $k \geq n_0$ and all $(\nu_1, \dots, \nu_k) \in (A_q)^k$. Here $\operatorname{diam}(C)$ denotes the diameter of C. For $\sigma \in \operatorname{Sin}_q(X)$, $\mathscr{W} = \{\sigma^{-1}(U_\alpha) | \alpha \in I\}$ is an open covering of Δ^q. This being compact there exists an $\epsilon_\sigma > 0$ such that for $C \subseteq \Delta^q$ of diameter less than ϵ_σ, there exist an index α with $C \in \sigma^{-1}(U_\alpha)$ (ϵ_σ is the Lebesgue number of the covering $\mathscr{W}$). Choose n_σ such that for all $k \geq n_\sigma$ and all $(\nu_1, \dots, \nu_k) \in (A_q)^k$ we have the implications

$$\operatorname{diam}(\beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1}(\imath_q)(\Delta^q)) < \epsilon_\sigma \Rightarrow \exists \alpha \in I : \beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1}(\imath_q)(\Delta^q) \subseteq \sigma^{-1}(U_\alpha)$$
$$\Rightarrow \exists \alpha \in I : \beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1}(\sigma) \in W_\alpha^q.$$

Now let $k \geq n_\sigma$ and $(\nu_1, \dots, \nu_k) \in (A_q)^k$ and choose $\alpha \in I$ such that $\beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1}(\sigma) \in W_\alpha^q$. Since W_α^q is an open subset of $\operatorname{Sin}_q(X)$ and $\beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1} : \operatorname{Sin}_q(X) \to \operatorname{Sin}_q(X)$ is continuous, it follows that there is an open neighborhood $U_\sigma^{\nu_1 \dots \nu_k}$ of σ in $\operatorname{Sin}_q(X)$ such that $\beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1}(U_\sigma^{\nu_1 \dots \nu_k}) \subseteq W_\alpha^q$. Set $U_\sigma^k = \bigcap_{\nu_1 \in A_q} \cdots \bigcap_{\nu_k \in A_q} U_\sigma^{\nu_1 \dots \nu_k}$. It is an open neighboorhood of σ in $\operatorname{Sin}_q(X)$ and for all $(\nu_1, \dots, \nu_k) \in (A_q)^k$ there is an index α with $\beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1}(U_\sigma^k) \subseteq W_\alpha^q$. Let $\mu \in \mathscr{C}_q(X)$ be a chain with $\operatorname{Supp}(\mu) \subseteq K$ where $K \subseteq \operatorname{Sin}_q(X)$ is compact and set $O_\sigma = U_\sigma^{n_\sigma}$. Since $\{O_\sigma\}_{\sigma \in \operatorname{Sin}_q(X)}$ is a covering of K with open subsets of $\operatorname{Sin}_q(X)$ we can find $\sigma_1, \dots, \sigma_l \in \operatorname{Sin}_q(X)$ such that $K \subseteq O_{\sigma_1} \cup \ldots \cup O_{\sigma_l}$. Set $n_j = n_{\sigma_j}$ for $j \in \{1, \dots, l\}$ and set $n = \max\{n_1, \dots, n_l\}$ and let $\tau \in K$. Choose a $j \in \{1, \dots, l\}$ such that $\tau \in O_{\sigma_j}$. Then for all $(\nu_1, \dots, \nu_{n_j}) \in (A_q)^{n_j}$ there is an index α with $\beta_q^{\nu_{n_j}} \circ \cdots \circ \beta_q^{\nu_1}(\tau) \in W_\alpha^q$. Now let $k \geq n_j$ and $(\nu_1, \dots, \nu_k) \in (A_q)^k$. Choose $\alpha \in I$ such that $\beta_q^{\nu_{n_j}} \circ \cdots \circ \beta_q^{\nu_1}(\tau) \in W_\alpha^q$. Then $\beta_q^{\nu_k} \circ \cdots \circ \beta_q^{\nu_1}(\tau) \in W_\alpha^q$. (Let $\omega \in W_\alpha^q$. Then $\omega(\Delta^q) \subseteq U_\alpha$. We therefore have

$$\beta_q^{\nu}(\omega)(\Delta^q) = \omega \circ \sigma_q^{\nu}(\Delta^q) \subseteq \omega(\Delta^q) \subseteq U_\alpha$$

for $\nu \in A_q$ so $\beta_q^{\nu}(\omega) \in W_\alpha^q$.) Thus to each $(\nu_1, \dots, \nu_n) \in (A_q)^n$ and $\tau \in K$ we can find an index α with $\beta_q^{\nu_n} \circ \cdots \circ \beta_q^{\nu_1}(\tau) \in W_\alpha^q$. Since $\beta_q^{\nu_n} \circ \cdots \circ \beta_q^{\nu_1}$ is continuous $L = \beta_q^{\nu_n} \circ \cdots \circ \beta_q^{\nu_1}(K)$ is a compact subset of $\operatorname{Sin}_q(X)$; actually $L \subseteq W^q = \bigcup_{\alpha \in I} W_\alpha^q$. The support of $\lambda = (\beta_q^{\nu_n} \circ \cdots \circ \beta_q^{\nu_1})_*(\mu)$ is contained in L. Choose $\alpha_1, \dots, \alpha_m \in I$ such that $L \subseteq W_{\alpha_1}^q \cup \ldots \cup W_{\alpha_m}^q$ and let $V_j = W_{\alpha_j}^q \cap L$. Since $V_j \in \mathscr{B}(\operatorname{Sin}_q(X))$ it follows that $\mathscr{B}(V_j) \subseteq \mathscr{B}(\operatorname{Sin}_q(X))$.

The restriction λ_j of λ to $\mathscr{B}(V_j)$ defines a real Borel measure on V_j, $j = 1, 2, \ldots, m$. $\{V_j\}_{j=1}^m$ is an open covering of L and L is normal since it is a closed subset of the normal space $\mathrm{Sin}_q(X)$. We can therefore choose a partition of unity $\{\rho_j\}_{j=1}^m$ subordinated to the covering $\{V_j\}_{j=1}^m$. The maps $\rho_j : V_j \to \mathsf{R}$ are continuous and therefore Borel measurable. Now let

$$\eta_j(B) = \int_B \rho_j \, d(\lambda_j), \quad B \in \mathscr{B}(V_j), \; j = 1, 2, \ldots, m.$$

Since $\rho_j(L) \subseteq [0, 1]$ it follows that $\rho_j \in L^1(\lambda_j)$. This implies that η_j is a signed Borel measure on V_j of bounded total variation, and we can define μ_j by

$$\mu_j(B) = \eta_j\Big((i_{\alpha_j})_\#(B) \cap L\Big), \quad B \in \mathscr{B}(\mathrm{Sin}_q(U_{\alpha_j})), \; j = 1, 2, \ldots, m.$$

Since $V_j \in \mathscr{B}(W^q_{\alpha_j})$ we have

$$\mathscr{B}(V_j) = \Big\{B \cap V_j | B \in \mathscr{B}(W^q_{\alpha_j})\Big\} = \Big\{B \cap L | B \in \mathscr{B}(W^q_{\alpha_j})\Big\}.$$

Now $(i_{\alpha_j})_\# : \mathrm{Sin}_q(U_{\alpha_j}) \to W^q_{\alpha_j}$ is a homeomorphism, so induces a bijection $\mathscr{P}\Big((i_{\alpha_j})_\#\Big) : \mathscr{B}(\mathrm{Sin}_q(U_{\alpha_j})) \to \mathscr{B}(W^q_{\alpha_j})$. Thus μ_j is a well defined real valued Borel measure on $\mathrm{Sin}_q(U_{\alpha_j})$. If we put $L_j = \mathrm{Supp}_L(\rho_j) \subseteq V_j \subseteq W^q_{\alpha_j}$ then

$$\begin{aligned}\eta_j(B) &= \int_B \rho_j \, d(\lambda_j) = \int_{V_j} \chi_B \rho_j \, d(\lambda_j) = \int_{V_j} \chi_B \chi_{L_j} \rho_j \, d(\lambda_j) \\ &= \int_{V_j} \chi_{B \cap L_j} \rho_j \, d(\lambda_j) = \int_{B \cap L_j} \rho_j \, d(\lambda_j) = \eta_j(B \cap L_j)\end{aligned}$$

for $B \in \mathscr{B}(V_j)$, where χ_B is the characteristic function of B etc. Observe here that L_j is a closed subset of L hence of $\mathrm{Sin}_q(X)$, so $L_j \in \mathscr{B}(\mathrm{Sin}_q(X))$. Moreover $B \in \mathscr{B}(V_j) \subseteq \mathscr{B}(\mathrm{Sin}_q(X))$ so $B \cap L_j \subseteq \mathscr{B}(\mathrm{Sin}_q(X))$. But then $B \cap L_j \subseteq \mathscr{B}(V_j)$ since $B \cap L_j \subseteq V_j \in \mathscr{B}(\mathrm{Sin}_q(X))$. This shows that the above calculations are allowed. Also $M_j = (i_{\alpha_j})^{-1}_\#(L_j)$ is a compact subset of $\mathrm{Sin}_q(U_{\alpha_j})$ and we have

$$\begin{aligned}\mu_j(D) &= \eta_j\Big((i_{\alpha_j})_\#(D) \cap L\Big) = \eta_j\Big((i_{\alpha_j})_\#(D) \cap L \cap L_j\Big) \\ &= \eta_j\Big((i_{\alpha_j})_\#(D) \cap L \cap (i_{\alpha_j})_\#(M_j)\Big) \\ &= \eta_j\Big((i_{\alpha_j})_\#(D \cap M_j) \cap L\Big) = \mu_j(D \cap M_j)\end{aligned}$$

for all $D \in \mathscr{B}(\mathrm{Sin}_q(U_{\alpha_j}))$, i.e. $\mu_j \in \mathscr{C}_q(U_{\alpha_j})$. If $B \in \mathscr{B}(\mathrm{Sin}_q(X))$ we have

$$\begin{aligned}\sum_{j=1}^{m} i_{\alpha_j}(\mu_j)(B) &= \sum_{j=1}^{m} \mu_j\Big((i_{\alpha_j})_{\#}^{-1}(B \cap W_{\alpha_j}^q)\Big) = \sum_{j=1}^{m} \eta_j(B \cap W_{\alpha_j}^q \cap L) \\ &= \sum_{j=1}^{m} \int_{B\cap V_j} \rho_j \, d(\lambda_j) = \sum_{j=1}^{m} \int_{V_j} \chi_{B\cap V_j}\rho_j \, d(\lambda_j) \\ &= \sum_{j=1}^{m} \int_{\mathrm{Sin}_q(X)} \chi_{B\cap V_j}\xi_j \, d\lambda,\end{aligned}$$

where $\xi_j : \mathrm{Sin}_q(X) \to \mathsf{R}$ is the Borel function defined by $\xi_j(x) = \rho_j\chi_L(x)$, $x \in \mathrm{Sin}_q(X)$. Now $\chi_{L\cap B} = \chi_{B\cap V_j} + \chi_{(L\cap B)\setminus V_j}$ and $\xi_j = 0$ on $(L \cap B) \setminus V_j$ so $\chi_{B\cap V_j}\xi_j = \chi_{L\cap B}\xi_j$. Since $\sum \xi_j \equiv 1$ on L,

$$\begin{aligned}\sum_{j=1}^{m} i_{\alpha_j}(\mu_j)(B) &= \sum_{j=1}^{m} \int_{\mathrm{Sin}_q(X)} \chi_{B\cap L}\xi_j \, d(\lambda) = \int_{\mathrm{Sin}_q(X)} \chi_{B\cap L} \, d(\lambda) \\ &= \lambda(B \cap L) = \lambda(B).\end{aligned}$$

It follows that $\lambda = \sum_{j=1}^{m} i_{\alpha_j}(\mu_j)$ and we conclude that

$$(\beta_q)^n(\mu) = \sum_{\nu_1\in A_q} \cdots \sum_{\nu_n\in A_q} r_{\nu_1\ldots\nu_n}(\beta_q^{\nu_n} \circ \cdots \circ \beta_q^{\nu_1})_*(\mu) \in \mathscr{C}_q^{\mathscr{U}}(X).$$

For $q = 0$ the result easily follows by the preceding. Observe that $W^0 = \bigcup_{\alpha\in I} W_\alpha^0 = \mathrm{Sin}_0(X)$ and skip the first part of the proof and let $\lambda = \mu$ and $L = K$ in the last part the proof.

Proof of theorem 3.1. Set $\mathscr{C}_*(X;\mathscr{U}) = \mathscr{C}_*(X)/\mathscr{C}_*^{\mathscr{U}}(X)$, giving the exact sequence

$$0 \longrightarrow \mathscr{C}_*^{\mathscr{U}}(X) \xrightarrow{\ i\ } \mathscr{C}_*(X) \xrightarrow{\ \pi\ } \mathscr{C}_*(X;\mathscr{U}) \longrightarrow 0.$$

We must show that $H_*(\mathscr{C}_*(X;\mathscr{U})) = 0$. Let $\mu + \mathscr{C}_q^{\mathscr{U}}(X) \in Z_q(\mathscr{C}_*(X;\mathscr{U}))$. Then

$$\bar{\partial}(\mu + \mathscr{C}_q^{\mathscr{U}}(X)) = \partial\mu + \mathscr{C}_{q-1}^{\mathscr{U}}(X) = 0 + \mathscr{C}_{q-1}^{\mathscr{U}}(X) \Leftrightarrow \partial\mu \in \mathscr{C}_{q-1}^{\mathscr{U}}(X).$$

By lemma 4.4 we can choose $n \in \mathsf{N}$ such that $(\beta_q)^n(\mu) \in \mathscr{C}_q^{\mathscr{U}}(X)$. Let c be the chain homotopy between β^n and $id_{\mathscr{C}_*(X)}$, see lemma 4.3. Set $y = (\beta_q)^n(\mu) - c_{q-1}(\partial\mu)$ and $x = -c_q(\mu)$. Since $\partial\mu \in \mathscr{C}_{q-1}^{\mathscr{U}}(X)$ it follows from lemma 4.3 that $c_{q-1}(\partial\mu) \in \mathscr{C}_q^{\mathscr{U}}(X)$ which implies that $y \in \mathscr{C}_q^{\mathscr{U}}(X)$. Moreover we have that $x \in \mathscr{C}_{q+1}(X)$ and $\mu = id(\mu) = (\beta_q)^n(\mu) - c_{q-1}(\partial\mu) - \partial(c_q(\mu)) = y + \partial x$. This implies that

$$\mu - \partial x \in \mathscr{C}_q^{\mathscr{U}}(X) \Leftrightarrow \mu + \mathscr{C}_q^{\mathscr{U}}(X) = \partial x + \mathscr{C}_q^{\mathscr{U}}(X) = \bar{\partial}(x + \mathscr{C}_{q+1}^{\mathscr{U}}(X)).$$

But then $\mu + \mathscr{C}_q^{\mathscr{U}}(X) \in B_q(\mathscr{C}_*(X;\mathscr{U}))$.

5. Measure homology and CW-complexes.

In this section we show that the measure homology groups are isomorphic to the singular homology groups on the category of CW-complexes. The problem is that not all CW-complexes are metrizable so we need the following result. Let X be a topological space and let $\{X_\alpha\}_{\alpha\in I}$ be the family of compact subsets of X partially ordered by inclusion. Let $i_\alpha^\beta : X_\alpha \to X_\beta$ be the inclusion when $X_\alpha \subseteq X_\beta$. Then $\{H_*^\mu(X_\alpha)\}_{\alpha\in I}$ forms a direct system of real vector spaces with the linear maps $f_\alpha^\beta = H_*^\mu(i_\alpha\beta) : H_*^\mu(X_\alpha) \to H_*^\mu(X_\beta)$ induced by the inclusion maps. We now have

Proposition 5.1.

$$H_*^\mu(X) \cong \varinjlim_\alpha H_*^\mu(X_\alpha)$$

Proof. Let $i_\alpha : X_\alpha \to X$ be the inclusion maps, $f_\alpha = H_*^\mu(i_\alpha) : H_*^\mu(X_\alpha) \to H_*^\mu(X)$ and let $f = \bigoplus_{\alpha\in I} f_\alpha : \bigoplus_{\alpha\in I} H_*^\mu(X_\alpha) \to H_*^\mu(X)$. If $\sum_{i=1}^n x_{\alpha_i} \in \bigoplus_{\alpha\in I} H_*^\mu(X_\alpha)$ and $\beta \in I$ is such that $X_{\alpha_i} \subseteq X_\beta$ for $i = 1, 2, \dots, n$ and $\sum_{i=1}^n f_{\alpha_i}^\beta(x_{\alpha_i}) = 0$, then $0 = f_\beta\left(\sum_{i=1}^n f_{\alpha_i}^\beta(x_{\alpha_i})\right) = \sum_{i=1}^n f_{\alpha_i}(x_{\alpha_i}) = f\left(\sum_{i=1}^n x_{\alpha_i}\right)$. Thus f induces a linear map

$$\tilde{f} : \varinjlim_\alpha H_*^\mu(X_\alpha) \to H_*^\mu(X).$$

Now let $[\mu] \in H_q^\mu(X)$ and choose a compact subset $K \subseteq \mathrm{Sin}_q(X)$ with $\mathrm{Supp}(\mu) \subseteq K$. The evaluation map $\omega : \mathrm{Sin}_q(X) \times \Delta^q \to X$ defined by $\omega(\sigma, x) = \sigma(x)$ is continuous since Δ^q is compact so $A = \omega(K \times \Delta^q)$ is a compact subset of X. Now if $\sigma \in K$ we have that $\sigma(\Delta^q) \subseteq A$, and it follows that $K \subseteq j_\#(\mathrm{Sin}_q(A))$ where $j : A \to X$ is the inclusion. But then $L = j_\#^{-1}(K)$ is a compact subset of $\mathrm{Sin}_q(A)$. The homeomorphism $j_\# : L \to K$ induces a bijection $Q = \mathscr{P}(j_\#) : \mathscr{B}(L) \to \mathscr{B}(K)$. Since $K \in \mathscr{B}(\mathrm{Sin}_q(X))$ we have that $\mathscr{B}(K) \subseteq \mathscr{B}(\mathrm{Sin}_q(X))$. Moreover $\mathscr{B}(L) = \{L \cap D \mid D \in \mathscr{B}(\mathrm{Sin}_q(A))\}$, so we can define a signed Borel measure ν on $\mathrm{Sin}_q(A)$ of bounded total variation by $\nu(D) = \mu \circ Q(L \cap D)$. By definition $\mathrm{Supp}(\nu) \subseteq L$ so $\nu \in \mathscr{C}_q(A)$. If $B \in \mathscr{B}(\mathrm{Sin}_q(X))$ we have

$$\begin{aligned} j(\nu)(B) &= \nu(j_\#^{-1}(B)) = \mu \circ Q(L \cap j_\#^{-1}(B)) = \mu \circ Q(j_\#^{-1}(K \cap B)) \\ &= \mu(K \cap B) = \mu(B). \end{aligned}$$

Now $0 = \partial\mu = j(\partial\nu)$ and by lemma 2.1 $j : \mathscr{C}_*(A) \to \mathscr{C}_*(X)$ is injective so $\partial\nu = 0$. All in all we see that $[\mu] = H_q^\mu(j)([\nu])$ so f and thus $\tilde{f}$ is surjective.

Suppose that $\sum_{i=1}^n [\mu_{\alpha_i}] \in \bigoplus_{\alpha\in I} H_*^\mu(X_\alpha)$ with $f\left(\sum_{i=1}^n [\mu_{\alpha_i}]\right) = 0$. Since

$$f\left(\sum_{i=1}^{n} [\mu_{\alpha_i}]\right) = \sum_{i=1}^{n} f_{\alpha_i}([\mu_{\alpha_i}]) = \sum_{i=1}^{n} [i_{\alpha_i}(\mu_{\alpha_i})] = \left[\sum_{i=1}^{n} i_{\alpha_i}(\mu_{\alpha_i})\right]$$

we can choose $\nu \in \mathscr{C}_{q+1}(X)$ with $\partial\nu = \sum_{i=1}^{n} i_{\alpha_i}(\mu_{\alpha_i})$. Let $K \subseteq \mathrm{Sin}_{q+1}(X)$ be a compact support of ν and let $A = \omega(K \times \Delta^{q+1})$ where $\omega : \mathrm{Sin}_{q+1}(X) \times \Delta^{q+1} \to X$ is the continuous evaluation map. Then $X_\beta = A \cup \bigcup_{i=1}^{n} X_{\alpha_i}$ is a compact subset of X. For $\sigma \in K$, $\sigma(\Delta^{q+1}) \subseteq A \subseteq X_\beta$, so we can choose $\lambda \in \mathscr{C}_{q+1}(X_\beta)$ with $i_\beta(\lambda) = \nu$. Since

$$i_\beta(\partial\lambda) = \partial i_\beta(\lambda) = \partial\nu = \sum_{i=1}^{n} i_{\alpha_i}(\mu_{\alpha_i}) = i_\beta\left(\sum_{i=1}^{n} i_{\alpha_i}^{\beta}(\mu_{\alpha_i})\right)$$

and since $i_\beta : \mathscr{C}_q(X_\beta) \to \mathscr{C}_q(X)$ is injective, $\sum_{i=1}^{n} i_{\alpha_i}^{\beta}(\mu_{\alpha_i})$ is a boundary in $\mathscr{C}_q(X_\beta)$. Hence $\sum_{i=1}^{n} f_{\alpha_i}^{\beta}([\mu_{\alpha_i}]) = \sum_{i=1}^{n} \left[i_{\alpha_i}^{\beta}(\mu_{\alpha_i})\right] = 0$.

Corollary 5.2. *The measure homology groups and the singular homology groups are isomorphic on the category of CW-complexes.*

Proof. This follows by the fact that a compact subset of a CW-complex is metrizable.

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MATEMATISK INSTITUT
AARHUS UNIVERSITET
BYGNING 530
8000 AARHUS C
DENMARK
email: kold@mi.aau.dk