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LOGARITHMIC DIFFERENTIAL FORMS AND THE COHOMOLOGY OF THE COMPLEMENT OF A DIVISOR

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Introduction.

Let D be a divisor in the complex manifold X, let $U = X \setminus D$, and denote the inclusion $U \hookrightarrow X$ by j. Recall the Grothendieck Comparison Theorem ([Gro]):

THEOREM. *Let $\Omega^\bullet(*D)$ denote the complex of differential forms on X with meromorphic poles along D. Then the de Rham morphism $\Omega^\bullet(*D) \to \mathsf{R}j_*\mathsf{C}_\mathsf{U}$ is a quasi-isomorphism.*

Thus, if X is a Stein space (for example, C^n) then for each cohomology class $c \in H^p(X \setminus D; \mathsf{C})$, there is a differential form ω with pole of finite order along D, such that for any p-cycle σ on $X \setminus D$ one has $c(\sigma) = \int_\sigma \omega$.

It is natural to ask what one can say about the order of the pole of ω along D. For example, if D is a complex submanifold then the order of pole can be taken to be 1; this generalises the elementary fact that the cohomology of the complement of $\{0\}$ in C is generated by the 1-form dz/z. The question of the order of the pole goes back to Griffiths [Gri]; see also [DD], [D], [K].

In case the divisor D is nonsingular, one can make a stronger statement than merely that the order of pole is 1: the differential form ω can be assumed to have a *logarithmic* pole along D. That is, not only does ω have a pole of order at most 1, but so also does its exterior derivative, $d\omega$. Differential forms with logarithmic poles evidently form a subcomplex of $\Omega^\bullet(*D)$, which is denoted by $\Omega^\bullet(\log D)$. This fact was exploited by Castro, Narváez and Mond in [CMN] to prove a version of the Grothendieck comparison theorem in which $\Omega^\bullet(*D)$ is replaced by $\Omega^\bullet(\log D)$, for a rather special class of divisors, namely locally quasihomogenenous free divisors. A divisor D is *free* if each term $\Omega^p(\log D)$ in the logarithmic complex is locally free as an $\mathcal{O}_X$-module; *local quasihomogeneity* is the property that in some neighbour-

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hood of each point $x \in D$, there is a good $\mathbb{C}^*$-action, centred at x, preserving D.

The theorem of [CMN] can be summarised as saying that the *logarithmic comparison theorem* holds for locally quasihomogenenous free divisors.

In this paper we generalise the method of [CMN] to cover a wider class of locally quasihomogeneous divisors. We construct a spectral sequence converging to $H^\bullet(X \setminus D; \mathbb{C})$, whose E_1 term involves local cohomology with coefficients in the $\Omega^p(\log D)$, and give conditions under which the sequence collapses to the complex $\Omega^\bullet(\log D)$. As a consequence of our main results (Theorems 2.3 and 2.4), we prove, for example, that the logarithmic comparison theorem holds if either D is a surface with only simple (i.e. du Val) singularities, or if D is a surface with smooth normalisation $\widetilde{D}$, such that the projection $\pi : \widetilde{D} \to D$ has only simple singularities in the sense of [Mo1]. Another consequence of our results (see 2.6) is that the logarithmic comparison theorem holds for some but not all simple isolated hypersurface singularities in $\mathbb{C}^n$, for $n \geq 3$. In fact, it holds precisely for those in the table below.

$$\begin{array}{ll} A_k : x_1^{k+1} + x_2^2 + \cdots + x_n^2, k \geq 1 & \text{except } k \text{ odd}, n \text{ even} \\ D_k : x_1^{k-1} + x_1x_2^2 + x_3^2 + \cdots + x_n^2, k \geq 4 & n \text{ odd} \\ E_6 : x_1^4 + x_2^3 + x_3^2 + \cdots + x_n^2 & \\ E_7 : x_1^3x_2 + x_2^3 + x_3^2 + \cdots + x_n^2 & n \text{ odd} \\ E_8 : x_1^5 + x_2^3 + x_3^2 + \cdots + x_n^2. & \end{array}$$

Our main technical result concerning isolated singularities is:

THEOREM. *Suppose that D is a divisor with isolated singularity at x in the n-dimensional complex manifold X, and that D is weighted homogenenous at x with respect to positive integer weights $w_1, \ldots, w_n$. Let $h \in \mathcal{O}_{X,x}$ be a reduced defining equation for D of weight r. Then the following statements are equivalent:*

(a) the logarithmic comparison theorem holds at x;
(b) $(R^ij_*\mathbb{C}_U)_x = 0$ for $i \geq 2$;
(c) $(\mathcal{O}_{X,x}/J_h)_{ir-\sum w_j} = 0$, for $1 \leq i \leq n-2$;
(d) the link of x in D is a $\mathbb{Q}-$homology sphere.

Furthermore, each of these statements implies that $\mathcal{H}^i(\Omega^\bullet(\log D))_x = 0$ for $i \geq 2$, and if $n = 3$ the reverse implication also holds.

1. The logarithmic defect.

1.1. A polynomial $h \in \mathbb{C}[x_1, \ldots, x_n]$ is *quasihomogeneous* with respect to (positive integer) weights $w_1, \ldots, w_n$ if it is homogeneous when the variable x_i is considered to have degree w_i, for $1 \leq i \leq n$. An analytic divisor $D \subset U$, where U is a non-empty open subset of $\mathbf{C}^n$, is *quasihomogeneous* (with respect to these weights) if it is the zero locus in U of a quasihomogeneous polynomial.

An analytic divisor D in an n-dimensional complex manifold X is *locally quasihomogeneous* if for each point $y \in D$ there exist local coordinates $(U; x_1, \ldots, x_n)$ centred at y (i.e. $x_i(y) = 0$, for $1 \leq i \leq n$) and weights $w_1, \ldots, w_n$ (depending on y) with respect to which $D \cap U \subset U$ is quasihomogeneous. Note that quasihomogeneity does not imply local quasihomogenenity: it is shown in [CMN2] that the quasihomogeneous hypersurface D in $\mathbf{C}^3$ defined by $x^5z + x^3y^3 + y^5z$ is not locally quasihomogeneous. It is singular at each point $(0, 0, z)$, but only when $z = 0$ is there a good $\mathbf{C}^*$-action centred at $(0, 0, z)$ preserving D.

However, it is evident that if h is a quasihomogeneous polynomial defining a hypersurface D with an isolated singularity at the origin then $D \subset \mathbf{C}^n$ is locally quasihomogeneous.

Let D be a divisor in a complex manifold X. Saito in [Sa] defines the coherent sheaf $\Omega^p(\log D)$ of logarithmic differential p-forms on X to be those meromorphic differential forms ω such that both ω and $d\omega$ have a pole of order at most one along D. Off the divisor $\Omega^p(\log D)$ coincides with Ω^p_X. He points out that exterior differentiation and wedge product make $\Omega^\bullet(\log D)$ into a differential graded algebra and observes that contraction by vector fields in $\mathrm{Der}(\log D)$ leaves $\Omega^\bullet(\log D)$ stable. Here, $\mathrm{Der}(\log D)$ denotes the sheaf of vector fields on X that stabilise $\mathscr{I}_D$.

1.2. In fact, it will be more convenient in this section to work locally. Fix, *throughout this section*, $D \subset U$ a quasihomogeneous divisor in an open subset U of $\mathbf{C}^n$. Let $S = \mathbf{C}[x_1, \ldots, x_n]$ and suppose that the variables have weights $w_1, \ldots, w_n$. Let us say that $h \in S$, a polynomial of weighted degree r, defines D at 0. Throughout this section we simplify notation by not distinguishing notationally between $\Omega^p(\log D)$ and the stalk of this sheaf at 0. We enforce a similar convention for $\mathrm{Der}(\log D)$.

The assumption that h is quasihomogeneous evidently leads to filtrations on $\Omega^p(\log D)$ and $\mathrm{Der}(\log D)$ etc. Perhaps the clearest way to understand this is as follows. The sheaf of algebraic logarithmic differential p-forms for an algebraic divisor is defined analogously to $\Omega^p(\log D)$. Let $\Omega^p_{\mathrm{alg}}(\log D)$ denote the the sheaf of algebraic logarithmic p-differential forms for the divisor in $\mathbf{C}^n$ defined by the quasihomogeneous polynomial h. Evidently $\Gamma(\mathbf{C}^n, \Omega^p_{\mathrm{alg}}(\log D))$ is a graded S-module. Further, it is easy to see that, in fact, the natural map $\Gamma(\mathbf{C}^n, \Omega^p_{\mathrm{alg}}(\log D)) \otimes_S \mathcal{O}_{\mathbf{C}^n, 0} \to \Omega^p(\log D)$ is an isomorphism. In general, if N is a graded S-module we write N_k for the part of degree k and $N[a]$ for the graded S-module with $N[a]_k = N_{a+k}$. If $M = N \otimes_S \mathcal{O}_{\mathbb{C}^n,0}$ then we use the notation $M[a] := N[a] \otimes_S \mathcal{O}_{\mathbb{C}^n,0}$.

It is shown in [Sa] that the natural contraction map $\Omega^1(\log D) \times \mathrm{Der}(\log D) \to \mathcal{O}_{\mathbb{C}^n,0}$ is a perfect pairing. Also, there is an isomorphism $\mathrm{Der}(\log D)[r - \sum w_k] \to \Omega^{n-1}(\log D)$ defined by $\mathscr{V} \mapsto \iota_{\mathscr{V}}(dx_1 \wedge \ldots \wedge dx_n/h)$. In the sequel, we often write dx/h for $dx_1 \wedge \ldots \wedge dx_n/h$.

Note that the *Euler field* $\chi_e = \sum w_i x_i \partial/\partial x_i$ is in $\mathrm{Der}(\log D)$. We define $\Omega^p(\log h)$ to be the submodule $\mathrm{Ker}\, \iota_{\chi_e}$ of $\Omega^p(\log D)$.

An easy but important observation is that for $\omega \in \Omega^p(\log D)$:

$$r\omega = \iota_{\chi_e}((dh/h) \wedge \omega) + (dh/h) \wedge \iota_{\chi_e}\omega.$$

It follows from this that $(\Omega^\bullet(\log D), \iota_{\chi_e})$ and $(\Omega^\bullet(\log D), (dh/h) \wedge \_)$ are split exact. In particular, one has that

$$\Omega^p(\log D) = \Omega^p(\log h) \oplus (dh/h) \wedge \Omega^{p-1}(\log h) \cong \Omega^p(\log h) \oplus \Omega^{p-1}(\log h). \tag{1.3}$$

Likewise, $\mathrm{Der}(\log D) = \mathcal{O}_{\mathbb{C}^n,0}(\chi_e) \oplus \mathrm{Der}(\log h)$, where $\mathrm{Der}(\log h)$ denotes the submodule of $\mathrm{Der}(\log D)$ vanishing on h. Putting this together with the isomorphism $\mathrm{Der}(\log D) \to \Omega^{n-1}(\log D)$ one deduces that the above decomposition of $\Omega^{n-1}(\log D)$ is:

$$\begin{aligned} \Omega^{n-1}(\log D) &= \mathcal{O}_{\mathbb{C}^n,0}\iota_{\chi_e}(dx/h) \oplus (dh/h) \wedge \Omega^{n-2}(\log h) \\ &\cong \mathcal{O}_{\mathbb{C}^n,0}[r - \sum w_k] \oplus \Omega^{n-2}(\log h). \end{aligned} \tag{1.4}$$

PROPOSITION 1.5. *Suppose that D is locally quasihomogeneous. Then*

(a) *The module* $\Omega^p(\log D)$ *is reflexive. In fact,*

$$\Omega^p(\log D) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n,0}}(\wedge^p \mathrm{Der}(\log D), \mathcal{O}_{\mathbb{C}^n,0}).$$

(b) *The wedge product gives a perfect pairing*

$$\Omega^p(\log D) \times \Omega^{n-p}(\log D) \to \Omega^n(\log D) \cong \mathcal{O}_{\mathbb{C}^n,0}[r - \sum w_j].$$

PROOF. (a) It follows from [Sa, Lemma 1.5] that there is a natural map $\Phi : \Omega^p(\log D) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n,0}}(\wedge^p \mathrm{Der}(\log D), \mathcal{O}_{\mathbb{C}^n,}0)$. This is easily seen to an isomorphism if $n = 1$. Since D is locally quasihomogeneous, and not merely quasihomogeneous, in a neighbourhood of any point distinct from the origin it is isomorphic to a product $\mathbb{C} \times D'$ for a locally quasihomogeneous divisor D' of lower dimension (see [CMN2, Lemmas 2.2, 2.3]). Thus, by induction on dimension, we may assume that Φ is an isomorphism at all points of a neighbourhood of 0, except possibly at 0 itself. It remains to establish the isomorphism at 0. Consider the short exact sequence:

$$0 \to \Omega^p(\log D) \xrightarrow{\Phi} \mathrm{Hom}_{\mathcal{O}_{\mathbf{C}^n,0}}(\wedge^p \operatorname{Der}(\log D), \mathcal{O}_{\mathbf{C}^n,0}) \to \operatorname{Cok}\Phi \to 0.$$

By assumption, the right-most term is supported at 0. The middle term is reflexive and so has depth at least two. We claim that the same is also true of the first term. This will certainly ensure that the right-most term is zero, as we need. In fact, this claim follows from the argument of [OT, Lemma 5.14].

(b) This is proved similarly to (a).

1.6. There is natural structure of a graded S-module on $H_0^q(\Omega_{\mathrm{alg}}^p(\log D))$, the qth cohomology group with supports in the origin and coefficients in $\Omega_{\mathrm{alg}}^p(\log D)$, as this identifies with the local cohomology group $H_{S_+}^q(\Gamma(\mathbf{C}^n, \Omega_{\mathrm{alg}}^p(\log D)))$. Consider now the analogous analytic situation. Unfortunately, the qth cohomology group with supports in the origin $H_0^q(\Omega^p(\log D))$ is, in general, bigger than the local cohomology group $H_{m_{\mathbf{C}^n,0}}^q(\Omega^p(\log D))$ (which coincides with $H_0^q(\Omega_{\mathrm{alg}}^p(\log D))$).

However, because of the quasihomogeneity of D we can exploit an eigenspace decomposition that will allow us to reduce computations to the algebraic case. Precisely, the Lie derivative L_{χ_e} defines an endomorphism of $\Omega^p(\log D)$ and hence an induced one of $H_0^q(\Omega^p(\log D))$. Let us write $H_0^q(\Omega^p(\log D))_m$ for the eigenspace of this endomorphism corresponding to the eigenvalue m. We will show in a moment that this eigenspace is the same as the corresponding algebraic one and, further, that the complex $H_0^q(\Omega^\bullet(\log D))$ has all its cohomology concentrated in degree zero. These facts are certainly implicit in [CMN2], see especially the discussion after [CMN2, Lemma 2.1].

Lemma. *One has that $H_0^q(\Omega^p(\log D))_m = H_0^q(\Omega_{\mathrm{alg}}^p(\log D))_m$, for all m. Further, $h^p(H_0^q(\Omega^\bullet(\log D))) = h^p(H_0^q(\Omega^\bullet(\log D))_0)$, for all q, p.*

Proof. This will be important later on in the paper so let us explain the details. Obviously, we may suppose that $q > 0$. Let $\mathfrak{U}$ denote the cover of $\mathbf{C}^n \setminus 0$ by the complements of the coordinate hyperplanes. There is a split injection of complexes from the weight m part of the Čech complex to the full Čech complex

$$\check{C}^\bullet(\mathfrak{U}, \Omega^p(\log D))_m \to \check{C}^\bullet(\mathfrak{U}, \Omega^p(\log D)).$$

Of course, we have the equality $\check{C}^\bullet(\mathfrak{U}, \Omega^p(\log D))_m = \check{C}^\bullet(\mathfrak{U}, \Omega_{\mathrm{alg}}^p(\log D))_m$. Furthermore, since $\check{C}^\bullet(\mathfrak{U}, \Omega_{\mathrm{alg}}^p(\log D))$ is completely reducible under the $\mathbf{C}^*$-action, the cohomology of $\check{C}^\bullet(\mathfrak{U}, \Omega_{\mathrm{alg}}^p(\log D))_m$ is exactly $H_0^\bullet(\Omega_{\mathrm{alg}}^p(\log D))_m$. It follows that there is a (split) injection $H_0^q(\Omega_{\mathrm{alg}}^p(\log D))_m \to H_0^q(\Omega^p(\log D))_m$, for every m. We now show that it is surjective. Let ω be a Čech cocycle representing a cohomology class in $H_0^q(\Omega^p(\log D))$ which has degree m. Note that this does *not* yield immediately that ω has degree m. As soon as we show that there is a cocycle of weight m representing the same class as ω the lemma will be proved.

We can represent ω as an infinite sum of homogeneous pieces $\omega = \sum_{k \in \mathbf{Z}} \omega_k$, where ω_k has weight k. As $\check{d}\omega = 0$, by assumption, we have $\check{d}\omega_k = 0$, for all k. As the class of ω has weight m we have $L_{\chi_e}\omega - m\omega = \check{d}\eta$, for some $\eta \in \check{C}^{q-1}(\mathfrak{U}, \Omega^p(\log D))$. Note that, since h and dh are weighted homogeneous, each η_k lies in $\check{C}^{q-1}(\mathfrak{U}, \Omega^p(\log D))$. In particular,

$$(k-m)\omega_k = L_{\chi_e}\omega_k - m\omega_k = \check{d}\eta_k.$$

Consider, now, $\nu = \sum_{k \neq m} \frac{1}{k-m}\eta_k$. This sum converges because $\sum \eta_k$ does. Further, $\nu \in \check{C}^{q-1}(\mathfrak{U}, \Omega^p(\log D))$ because each ν_k is.

Now $\omega - \check{d}\nu$ represents the same cohomology class as ω and

$$\omega - \check{d}\nu = \sum_k \omega_k - \sum_{k \neq m} \omega_k = \omega_m$$

does have weight m, as required.

1.7. In fact, if D is locally quasihomogeneous (or free) the degree zero part of this cohomology vanishes for $q = 0$, $q = 1$ and $q = n$. The first two of these statements follow for reasons of depth (1.5). The last is a consequence of local duality (see [BH, 3.6.19]): if N is a finitely generated graded S-module then there is an isomorphism of graded S-modules

$$H_0^q(N)^\vee \cong \mathrm{Ext}^{n-q}(N, \omega_S).$$

Here, if N is any graded S-module, then $N^\vee$ is the graded S-module with $(N^\vee)_k = \mathrm{Hom}_{\mathbf{C}}(N_{-k}, \mathbf{C})$. Note also that, as graded S-modules, one has $\omega_S \cong S[-\sum w_k]$.

1.8. Corollary. *Suppose that D is locally quasihomogeneous. Then $H_0^n(\Omega^p(\log D))_0 = 0$.*

Proof. *By 1.5 and 1.7,*

$$H_0^n(\Omega^p(\log D))_0^\vee \cong \mathrm{Hom}(\Omega^p_{\mathrm{alg}}(\log D), S[-\sum w_k])_0 \cong \Omega^{n-p}_{\mathrm{alg}}(\log D)_{-r}$$

and the latter is is easily seen to be zero.

1.9. We say that the quasihomogeneous divisor D is *log acyclic* if $H_0^q(\Omega^p(\log D))_0 = 0$, for all q, p.

More generally, a divisor in a complex manifold is said to be *locally log acyclic* if it is locally quasihomogeneous and each point of the divisor is log acyclic. The starting point for this paper was the fact, noted in [CMN] (see also [CMN2]), although not in this language, that locally quasihomogeneous free divisors are locally log acyclic. Note, also, that a quasihomogeneous free divisor is log acyclic.

THEOREM. *If each $\Omega^p(\log D)$ has a graded free resolution*

$$0 \to \oplus_i \mathcal{O}_{\mathbf{C}^n,0}[-r_{is}]^{\beta_{is}} \to \cdots \to \oplus_i \mathcal{O}_{\mathbf{C}^n,0}[-r_{i0}]^{\beta_{i0}} \to 0$$

in which $r_{ij} < \sum_k w_k$ then D is log acyclic.

PROOF. Fix some q, p. By local duality (1.7), in order to show that $H_0^q(\Omega^p(\log D))_0 = 0$ it is enough to show that $\mathrm{Ext}^{n-q}(\Omega^p_{\mathrm{alg}}(\log D), S[-\sum w_k])_0 = 0$. But the latter is computed by applying $\mathrm{Hom}(\_, S[-\sum w_k])_0$ to the free resolution of $\Omega^p_{\mathrm{alg}}(\log D)$. The hypothesis that $r_{ij} < \sum w_k$ ensures that $\mathrm{Hom}(S[-r_{ij}], S[-\sum w_k])_0 = 0$, completing the proof, as $\mathrm{Ext}^{n-q}(\Omega^p_{\mathrm{alg}}(\log D), S[-\sum w_k])_0$ is a subquotient of this.

1.10. DEFINITION. Suppose that D is locally quasihomogeneous and that either $n = 3$ or that 0 is an isolated singularity. We define the log defect *of D to be*

$$\sum_{i=1}^{n-2} \dim H_0^{n-i}(\Omega^i(\log h))_0.$$

We denote it by $\delta(\log D)$. When we work globally with a locally quasihomogeneous divisor E in a complex manifold X, such that either X is three-dimensional or the singularities of E are all isolated then we use the notation $\delta(\log E)_x$ to denote the log defect at a point x of the divisor E.

Suppose that $n = 3$. It is worth pointing out that by virtue of 1.3 and 1.4 the modules $\Omega^1(\log D)$ and $\Omega^2(\log D)$ are identical up to a free summand. Precisely,

$$\Omega^1(\log D) \cong \Omega^1(\log h) \oplus \mathcal{O}_{\mathbf{C}^3,0}[0] \quad \text{and} \quad \Omega^2(\log D) \cong \mathcal{O}_{\mathbf{C}^3,0}[r - \sum w_k] \oplus \Omega^1(\log h).$$

In particular,

$$\delta(\log D)_0 = \dim H_0^2(\Omega^1(\log h))_0 = \dim H_0^2(\Omega^1(\log D))_0 = \dim H_0^2(\Omega^2(\log D))_0.$$

Thus, by the definition of log acyclicity and 1.6 we obtain the following result.

PROPOSITION. *Suppose that $n = 3$. The locally quasihomogeneous divisor D is log acyclic if and only if it has zero log defect.*

1.11. In our next result (only) we relax one of the standing hypotheses of this section and consider a not necessarily quasihomogeneous divisor.

LEMMA. *Suppose that D is a not necessarily quasihomogeneous divisor in an* *open subset U of $\mathbf{C}^n$ with a unique singuarity at the origin and suppose that h is a local equation of D at* 0.

(a) *There is a short exact sequence*

$$0 \to \Omega^{p-1}(\log D) \xrightarrow{i} \Omega^{p-1} \oplus \Omega^p \xrightarrow{\pi} M^p := (dh/h) \wedge \Omega^{p-1} + \Omega^p \to 0.$$

Here $i\omega = (h\omega, dh \wedge \omega)$ and $\pi(\omega_1, \omega_2) = (dh/h) \wedge \omega_1 - \omega_2$. Note that, if D is quasihomogeneous then i has degree r and π has degree 0.

(b) *For $p \leq n-2$, we have $M^p = \Omega^p(\log D)$.*

(c) *For $n \geq 3$ and $i \geq 2$, the inclusion $M^p \subseteq \Omega^p(\log D)$ induces an isomorphism $H_0^i(M^p) \cong H_0^i(\Omega^p(\log D))$.*

(d) *If $n \leq 2$ then all the $\Omega^p(\log D)$ are free. If $n \geq 3$ then $\Omega^p(\log D)$ has depth $n-p$, for $p \leq n-2$, depth 2, for $p = n-1$, and depth n, for $p = n$.*

Proof. (a) This is really just the definition of $\Omega^p(\log D)$. If $\omega \in \Omega^{p-1}(*D)$ then $\omega \in \Omega^{p-1}(\log D)$ if and only if $h\omega \in \Omega^{p-1}$ and $dh \wedge \omega \in \Omega^p$. Certainly then i has image inside $\ker\pi$. On the other hand, if (ω_1, ω_2) is in the kernel of π then $dh \wedge (\omega_1/h) = \omega_2$. Thus, we see that $(\omega_1, \omega_2) = i(\omega_1/h)$.

(d) The claim when $n \leq 2$ follows from Saito's results (1.2). So suppose that $n \geq 3$. Note that the short exact sequence

$$0 \to \mathrm{Der}(\log D) \to \mathrm{Der}\mathcal{O}_{\mathbf{C}^n,0} \xrightarrow{\bullet h} J_h + h\mathcal{O}_{\mathbf{C}^n,0}/h\mathcal{O}_{\mathbf{C}^n,0} \to 0$$

shows that $\mathrm{Der}(\log D)$ has depth two. Suppose that $0 \leq p \leq n-2$. We prove the result by induction on p, the case $p = 0$ being trivial. Consider the short exact sequence of (a) in the case $p = 1$. Clearly, M_1 is either free or has projective dimension one. On the other hand, M_1 agrees with $\Omega^1(\log D)$ off D and at the smooth points of D. Thus, M_1 must coincide with $\Omega^1(\log D)$. If M_1 is free then D is free, a contradiction. Thus, $M_1 = \Omega^1(\log D)$ has projective dimension one.

Consider the short exact sequence of (a) in the case $2 \leq p \leq n-2$. By induction, $\Omega^{p-1}(\log D)$ has projective dimension $p-1$. It follows that M^p has projective dimension p and hence (as $p \leq n-2$) depth at least 2. Evidently, M^p agrees with $\Omega^p(\log D)$ off D and at the smooth points of D. Thus, M^p must coincide with $\Omega^p(\log D)$ (proving (b)). Now, induction finishes the proof of (d).

Note that (c) is clear if $p \leq n-2$. If $p = n-1$ or n then the inclusion of M^p in $\Omega^p(\log D)$ has finite-dimensional cokernel supported at the origin. But such a module has non-zero zeroth cohomology with supports in the origin and all its other cohomology groups, with supports in the origin, are zero. The result is now clear.

1.12. We now return to our standing assumptions of this section. For a quasihomogeneous divisor with isolated singularity we can obtain a more precise result than that of (1.11).

PROPOSITION. *Suppose that the origin is an isolated singularity of D. Let $p \leq n-2$. Then*

(a) *There is a short exact sequence*

$$0 \to \Omega^{p-1}(\log h)[-r] \xrightarrow{dh\wedge} \Omega^p \xrightarrow{\iota_{\chi_e}(dh/h)\wedge} \Omega^p(\log h) \to 0.$$

(b) *In particular,*

$$H_0^q(\Omega^p(\log h))_0^\vee = \begin{cases} (\mathcal{O}_{\mathbf{C}^n,0}/J_h)_{pr-\Sigma w_j} if q = n-p \\ 0 \qquad \text{otherwise.} \end{cases}$$

PROOF. (a) The given sequence is clearly a complex. That the map on the right is surjective follows from Lemma 1.11. That the map on the left is injective follows from the fact that if $dh \wedge \omega = 0$ then, applying, ι_{χ_e}, one obtains $rh\omega = 0$.

Finally, exactness in the middle. If $\omega \in \Omega^p$ and $\iota_{\chi_e}((dh/h) \wedge \omega) = 0$ then $\omega = dh \wedge (\iota_{\chi_e}\omega/rh)$. It follows from this that $\iota_{\chi_e}\omega/rh$ is a logarithmic differential form and, hence, the result.

(b) Note that we may suppose that $q < n$, by virtue of 1.8. The short exact sequence shows that $\Omega^p(\log h)$ is a $n-p$th syzygy in the Koszul resolution of $\mathcal{O}_{\mathbf{C}^n,0}/J_h$ corresponding to the regular sequence $(\partial h/\partial x_1, \ldots, \partial h/\partial x_n)$. By local duality (1.7) we can compute

$$\begin{aligned} H_0^q(\Omega^p(\log h))_0 &= \operatorname{Ext}^{n-q}(\Omega^p_{\text{alg}}(\log h), S[-\textstyle\sum w_j])_0 \\ &= \operatorname{Ext}^{n-q+n-p}((S/J_h)[(n-p)r], S[-\textstyle\sum w_j])_0. \end{aligned}$$

The second equality following by dimension shifting. Finally, it is easy to see that the above vector space is zero, for $2n-q-p \neq n$, and equals $(S/J_h)_{pr-\sum w_j}$, for $q = n-p$.

1.13. LEMMA. *Let D be a quasihomogeneous divisor in $\mathbf{C}^n$ with an isolated singularity at the origin. Then*

$$\delta(\log D)_0 = \sum_{i=1}^{n-2} \dim\big(\mathcal{O}_{\mathbf{C}^n,0}/J_h\big)_{ir-\Sigma w_j}.$$

In particular, if $D \subset \mathbf{C}^3$ is a quasihomogeneous divisor with an isolated singularity at 0, then

$$\delta(\log D)_0 = |\{(i_1, i_2, i_3) \in \mathbf{Z}^3 : i_1, i_2, i_3 > 0 \quad \text{and} \quad i_1 w_1 + i_2 w_2 + i_3 w_3 = r\}|.$$

PROOF. This follows immediately from the Proposition. Note that when $n = 3$ then

$$\left(\mathcal{O}_{\mathbf{C}^n,0}/J_h\right)_{ir-\sum w_j} = S_{ir-\Sigma w_j}.$$

1.14. EXAMPLES. Let $D \subset \mathbf{C}^3$ be a quasihomogeneous divisor with an isolated singularity at 0. It is easy to use the above lemma to compute the logarithmic defect at the origin. For example:

$\delta(\log D)_0 = 0$ if the origin is a rational double point.

$\delta(\log D)_0 = 0$ if the origin is a triangle singularity. That is, in Arnold's notation [AGV, p.185], it has one of the 14 types: Q_{10}-Q_{12}, Z_{11}-Z_{13}, S_{11}, S_{12}, W_{12}, W_{13}, E_{12}-E_{14}, U_{12}.

$\delta(\log D)_0 = 1$ for a simple elliptic singularity of type $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ (in the notation of [Sa1]).

Of course, it is already well known that the link of each of these singularities (except for the last three) is a rational homology sphere.

2. Proof of the main results.

REMARK. If D is a locally quasihomogeneous divisor in a complex manifold X of dimension three and x is a non-isolated singularity of D then it can easily happen that $\delta(\log D)_x = 0$ and $(R^3 j_* \mathbf{C}_U)_x \neq 0$. The reader will easily supply examples after reading the results of Section 3.

2.1. We now establish the technical results about the spectral sequence mentioned in the introduction that will be used in the proofs of the theorem of the introduction. Throughout this section we assume that D is a locally quasihomogeneous divisor in a complex manifold X of dimension n. Further, we assume that D is log acyclic except at isolated points.

Note that this latter condition will hold in either of the two cases (a) $\dim X = 3$ or (b) D has only isolated singularities. If D had only isolated singularities then the divisor D will be free away from these singular points If $\dim X = 3$ then again D will be free away from some isolated points (see [CMN2, Proposition 2.4]). Thus in either case the main theorem of [CMN] applies.

PROPOSITION *Let $x \in D$. For any sufficiently small Stein open neighbourhood V of x there is a spectral sequence with E_1 term*

$$E_1^{p,q} = H^q(V \setminus x, \Omega^p(\log D))$$

which converges to $h^\bullet(\Gamma(V \setminus V \cap D, \Omega^\bullet_{V \cap D}))$. Further, $E_1^{p,0} = \Gamma(V, \Omega^p(\log D))$, for all p.

PROOF. It follows from the arguments of [CMN2, proof of theorem 1.1] that we may choose a Stein open neighbourhood V of x with the following properties : (1) For suitable local coordinates centred at x, the divisor $D \cap V \subset V$ is quasihomogeneous; that is, defined by quasihomogeneous $h \in S$. (2) Writing $U = V \setminus V \cap D$, and with the inclusion $j : U \to V$, the natural morphism $\Omega^\bullet(\log D) \to \mathsf{R}j_*\mathsf{C}_U$ is a quasi-isomorphism away from $x = 0$.

To simplify notation let us replace D by $D \cap V$. We claim that the argument of [CMN2, proof of theorem 1.1] shows that there is a spectral sequence with E_1 term

$$E_1^{p,q} = H^q(V \setminus x, \Omega^p(\log D))$$

converging to $h^\bullet(\Gamma(V, \Omega_U^\bullet))$. For the convenience of the reader we will explain this here.

Let $V_i = V \cap \{x_i \neq 0\}$ and let $V_i' = V_i \setminus V_i \cap D$. Thus, $\{V_i\}$ and $\{V_i'\}$ are Stein open covers of $V \setminus 0$ and $V \setminus D$. Consider the double complexes

$$K^{p,q} = \bigoplus_{1 \le i_0 < \ldots < i_q \le t} \Gamma\Big(\bigcap_{j=0}^{q} V_{i_j}, \Omega^p(\log D)\Big),$$

$$\tilde{K}^{p,q} = \bigoplus_{1 \le i_0 < \ldots < i_q \le t} \Gamma\Big(\bigcap_{j=0}^{q} V_{i_j}', \Omega^p\Big),$$

each of which is equipped with the Čech differential $\check{d}$ and exterior derivative d. The restriction morphism $\rho_0^{\bullet,\bullet} : K^{\bullet,\bullet} \to \tilde{K}^{\bullet,\bullet}$ commutes with both differentials; hence it induces morphisms of the spectral sequences arising from the standard filtrations on the total complexes of these double complexes. Denoting the spectral sequences for $K^\bullet$ by ${}^I E$ and ${}^{II} E$ and those for $\tilde{K}^\bullet$ by ${}^I \tilde{E}$ and ${}^{II} \tilde{E}$ we have

$${}^I E_1^{p,q} = h^p(K^{\bullet,q}) = h^p\Big(\bigoplus_{1 \le i_0 < \ldots < i_q \le t} \Gamma\Big(\bigcap_{j=0}^{q} V_{i_j}, \Omega^\bullet(\log D)\Big)\Big),$$

$${}^I \tilde{E}_1^{p,q} = h^p\Big(\bigoplus_{1 \le i_0 < \ldots < i_q \le t} \Gamma\Big(\bigcap_{j=0}^{q} V_{i_j}', \Omega^\bullet\Big)\Big) = \bigoplus_{1 \le i_0 < \ldots < i_q \le t} H^p\Big(\bigcap_{j=0}^{q} V_{i_j}'; \mathsf{C}\Big),$$

$${}^{II} E_1^{p,q} = h^q(K^{p,\bullet}) = H^q(V \setminus 0, \Omega^p(\log D)), \text{ and}$$

$${}^{II} \tilde{E}_1^{p,q} = h^q(\tilde{K}^{p,\bullet}) = H^q(V \setminus D, \Omega^p).$$

By the first paragraph of the proof ${}^I \rho_1^{p,q} : {}^I E_1^{p,q} \to {}^I \tilde{E}_1^{p,q}$ is an isomorphism, for all p, q. Hence, ${}^I \rho_\infty^{p,q}$ is an isomorphism also.

Now, ${}^{II} \tilde{E}$ evidently collapses at E_2 because $V \setminus D$ is Stein. Thus, ${}^{II} \tilde{E}$ converges to $h^p(\Gamma(U, \Omega^\bullet))$. Of course, ${}^I \tilde{E}$ converges to the same thing. On the other hand, the isomorphism ${}^{I}\rho_{\infty}^{p,q}$ shows that ${}^{I}E$ also converges to this. Finally, ${}^{II}E$ converges to the same limit as ${}^{I}E$.

This establishes the first statement of the proposition, with $E = {}^{II}E$. For the second, recall from Proposition 1.5 that $\Omega^p(\log D)$ has depth at least two. Thus, the final row in ${}^{II}E_1$ is simply $\Gamma(V, \Omega^\bullet(\log D))$.

2.2. For the rest of this section we assume that either $n = 3$ or D has only isolated singularities.

THEOREM. *The above spectral sequence degenerates at E_2 page which is:*

$$\begin{array}{ccccccc}
0 & 'G^1 & ''G^1 & 0 & 0 & \dots & 0 \\
0 & 0 & 'G^2 & ''G^2 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \\
0 & 0 & \dots & 0 & 'G^{n-2} & ''G^{n-2} & 0 \\
H0 & H^1 & \dots & H^{n-3} & H^{n-2} & H^{n-1} & H^n
\end{array}$$

Here, $H^i = h^i(\Gamma(V, \Omega^\bullet(\log D)))$, for $i \geq 0$; $'G^i = H_0^{n-i}(\Omega^i(\log h))_0$ and $''G^i = H_0^{n-i}((dh/h) \wedge \Omega^i(\log h))_0$, for $1 \leq i \leq n-2$.

In particular, the natural map from the cohomology of the complex $\Gamma(V, \Omega^\bullet(\log D))$ to the cohomology of $V \setminus D$ is an isomorphism if and only if $\delta(\log D)_0 = 0$.

PROOF. Note that the quasihomogeneous polynomial h defines a divisor not just in V but in all of C^n. Abusing notation we will also call this divisor D. Let $\mathfrak{U}$ be the cover of $\mathsf{C}^n \setminus 0$ by the complements of the coordinate hyperplanes. We consider the spectral sequence F with F_1 term

$$F_1^{p,q} = H^q(\mathsf{C}^n \setminus 0, \Omega^p_{\mathrm{alg}}(\log D)).$$

It is associated to the second filtration on the double complex $L^{\bullet,\bullet}$ with $L^{p,q} = \check{C}^q(\mathfrak{U}, \Omega^p_{\mathrm{alg}}(\log D))$.

Notice that for $q > 0$ we have $(E_1^{p,q})_0 = (F_1^{p,q})_0$. This is because of excision:

$$H^q(\mathsf{C}^n \setminus 0, \Omega^p_{\mathrm{alg}}(\log D))_0 = H_0^{q+1}(\Omega^p_{\mathrm{alg}}(\log D))_0 = H_0^{q+1}(\Omega^p(\log D))_0 = H^q(\mathsf{C}^n \setminus 0, \Omega^p(\log D))_0.$$

It follows from this and 1.6 that if we can prove that for all $r \geq 2$ the differentials of F_r pointing to the $q = 0$ row are all zero then the same will be true in E_r and that $E_r^{p,q} = F_r^{p,q}$, for $q \geq 1$.

Furthermore, as we know that the cohomology of the differentials in F_1 is concentrated in degree zero, that is, $(F_2^{p,q})_0 = F_2^{p,q}$, if we let G be the spectral sequence associated to the double complex $M^{\bullet,\bullet} := (L^{\bullet,\bullet})_0$ then $G_2 = F_2$.

Thus, we may as well concentrate our attention on the spectral sequence G. The first thing to notice is that, by (1.3),

$$M^{p,q} = \check{C}^q(\mathfrak{U}, \Omega^p_{\mathrm{alg}}(\log D))_0 = \check{C}^q(\mathfrak{U}, \Omega^p_{\mathrm{alg}}(\log h))_0 \oplus \check{C}^q(\mathfrak{U}, (dh/h) \wedge \Omega^{p-1}_{\mathrm{alg}}(\log h))_0$$
$$:= {}'M^{p,q} \oplus {}''M^{p,q}.$$

Now, $\check{d}$ obviously respects the decomposition. Further, if $(\omega_i) \in \check{C}^q(\mathfrak{U}, \Omega^p_{\mathrm{alg}}(\log D))_0$, say

$$(\omega_i) = ({}'\omega_i) + ({}''\omega_i) \in {}'M^{p,q} \oplus {}''M^{p,q}$$

then $({}''\omega_i) = ((dh/h) \wedge {}'\eta_i)$, for some $({}'\eta_i) \in {}'M^{p-1,q}$. Now,

$$d(\omega_i) = d({}'\omega_i) + d((dh/h) \wedge {}'\eta_i) = (d'\omega_i) - ((dh/h) \wedge d'\eta_i).$$

Since $d\iota_{\chi_e} = -\iota_{\chi_e} d$ on $\Omega^p_{\mathrm{alg}}(\log D)_0$, we see that d also respects the decomposition. In other words, this decomposition makes

$$M^{\bullet,\bullet} = {}'M^{\bullet,\bullet} \oplus {}''M^{\bullet,\bullet}$$

into a direct sum of double complexes. Even more is true however, because it is clear from the above discussion that there is an isomorphism of double complexes ${}''M^{\bullet,\bullet} \cong {}'M^{\bullet-1,\bullet}$.

Thus, we obtain that the spectral sequence G is a direct sum of spectral sequences $G = {}'G \oplus {}''G$ and that

$$G_r^{p,q} = {}'G_r^{p,q} \oplus {}''G_r^{p,q} \cong {}'G_r^{p,q} \oplus {}'G_r^{p-1,q}.$$

We claim that G_1 is:

$$\begin{array}{cccccccc}
0 & {}'G_1^{1,n-2} & {}''G_1^{2,n-2} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & {}'G_1^{2,n-3} & {}''G_1^{3,n-3} & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & & 0 & {}'G_1^{n-2,1} & {}''G_1^{n-1,1} & 0 \\
G_1^{0,0} & G_1^{1,0} & \dots & & & G_1^{n-2,0} & G_1^{n-1,0} & {}''G_1^{n,0}
\end{array}$$

To see this, firstly note that $G^{p,n-1} = 0$, by Corollary 1.8. Also, $\Omega^n(\log D) = (dh/h) \wedge \Omega^{n-1}(\log D)$ which accounts for the fact that $G_1^{n,0} = {}''G_1^{n,0}$. Now, $\Omega^0(\log D)$ and $\Omega^n(\log D)$ are both free, which shows that the initial and final columns are zero above the $q = 0$-axis. In the case $n = 3$, this last fact ensures that $G_1^{n-2,1} = {}'G_1^{n-2,1}$ and that $G_1^{n-1,1} = {}''G_1^{n-1,1}$, and it follows that G_1 is as claimed. So suppose that we are in the isolated case. Now, by Proposition 1.12, for $1 \le p \le n-2$ and $1 \le q \ne n-p-1$, we have ${}'G_1^{p,q} = 0$. Recalling that ${}'G_1^{p-1,q} = 0$ implies that ${}''G_1^{p,q} = 0$ and, likewise, that ${}''G_1^{p+1,q} = 0$ implies that ${}'G_1^{p,q} = 0$, we obtain the G_1-page is as claimed.

Next we claim that the differentials (above the $(q = 0)$-axis) on the G_1-page are all zero. This follows because the only possibly non-zero ones map from the $(p + q = n - 1)$-diagonal to the $(p + q = n)$-diagonal. The former is pure $'$ and the latter is pure $''$. Similarly, the only possibly non-zero differentials on the G_2-page map from the $(p+q=n-1)$-diagonal to the $(p+q=n)$-diagonal and so, in fact, are zero.

2.3. We can now give a criterion for the cohomology of the complement to be logarithmic.

THEOREM. *Suppose that D is a locally quasihomogeneous divisor in X a complex manifold. Further suppose that X has dimension three or that D has only isolated singularities. Let $U = X \setminus D$ and let $j : U \to X$ be the inclusion.*

Then, $\Omega^\bullet(\log D) \to \mathsf{R}j_*\mathsf{C}_U$ is a quasi-isomorphism if and only if $\delta(\log D)_x = 0$, for every $x \in D$.

PROOF. Note that if V is any sufficiently small open Stein neighbourhood of x then $\delta(\log D)_x$ is zero if and only if the natural map $h^q(\Gamma(V, \Omega^\bullet(\log D))) \to H^q(V \setminus D; \mathsf{C})$ is an isomorphism. On the other hand, $(R^q j_*\mathsf{C}_U)_x$ is the limit, over sufficiently small Stein open neighbourhoods V of x, of $H^q(V \setminus D; \mathsf{C})$ and, similarly, $\mathscr{H}^q(\Omega^\bullet(\log D))_x$ is the limit of $h^q(\Gamma(V, \Omega^q(\log D)))$.

2.4. Finally we prove the theorem of the introduction.

THEOREM. *Suppose that D is a locally quasihomogeneous divisor with only isolated singularities in a complex manifold X. Let $U = X \setminus D$ and let $j : U \to X$ denote the inclusion. Finally, for $x \in D$, let $w_1, \ldots, w_n$ be the system of weights at x, and r the degree of a weighted homogeneous local equation h of D at x. Consider the following statements:*

(a) $\delta(\log D)_x = 0$;
(b) $(R^i j_*\mathsf{C}_U)_x = 0$, for $i \geq 2$;
(c) $(\mathcal{O}_{X,x}/J_h)_{ir-\sum w_j} = 0$, for $1 \leq i \leq n-2$;
(d) The link of (D, x) is a Q-homology sphere.
(e) $\mathscr{H}^i(\Omega^\bullet(\log D))_x = 0$, for $i \geq 2$.

Then (a)–(d) *are equivalent and imply* (e). *If X has dimension three then* (e) *is equivalent to* (a)–(d).

PROOF. Let us begin by computing the cohomology of $\Gamma(\mathsf{C}^n, \Omega^\bullet(\log D))$. As usual, this coincides with $H^\bullet$, the cohomology of $\Gamma(\mathsf{C}^n, \Omega^\bullet_{\mathrm{alg}}(\log D))_0$. Further, we have the usual bigrading

$$H^p = {}'H^p \oplus {}''H^p \cong {}'H^p \oplus {}'H^{p-1}.$$

Now the Wang sequence shows that the only possible non-zero values of $H^\bullet(\mathsf{C}^n \setminus D; \mathsf{C})$ are for $\bullet = 0, 1, n-1, n$. It follows from Theorem 2.2 that $H^p = 0$, for $p \neq 0, 1, n-1, n$. Furthermore, this shows that $H^1 = {}''H^1$ and $H^{n-1} = {}'H^{n-1}$. On the other hand, looking at $\Omega^0_{\mathrm{alg}}(\log D)$ and $\Omega^n_{\mathrm{alg}}(\log D)$ we can see that $H^0 = {}'H^0$ and $H^n = {}''H^n$. Obviously we have $H^0 = \mathsf{C}$. It follows that $H^1 = \mathsf{C}(dh/h)$. As we have already remarked $H^p = 0$, for $2 \leq p \leq n-2$. Let us turn our attention to H^n. We have

$$H^n = \left(\frac{S(dx/h)}{(dh/h) \wedge d\Omega^{n-2}_{\mathrm{alg}}(\log h))}\right)_0.$$

Now the isomorphism $\mathrm{Der}(\log h)[r - \sum w_k] \to \Omega^{n-2}_{\mathrm{alg}}(\log h)$ together with the fact that $\mathrm{Der}(\log h)$ is generated by the vector fields $(\partial h/\partial x_i)\partial/\partial x_j - (\partial h/\partial x_j)\partial/\partial x_i$ for $1 \leq i < j \leq n$ shows that $\Omega^{n-2}_{\mathrm{alg}}(\log h)$ has a system of generators with degrees $\sum_{k \neq i,j} w_k$, for $1 \leq i \neq j \leq n$. In particular, $\Omega^{n-2}_{\mathrm{alg}}(\log h)_0 = 0$. It follows that $H^n = (S(dx/h))_0$. Similarly, $H^{n-1} = (S(\iota_{\chi_e}(dx/h)))_0 \cong H^n$.

(a) $\iff$ (c). This is immediate from Lemma 1.13

(c) $\Longrightarrow$ (b). Consider the spectral sequence of 2.1 in the case when $V = \mathsf{C}^n$. By (c) this spectral sequence collapses onto the bottom row which is $h^\bullet(\Gamma(\mathsf{C}^n, \Omega^\bullet(\log D)))$ and computes $H^\bullet(\mathsf{C}^n \setminus D; \mathsf{C})$. By the hypothesis (c) we certainly have $(\mathcal{O}_{\mathsf{C}^n,0}/J_h)_{r-\sum w_j} = 0$. On the other hand, J_h contains no element of degree less than $r - \sup w_j$. Thus, we have $S_{r-\sum w_j} = 0$. It follows, by the first paragraph of the proof, that $H^n = H^{n-1} = 0$. Thus, $H^i(\mathsf{C}^n \setminus D; \mathsf{C}) = 0$, for $i \geq 2$. Now let V be any small Stein neighbourhood of the origin such that $V \setminus V \cap D$ is a deformation retract of $\mathsf{C}^n \setminus D$. Clearly we have $H^i(V \setminus V \cap D; \mathsf{C}) = 0$, for $i \geq 2$. This completes the proof that (c) implies (b).

(b) $\Longrightarrow$ (c). On the other hand, (b) is easily seen to yield (c) by virtue of Theorem 2.2.

(b) $\iff$ (d). The equivalence of (b) and (d) follows by Alexander duality.

That (a)–(d) imply (e) is clear by Theorem 2.3 That (e) implies the other statements, if $n = 3$, can be seen by fine-tuning the proof above that (c) is equivalent to (b). Specifically, by the first paragraph of the proof $H^3 = (Sdx/h)_0$. This latter vector space has dimension $\delta(\log D)_0$, as required.

2.5. Let us make a few observations about the value of $\delta(\log D)_x$ in the isolated case and its connection with the mixed Hodge structure on the Milnor fibre.

REMARK. Suppose that D is a quasihomogeneous divisor in C^n with an isolated singularity at the origin. Further, suppose that h is an equation for D with weighted degree r. Let F denote the Milnor fibre, that is, the hypersurface defined by $h - 1$ in C^n. Then

(a) The defect $\delta(\log D)_0 = 0$ if and only if $h^{p,n-p} = 0$, for $p = 1, \ldots, n-1$, where the former are the mixed Hodge numbers of $H^{n-1}(F)$.

(b) Let $W_\bullet$ denote the weight filtration for the $n-1$st cohomology group of the Milnor fibre $H^{n-1}(F; \mathsf{R})$. Thus,

$$0 = W_{n-2} \subset W_{n-1} \subset W_n = H^{n-1}(F; \mathbf{R}).$$

Then $\delta(\log D)_0 = 0$ if and only if $W_{n-1} = H^{n-1}(F; \mathbf{R})$.

(c) If the degree r of h is odd then stabilisation in an odd number of variables (i.e. replacing h by $h + y_1^2 + \cdots + y_{2k+1}^2$) yields a divisor with $\delta(\log D)_0 = 0$.

(d) The log defect $\delta(\log D)_0$ is unchanged by stabilisation in an even number of variables. (i.e. replacing h by $h + y_1^2 + \cdots + y_{2k}^2$).

Proof. (a) Steenbrink [St.] has computed that

$$h^{p,n-p} = \dim(S/J_h)_{pr-\sum w_j}.$$

Now, by the theorem, the log defect is zero if and only $h^{p,n-p} = 0$, for $p = 1, \ldots, n-2$. On the other hand, $h^{n-1,1} = h^{1,n-1}$.

(b) The hypothesis that $\delta(\log D)_0 = 0$ is equivalent to $\mathrm{Gr}_n^W H_{\mathbf{C}} = 0$, by (a). On the other hand, the latter says that $W_n = W_{n-1}$.

(c) Let us assign the weights $r/2$ to the additional $2k+1$ variables. We have to consider the sequence of weights $(i - (2k+1)/2)r - \sum w_j$, for $1 \le i \le n + 2k - 1$. The weights are clearly not integers. On the other hand, $\mathcal{O}_{\mathbf{C}^n,0}/J_h$ clearly is concentrated in integer weights.

(d) Consider the log defect of the new equation. It is

$$\sum_{i=1}^{n+2k-2} \dim(S/J_h)_{ir-\Sigma w_j - kr} = \sum_{i=1}^{n+2k-2} h^{i-k,n-i+k}.$$

The increase on the old value is

$$\sum_{i=1}^{k} h^{i-k,n-i+k} + \sum_{i=n+k-1}^{n+2k-1} h^{i-k,n-i+k}$$

Using $h^{p,q} = h^{q,p}$ and the fact that $h^{i-k,n-i+k} = 0$, for $i \le k$, we obtain the result.

2.6. By way of illustration we give some examples. The reader will easily check the statements made using the above results.

Examples. Consider, for $n \ge 3$, the quasihomogeneous divisor D in $\mathbf{C}^n$ defined by the series

$A_k : x_1^{k+1} + x_2^2 + \cdots + x_n^2, k \ge 1$	except k odd, n even
$D_k : x_1^{k-1} + x_1 x_2^2 + x_3^2 + \cdots + x_n^2, k \ge 4$	n odd
$E_6 : x_1^4 + x_2^3 + x_3^2 + \cdots + x_n^2$	
$E_7 : x_1^3 x_2 + x_2^3 + x_3^2 + \cdots + x_n^2$	n odd
$E_8 : x_1^5 + x_2^3 + x_3^2 + \cdots + x_n^2.$	

In each case $\delta(\log D)_0 = 0$. If the condition in the second column fails then $\delta(\log D)_0 = 1$ in types D_k, if k odd, and in types A_k, E_7. In type D_k with n and k both even then $\delta(\log D)_0 = 2$.

2.7. Finally, let us observe, for use elsewhere, that the argument of 2.1 (which is taken directly from [CMN2]) yields the following result.

COROLLARY *Let D be a locally log acyclic divisor in a complex manifold X, let $U = X \setminus D$ and let $j : U \to X$ be the inclusion. Then the natural map $\Omega^\bullet(\log D) \to \mathsf{R}j_*\mathsf{C}_U$ is a quasi-isomorphism.*

3. Finitely determined quasihomogeneous map germs $\mathsf{C}^2, 0 \to \mathsf{C}^3, 0$.

3.1. Throughout this section we consider a quasihomogeneous map $f : \mathsf{C}^2 \to \mathsf{C}^3$ which is finitely determined at the origin. Let D denote the image of this map. Denote the coordinate functions on the domain by y_1, y_2 with weights d_1, d_2. The coordinate functions on C^3 are denoted by x_1, x_2, x_3 with weights w_1, w_2, w_3. Denote by h the quasihomogeneous polynomial defining D and write r for its degree. By [Mo2, Proposition 1,5(i)] we have $r = w_1w_2w_3/d_1d_2$. Now, $f(y_1, y_2) = (f_1, f_2, f_3)$ and we denote by $\mathscr{R}_f$ the ideal of $\mathscr{O}_{\mathsf{C}^2,0}$ generated by the maximal minors of df.

THEOREM. *One has the following formula for the log defect:*

$$\delta(\log D)_0 = \dim\big(N\mathscr{A}_e f\big)_{\frac{w_1w_2w_3}{d_1d_2}-(w_1+w_2+w_3)} + \dim\big(\mathscr{O}_{\mathsf{C}^2,0}/\mathscr{R}_f\big)_{\frac{w_1w_2w_3}{d_1d_2}-d_1-d_2}.$$

3.2. As a special case we obtain some more examples of divisors with zero log defect.

COROLLARY. *If f has a simple singularity at the origin then $\delta(\log D)_0 = 0$.*

3.3. The corollary follows immediately from the theorem and the following two lemmas.

LEMMA. *If f has a simple singularity at the origin then* $\dim(N\mathscr{A}_e f)_{\frac{w_1w_2w_3}{d_1d_2}-\sum w_i} = 0$.

PROOF. This follows, by a simple computation, from the explicit description of $N\mathscr{A}f$ which can be found in [Mo1].

3.4. Next we need a result that follows from the work of Scheja and Storch [SS].

LEMMA. *If f has corank one at the origin then*

$$\big(\mathscr{O}_{\mathsf{C}^2,0}/\mathscr{R}_f\big)_{\frac{w_1w_2w_3}{d_1d_2}-d_1-d_2} = 0.$$

Proof. Without loss of generality, we may take $f_1 = y_1$ and so $d_1 = w_1$. We may also assume without loss that $f_2 = y_2^k + \ldots$, with $k \geq 1$, and that $w_3 \geq d_2$.

Scheja and Storch prove in [SS] that if polynomials $t_1, \ldots, t_n$ define $0 \in \mathbf{C}^n$ as a 0-dimensional complete intersection then their Jacobian determinant generates the (1-dimensional) socle of the the algebra $\mathcal{O}_{\mathbf{C}^n,0}/(t_1, \ldots, t_n)$. It follows, in the graded case, that this is the element of highest weight. Let us call this weight N.

Now

$$df = \begin{bmatrix} 1 & 0 \\ \partial f_2/\partial y_1 & \partial f_2/\partial y_2 \\ \partial f_3/\partial y_1 & \partial f_3/\partial y_2 \end{bmatrix}$$

and so $t_1 = \partial f_2/\partial y_2$ and $t_2 = \partial f_3/\partial y_2$ generate the ramification ideal $\mathcal{R}_f$ (and hence define $0 \in \mathbf{C}^2$ as a complete intersection). Thus, we discover that $N = w_2 + w_3 - d_1 - 3d_2$. The result will follow if we can show that $r - d_1 - d_2 > w_2 + w_3 - d_1 - 3d_2$. Thus, we must demonstrate that $w_2 w_3/d_2 > w_2 + w_3 - 2d_2$. Now $w_2 = kd_2$ so this becomes $(k-1)w_3 > (k-2)d_2$. But this is true, since $w_3 \geq d_2$.

3.5. Now we can return to the proof of the theorem.

Proof. By local duality (1.7), using the isomorphism $\Omega^2(\log D) \cong \mathrm{Der}(\log D)[-\sum w_i + r]$, one has

$$H_0^2(\Omega^2(\log D))_0^\vee \cong \mathrm{Ext}^1(\mathrm{Der}(\log D)[-\sum w_i + r], \omega_{\mathbf{C}^3})_0.$$

The exact sequence:

$$0 \to \mathrm{Der}(\log D) \to \theta_{\mathbf{C}^3} \to (J_h/h)[r] \to 0$$

together with the equality $J_h \mathcal{O}_D = J_h/h$ shows that that

$$\mathrm{Ext}^1(\mathrm{Der}(\log D)[-\sum w_i + r], \omega_{\mathbf{C}^3})_0 \cong \mathrm{Ext}^2(J_h \mathcal{O}_D[-\sum w_i + 2r], \omega_{\mathbf{C}^3})_0.$$

But now if $\mathscr{C}$ is the conductor of $\mathcal{O}_{\mathbf{C}^2}$ into $\mathcal{O}_D$ then we have the exact sequence

$$0 \to J_h \mathcal{O}_D \to \mathscr{C} \to \mathscr{C}/j_h \mathcal{O}_D \to 0.$$

The conductor $\mathscr{C}$ is principal generated by $c = (-1)^i (\partial h/\partial x_i)/df_i$, where df_i denotes the ith maximal minor of df (see [Pi, Theorem 1, Example 1]). Since $\mathscr{C} = c\mathcal{O}_{\mathbf{C}^2}$, $\mathrm{Ext}^\bullet(\mathscr{C}, \mathcal{O}_{\mathbf{C}^3})$ is zero in degrees different from 1. Therefore

$$\begin{aligned} \mathrm{Ext}^2(J_h \mathcal{O}_D[-\textstyle\sum w_i + 2r], \omega_{\mathbf{C}^3})_0 &\cong \mathrm{Ext}^3(\mathscr{C}/J_h \mathcal{O}_D[-\textstyle\sum w_i + 2r], \omega_{\mathbf{C}^3})_0 \\ &= \mathrm{Ext}^3(\mathscr{C}/J_h \mathcal{O}_D, \omega_{\mathbf{C}^3})_{\Sigma w_i - 2r}. \end{aligned}$$

Furthermore, multiplying by c shows that there is an isomorphism $\mathscr{C}/J_h\mathscr{O}_{\mathbf{C}^2,0} \cong (\mathscr{O}_{\mathbf{C}^2,0}/\mathscr{R}_f)[-\deg c]$. Now, there is a short exact sequence of finite-dimensional modules

$$0 \to J_h\mathscr{O}_{\mathbf{C}^2}/J_h\mathscr{O}_D \to \mathscr{C}/J_h\mathscr{O}_D \to \mathscr{C}/J_h\mathscr{O}_{\mathbf{C}^2} \to 0.$$

Thus, by local duality (1.7), we have

$$\dim Ext^3(\mathscr{C}/J_h\mathscr{O}_D, \omega_{\mathbf{C}^3})_{\sum w_i - 2r} = \dim\big(J_h\mathscr{O}_{\mathbf{C}^2}/J_h\mathscr{O}_D\big)_{2r-\sum w_i} + \dim(\mathscr{O}_{\mathbf{C}^2,0}/\mathscr{R}_f)[-\deg c]_{2r-\sum w_i}.$$

By [Mo3, Proposition 2.1] one has that $J_h\mathscr{O}_{\mathbf{C}^2}/J_h\mathscr{O}_D \cong N\mathscr{A}_e f[-r]$. Finally, $\deg c = r - \sum w_i + d_1 + d_2$ and so we obtain that

$$\delta(\log D)_0 = \dim\big(N\mathscr{A}_e f\big)_{r-\sum w_i} + \dim\big(\mathscr{O}_{\mathbf{C}^2}/\mathscr{R}_f\big)_{r-d_1-d_2}.$$

REMARK. The simple singularities do not yield all the examples with $\delta(\log D)_0 = 0$. For example, consider a finitely determined map-germ of the form $(x,y) \mapsto (x, y^2, yp(x,y^2))$. A short computation together with Theorem 3.1 shows that $\delta(\log D)_0 = 0$.

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