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A NOTE ON THE LOCAL INVERTIBILITY OF SOBOLEV FUNCTIONS

ROBERTO VAN DER PUTTEN

Abstract.

We give some topological and analytical conditions in order that a continuous Sobolev function be a local homeomorphism. The results are obtained in the setting of the spaces $W^{1,n}(\Omega; \mathsf{R}^n)$ and $W^{2,p}(\Omega; \mathsf{R}^n)$.

1. Introduction.

In this paper we deal with the local invertibility of continuous mappings and, more precisely, with the properties of the branch set of such mappings; we recall that, if Ω is an open subset of R^n and $f : \Omega \to \mathsf{R}^n$ a continuous mapping, the branch set of f, denoted by B_f, is the set of all points $x \in \Omega$ where f does not define a local homeomorphism. It is well known that if $f \in C^1$, then $B_f \subset Z_f$ where $Z_f = \{x \in \Omega : Df(x) \text{ exists and } \det Df(x) = 0\}$, but the study of B_f becomes more difficult beyonds the class of smooth mappings. Some results have been obtained under topological assumptions: if f is light and sense-preserving (see below for definitions) then the topological dimension of B_f and $f(B_f)$ is not greater than $n-2$ and

$$B_f \subset Z_f \cup S_f \tag{1.1}$$

where $S_f = \{x \in \Omega : f \text{ is not weakly differentiable at } x\}$ ([11] and [3]).

However, it is not known under what analytical conditions a mapping is light and sense-preserving ; some results can be found in [7] (mappings with finite dilatation) and in the monograph of Rickman ([11]) on quasiregular mappings.

Invertibility has been studied also in the setting of nonlinear elasticity: in fact this requirement guarantees that interpenetration of matter does not occur. In this case Ball and Šverak ([2], [13]) have found analytical conditions which implies the global invertibility of Sobolev functions.

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In this paper we present three results in the setting of Sobolev spaces: the first two concern mappings belonging to $W^{1,n}(\Omega;\mathsf{R}^n)$ and they are slight improvements of the recalled result in [11] (Chap. I, Lemma 4.11); we prove that (1.1) holds if $\det Df \geq 0$ almost everywhere in Ω and f is either open or light. The topological degree is widely used in the proofs. The third theorem concerns mappings belonging to $W^{2,p}(\Omega;\mathsf{R}^n)$. First we prove that S_f is a set of zero capacity if $p > \frac{n(n-1)}{2n-1}$; then we use this result to show that (1.1) holds if $p > n-1, Z_f$ is a set of zero capacity and $\det Df > 0$ almost everywhere in Ω.

2. Notations and preliminaries.

Throughout this paper Ω is a nonempty, bounded and open set in R^n, with $n \geq 2$.

We write $\mathscr{L}^n$ for the Lebesgue measure in R^n and $\|\ \|$ for the norm in the same space. Given $x \in \mathsf{R}^n$ and $r > 0, B(x,r)$ is the open ball of center x and radius $r; Q(x,r)$ is the set $\{y \in \mathsf{R}^n : |x_i - y_i| < r,\ i \in \{1,...,n\}\}$, where $x = (x_1,....,x_n)$ and $y = (y_1,....,y_n)$. If $A \subset \mathsf{R}^n$, $D(A)$ will be the set of accumulation points of A.

For $1 \leq p \leq +\infty$ and $m \geq 1$, let $L^p(\Omega;\mathsf{R}^m)$ be the collection of all m-tuples $(f_1,....,f_m)$ of real functions in $L^p(\Omega)$. For $k \geq 1$, we say that $f \in W^{k,p}(\Omega;\mathsf{R}^m)$ if $f \in L^p(\Omega;\mathsf{R}^m)$ together with its derivatives (in the sense of distribution) up to kth order; Df will be the distributional Jacobian matrix of f.

Now we introduce the Bessel capacity. Let g be the Bessel kernel, that is the function whose Fourier transform is

$$(\breve{g})(x) = (2\pi)^{-\frac{n}{2}}(1 + \|x\|^2)^{-\frac{1}{2}};$$

for $p > 1$, we define the Bessel capacity for any set $A \subset \mathsf{R}^n$ as

$$B_{1,p}(A) = \inf\left\{\int_{\mathsf{R}^n} |f(x)|^p dx : f \in L^p(\mathsf{R}^n), g * f \geq 1 \text{ on } A, f \geq 0\right\},$$

where $g * f$ is the convolution of g and f (the elementary properties of Bessel capacity can be found in [15]).

Let $A \subset \mathsf{R}^n$. The Hausdorff dimension of A is defined by $\dim_H(A) = \sup\{\alpha \geq 0 : H^\alpha(A) > 0\}$ with the convention $\dim_H(\emptyset) = 0$, where H^α is the α-dimensional Hausdorff measure (see [4]).

Now let $f : \Omega \to \mathsf{R}^n$ be a continuous mapping. We say that f satisfies the condition (N) on Ω if $\mathscr{L}^n(f(A)) = 0$ whenever $A \subset \Omega$ is such that $\mathscr{L}^n(A) = 0$. If $A \subset \Omega$ and $y \in \mathsf{R}^n$, we denote by $N(f, A, y)$ the number (possibly infinite) of elements of the set $A \cap f^{-1}(y)$. The map f is said to be light if $f^{-1}(y)$ is totally disconnected for every $y \in \mathsf{R}^n$.

In the following A will be a domain such that $A \subset\subset \Omega$ and $y \in \mathsf{R}^n$.

Now we introduce the topological degree. Suppose that $y \notin f(\partial A)$. Then there exists $r \in \mathsf{R}$, $0 < r < 1$, r small enough, such that f induces a homomorphism of cohomology groups

$$f^* : H_{n+1}(\overline{B(y, r^{-1})}; \overline{B(y, r^{-1})} \backslash B(y, r)) \to H_{n+1}(\overline{A}; \partial A).$$

If g_1, g_2 are suitable generators of the cohomology groups, there exists an integer, we denote it $\mu(y,f,A)$, such that $f^*(g_1) = \mu(y,f,A)g_2$; $\mu(y,f,A)$ is called the topological degree of y with respect to the pair (f,A).

We say that f is sense-preserving (weakly sense-preserving) if $\mu(y,f,A) > 0(\mu(y,f,A) \geq 0)$ for every domain $A \subset\subset \Omega$ and $y \notin f(\partial A)$.

Now, resorting to the topological degree, we may define some multiplicity functions. Given a domain B such that $\overline{B} \subset A$, we say that B is a positive (negative) indicator domain for (y,f,A) if $y \notin f(\partial B)$ and $\mu(y,f,B) > 0(< 0)$. A finite (possibly empty) collection of pair-wise disjoint positive (negative) indicator domains for (y,f,A) is called a positive (negative) indicator system for (y,f,A) and is denoted by $\sigma^+(y,f,A)(\sigma^-(y,f,A))$. Finally we define the multiplicity functions

$$K^+(y,f,A) = \mathrm{Sup}\left\{ \sum_{B \in \sigma^+} \mu(y,f,B) : \sigma^+ = \sigma^+(y,f,A) \right\},$$

$$K^-(y,f,A) = \mathrm{Sup}\left\{ -\sum_{B \in \sigma^-} \mu(y,f,B) : \sigma^- = \sigma^-(y,f,A) \right\},$$

$$K(y,f,A) = K^+(y,f,A) + K^-(y,f,A)$$

with the convention that $K^+(y,f,A) = 0(K^-(y,f,A) = 0)$ if there is no positive (negative) nonempty indicator system. The multiplicity function K is related with the concept of essential maximal model continua (e.m.m.c.). We say that $C \subset \mathsf{R}^n$ is an e.m.m.c. for (y,f,A) if C is a component of $A \cap f^{-1}(y)$ which is a continuum and if for every open set D such that $C \subset D \subset A$ there exists a positive or negative indicator domain B for (y,f,A) such that $C \subset B \subset \overline{B} \subset D$. If either $K(y,f,A) \leq 1$ or $K(y,f,A) = +\infty$ then $K(y,f,A)$ agrees with the number of e.m.m.c. for (y,f,A) ([10], II.3.4., Thm. 3).

A sequence $\{B_k\}_{k \in \mathsf{N}}$ of nonempty domains is called a determining sequence for (y,f,A) if $\overline{B_{k+1}} \subset B_k \subset\subset A, y \in f(\overline{B_k}) \backslash f(\partial B_k)$, for every $k \in \mathsf{N}$ and $\lim_{k \to \infty} \mathrm{diam}(f(\overline{B_k})) = 0$.

Now let $x_0 \in \Omega$; we say that f has a weak differential at x_0 if there exists a linear mapping $L : \mathsf{R}^n \to \mathsf{R}^n$ and a set $B \subset \mathsf{R}$ such that 0 is a point of right density of B satisfying

$$\limsup_{\substack{t\to 0^+\\ t\in B}}\left\{\left\|\frac{f(x_0+tz)-f(x_0)}{t}-L(z)\right\| : z\in\partial Q(0,1)\right\}=0.$$

The following result by Goffman and Ziemer ([6] ,Thm. 3.4) states the weak differentiability properties of Sobolev functions.

THEOREM 2.1. *If $f \in W^{1,p}(\Omega;\mathsf{R}^n)$ with $p > n-1$, then f is weakly differentiable almost everywhere in Ω.*

3. Local invertibility of Sobolev functions.

THEOREM 3.1. *Let $f \in W^{1,n}(\Omega;\mathsf{R}^n)$ be a continuous, open mapping such that* $\det Df \geq 0$ *almost everywhere in Ω. Then $B_f \subset Z_f \cup S_f$.*

PROOF. Let $x_0 \notin Z_f \cup S_f$; we shall prove that $x_0 \notin B_f$. We use the link between the weak differential and the topological degree ([10], p. 329); if $x_0 \notin Z_f \cup S_f$, there exist $r_1, r_2 > 0$ such that $Q(x_0, r_1) \subset\subset \Omega$ and $\mu(y,f,Q(x_0,r_1)) = 1$ for every $y \in B(f(x_0), r_2)$.

We show that $N(y,f,Q(x_0,r_1)) = 1$ for every $y \in B(f(x_0), r_2)$. Since f is weakly sense-preserving ([11], Ch. VI, Lemma 5.1), we have $\sigma^-(y,f,Q(x_0,r_1)) = \emptyset$ for every $y \in \mathsf{R}^n$; therefore $K^-(y,f,Q(x_0,r_1)) = 0$ and $K(y,f,Q(x_0,r_1)) = K^+(y,f,Q(x_0,r_1))$ for every $y \in \mathsf{R}^n$. Furthermore

$$K^+(y,f,Q(x_0,r_1)) = \mu(y,f,Q(x_0,r_1)) = 1 \tag{3.1}$$

for every $y \in B(f(x_0), r_2)$ such that $K(y,f,Q(x_0,r_1)) < +\infty$ ([10], II. 3.4, Thm. 2 and Thm. 4). Now we note that f satisfies the condition (N) ([8], Corollary B) and, by Theorem 2.1, f is weakly differentiable almost everywhere in Ω. This implies that $K(\cdot,f,Q(x_0,r_1)) \in L^1(\mathsf{R}^n)$ ([10], V. 3.3., Thm. 5) and then $K(y,f,Q(x_0,r_1)) < +\infty$ for almost every $y \in \mathsf{R}^n$. By (3.1), $K(y,f,Q(x_0,r_1)) = 1$ for almost every $y \in B(f(x_0), r_2)$ and, therefore, also $N(y,f,Q(x_0,r_1)) = 1$ for almost every $y \in B(f(x_0), r_2)$ ([10], V. 3.3. Thm. 2). Since f is open , then $N(\cdot,f,Q(x_0,r_1))$ is lower semicontinuous in R^n ([5], Chap. 5, Thm. 1.3) and this implies that $N(\cdot,f,Q(x_0,r_1)) \leq 1$ everywhere in $B(f(x_0), r_2)$. On the other hand $B(f(x_0), r_2) \subset f(Q(x_0,r_1))$ because $\mu(y,f,Q(x_0,r_1)) = 1$ for every $y \in B(f(x_0), r_2)$ and then $N(\cdot,f,Q(x_0,r_1)) = 1$ everywhere in $B(f(x_0), r_2)$.

Finally, let $A_{x_0} = f^{-1}(B(f(x_0), r_2)) \cap Q(x_0, r_1)$; since $f(A_{x_0}) \subset B(f(x_0), r_2)$, the restriction of f to A_{x_0} is one-to-one and, as f is an open mapping, f is a homeomorphism from A_{x_0} to $f(A_{x_0})$, that is $x_0 \notin B_f$.

DEFINITION 3.2. Let $f : \Omega \to \mathsf{R}^n$ be a continuous mapping. For every $y \in \mathsf{R}^n$ we define $R_y = \{x \in f^{-1}(y)$: there exists a sequence $\{V_m\}_{m\in\mathsf{N}}$ of open neighbourhood of x such that $\operatorname{diam}(V_m) < \frac{1}{m}$, there exists $\mu(y,f,V_m)$ and it is positive for every $m \in \mathsf{N}\}$.

REMARK 3.3. If f is light, weakly sense-preserving and $A \subset \Omega$ is a domain, we have that $R_y \cap A$ is the set of the e.m.m.c. for (y,f,A).

In the following lemma , we recall a property of the set R_y for such mappings (see also [14]).

LEMMA 3.4. *Let $f : \Omega \to \mathsf{R}^n$ be a continuous, light and weakly sense-preserving mapping. Then $D(R_y) \cap \Omega = \emptyset$ for every $y \in \mathsf{R}^n$.*

PROOF. By contradiction we suppose that there exists $y \in \mathsf{R}^n$ such that $D(R_y) \cap \Omega \neq \emptyset$. Then let $x \in \Omega$ be a limit point of a sequence $\{x_m\}_{m\in\mathsf{N}}$ in R_y. Since f is continuous and light, then $x \in f^{-1}(y)$ and there exists an open neighbourhood B of x such that $B \subset \Omega$ and $f^{-1}(y) \cap \partial B = \emptyset$. Let $\mu(y,f,B) = \alpha \geq 0, k \in \mathsf{N}$ satisfying $k > \alpha$ and let $\{\overline{x}_1, \overline{x}_2, ..., \overline{x}_k\}$ be k points of the sequence $\{x_m\}_{m\in\mathsf{N}}$ belonging to B. Then, for every $i \in \{1,2,...,k\}$, there exists an open neighbourhood V_i of $\overline{x}_i$ such that $\overline{V}_i \subset B$, $\overline{V}_j \cap \overline{V}_l = \emptyset$ for every $j,l \in \{1,2,...,k\}$ and $\mu(y,f,V_i) \geq 1$.

Finally let $V = \cup_{i=1}^{k} V_i$ and $\Omega = B \setminus \overline{V}$. Then $f^{-1}(y) \cap \partial V = \emptyset$, $f^{-1}(y) \cap \partial W = \emptyset$ and therefore $f^{-1}(y) \cap B \subset \cup_{i=1}^{K} V_i \cup W$. Consequently

$$\alpha = \mu(y,f,B) = \sum_{i=1}^{K} \mu(y,f,V_i) + \mu(y,f,W) \geq k$$

and this is a contradiction.

THEOREM 3.5. *Let $f \in W^{1,n}(\Omega;\mathsf{R}^n)$ be a continuous, light mapping such that* $\det Df \geq 0$ *almost everywhere in Ω. Then $B_f \subset Z_f \cup S_f$.*

PROOF. As in Theorem 3.1, we consider $x_0 \notin Z_f \cup S_f$ and we prove that $x_0 \notin B_f$. Analogously we can show that there exist $r_1, r_2 > 0$ such that $K(y,f,Q(x_0,r_1)) = 1$ for every $y \in B(f(x_0),r_2)$ satisfying $K(y,f,Q(x_0,r_1)) < +\infty$.

Now we prove that $\{y \in B(f(x_0),r_2) : K(y,f,Q(x_0,r_1)) = +\infty\} = \emptyset$. By contradiction we suppose that there exists $y \in B(f(x_0),r_2)$ such that $K(y,f,Q(x_0,r_1)) = +\emptyset$. Then let $\{C_i\}_{i\in\mathsf{N}}$ be a sequence of e.m.m.c. for $(y,f,Q(x_0,r_1))$; since f is light, we have $C_i = \{x_i\}$ for every $i \in \mathsf{N}$, and $x_i \in R_y$ for every $i \in \mathsf{N}$. Therefore $R_y \cap Q(x_0,r_1)$ contains a bounded, infinite subset and this implies that $D(R_y) \cap \Omega \neq \emptyset$, which contradicts Lemma 3.4. Hence we have $K(y,f,Q(x_0,r_1)) = 1$ for every $y \in B(f(x_0),r_2)$ and, by Remark 3.3, this implies that $R_y \cap Q(x_0,r_1) = \{x_y\}$ for every $y \in B(f(x_0),r_2)$.

Now let $g(y) = x_y$ for every $y \in B(f(x_0),r_2)$. We prove that g is a homeomorphism and $g = f^{-1}$ in a neighbourhood of $f(x_0)$.

First we show that g is continuous on $B(f(x_0), r_2)$. Let $y \in B(f(x_0), r_2)$ be a limit point of a sequence $\{y_m\}_{m\in\mathsf{N}}$ in $B(f(x_0), r_2)$ and suppose for contradiction that there exists a subsequence $\{x_{y_{m_k}}\}_{k\in\mathsf{N}}$ which converges to $x \in \overline{Q(x_0, r_1)}$ and $x \neq x_y$. We have that $x \in f^{-1}(y)$ and therefore $x \in Q(x_0, r_1)$. Hence $x \notin R_y$ and, since f is light , this implies that there exists an open neighbourhood W of x such that $\mu(y,f, W) = 0$; consequently, there exists $h \in \mathsf{N}$ such that $\mu(y_{m_k},f, W) = 0$ for every $k \geq h$. On the other hand there exists an open neighbourhood V_m of x_{y_m} such that $\mathrm{diam}(V_m) < \frac{1}{m}$ and $\mu(y,f, V_m) \geq 1$; then, if we take $h^* \in \mathsf{N}$ such that $h^* \geq h$ and $V_{m_k} \subset\subset W$ for every $k \geq h^*$ we have

$$\mu(y_{m_k},f, W) = \mu(y_{m_k},f, V_{m_k}) + \mu(y_{m_k},f, W\backslash\overline{V}_{m_k}) = 0$$

for every $k \geq h^*$. Hence $\mu(y_{m_k},f, V_{m_k}) = 0$ and this is a contradiction. Therefore g is continuous on $B(f(x_0), r_2)$.

Now we observe that, by the definition of $R_y, f(g(y)) = y$ for every $y \in B(f(x_0), r_2)$ and then g is one-to-one. Therefore, by the invariance domain theorem, g is open and then it is a homeomorphism from $B(f(x_0), r_2)$ to $g(B(f(x_0), r_2))$; consequently $f = g^{-1}$ is a homeomorphism from $g(B(f(x_0), r_2))$ to $B(f(x_0), r_2)$. Finally we observe that $g(f(x_0)) = x_0$ ([10], pag 329) and $g(B(f(x_0), r_2))$ is an open neighbourhood of x_0. Therefore $x_0 \notin B_f$.

Remark 3.6. Theorems 3.1 and 3.5 are not true without the topological assumptions about f, i.e. if f is neither open nor light, as it is shown by the following example : let $k \in \mathsf{N}, I = (-2, 2), I_k = (k^{-1}, k^{-1} + k^{-2}), F = \cup_{k\in N} I_k$ and $E = F \cup (-F)$; it is easily verified that $x_0 = 0$ is a point of density 1 for E (see [4] for the definition). We set

$$g(x) = \int_0^x \chi_E(t)dt \text{ and } f(x, y) = (g(x), y) \text{ if } (x, y) \in I \times I$$

where χ_E is the characteristic function of E. Then $f \in W^{1,\infty}(I \times I; \mathsf{R}^2)$ and $\det Df(x, y) = \chi_E(x) \geq 0$ for almost every $(x, y) \in I \times I$.

Now we observe that $(0, 0) \in B_f\backslash(Z_f \cup S_f)$. Indeed, if $k \in \mathsf{N}$, let $J_k = ([k + 1]^{-1} + [k + 1]^{-2}, k^{-1})$; we note that $g'(x) = 0$ for every $x \in \cup_{k\in\mathsf{N}} J_k$ and therefore g is constant on each J_k. Then, for every $\delta > 0$ we may choose $m \in \mathsf{N}$ such that $J_m \subset (-\delta; \delta)$ and $f(J_m \times \{0\})$ is a point. This implies that $(0, 0) \in B_f$. On the other hand, since $x_0 = 0$ is a point of density 1 for E, we have that f is weakly differentiable in $(0, 0)$ and $\det Df(0, 0) = 1$, that is $(0, 0) \notin (Z_f \cup S_f)$. Finally, we note that f is neither open nor light ; indeed $f(J_k \times I) = \{g(k^{-1})\} \times I$ and $f^{-1}(g(k^{-1}), 0) = \overline{J}_k \times \{0\}$ for every $k \in \mathsf{N}$.

4. Local invertibility of Sobolev functions of higher order.

The following lemma concerns a local property of the fibers $f^{-1}(y)$; the proof is similar to the one of Theorem 1, (iv) in [2].

LEMMA 4.1. *Let $f \in W^{1,n}(\Omega; \mathsf{R}^n)$ be a continuous mapping such that $\det Df > 0$ almost everywhere in Ω and let $x_0 \notin Z_f \cup S_f$.*

Then there exist $r_1, r_2 > 0$ such that $f^{-1}(y) \cap Q(x_0, r_1)$ is a continuum for every $y \in B(f(x_0), r_2)$.

PROOF. Since $x_0 \notin Z_f \cup S_f$, there exist $r_1, r_2 > 0$ such that $Q(x_0, r_1) \subset\subset \Omega$ and $\mu(y, f, Q(x_0, r_1)) = 1$ for every $y \in B(f(x_0), r_2)$ (see the proof of Theorem 3.1).

First we observe that $f^{-1}(y) \cap Q(x_0, r_1)$ is closed. In fact for every $y \in B(f(x_0), r_2), f^{-1}(y) \cap \partial Q(x_0, r_1)$ is empty and since $Q(x_0, r_1) \subset\subset \Omega$ and $f^{-1}(y)$ is closed in Ω, we have $\overline{f^{-1}(y) \cap Q(x_0, r_1)} \subset Q(x_0, r_1)$ and this implies that $f^{-1}(y) \cap Q(x_0, r_1)$ is closed.

Now let's suppose by contradiction that $f^{-1}(y) \cap Q(x_0, r_1)$ is not connected and consider a partition of compact sets $\{C_i\}_{i=1,2}$ of $f^{-1}(y) \cap Q(x_0, r_1)$; besides let A_1, A_2 be disjoint open sets such that $C_i \subset A_i \subset Q(x_0, r_1)$ $(i = 1, 2)$. Since $y \notin f(\partial A_1) \cup f(\partial A_2)$ we have

$$1 = \mu(y, f, Q(x_0, r_1)) = \mu(y, f, A_1) + \mu(y, f, A_2).$$

Without loss of generality we can suppose $\mu(y, f, A_1) \leq 0$. Then there exists $\alpha > 0$ such that $B(y, \alpha) \cap [f(\partial A_1) \cup f(\partial A_2)]$ is empty and $\mu(z, f, A_1) \leq 0$ for every $z \in B(y, \alpha)$; now, f satisfies condition (N) on Ω ([9], Corollary 3.13) and therefore, as in Theorem 3.1, one can prove that $K(y, f, A_1) < +\infty$ for almost every $y \in \mathsf{R}^n$. Hence the following transformation formula holds ([10], II.3.4., Thm 2 and V.3.4., Thm 1)

$$\int_{A_1} \chi_{B(y,\alpha)}(f(x)) \det Df(x)\, dx = \int_{B(y,\alpha)} \mu(z, f, A_1)\, dz$$

and consequently $\mathscr{L}^n(A_1 \cap f^{-1}(B(y, \alpha))) = 0$. Since $A_1 \cap f^{-1}(B(y, \alpha))$ is open, it follows that $A_1 \cap f^{-1}(B(y, \alpha)) = \emptyset$ and therefore $C_1 \cap f^{-1}(y) = \emptyset$, which is a contradiction.

In the following we suppose that Ω has a Lipschitz boundary (see [1]).

LEMMA 4.2. *If $f \in W^{2,p}(\Omega; \mathsf{R}^n)$ with $p > \frac{n(n-1)}{2n-1}$, then $B_{1,p}(S_f) = 0$.*

PROOF. From the Sobolev inequalities it follows that $f \in W^{1,p}(\Omega; \mathsf{R}^n)$ with $p > n - 1$ and then, according to the proof of Theorem 3.4 in [6], we have

$$\Omega\backslash S_f \supset \left\{ x\in\Omega : Df(x) \text{ exists and } \lim_{t\to 0^+}\left[t^{-n}\int_{Q(x;2t)} ||Df(y)-Df(x)||^p\,dy + \right.\right.$$

$$\left.\left. + t^{-n-p}\int_{Q(x;2t)} ||f(y)-f(x)-Df(x)(y-x)||^p\,dy \right] = 0 \right\}.$$

Since $Df \in W^{1,p}(\Omega;\mathsf{R}^{n^2})$, there exists a set $E\subset\Omega$ such that $B_{1,p}(E)=0$ and

$$\lim_{t\to 0^+} t^{-n}\int_{Q(x;2t)} ||Df(y)-Df(x)||^p\,dy = 0$$

for every $x\in\Omega\backslash E$ ([15], Theorem 3.3.3.). Besides, there exists a set $F\subset\Omega$ such that $B_{1,p}(F)=0$ and

$$\lim_{t\to 0^+} t^{-n-p}\int_{Q(x;2t)} ||f(y)-f(x)-Df(x)(y-x)||^p\,dy = 0$$

for every $x\in\Omega\backslash F$ ([15], Theorem 3.4.2.). Finally, if we define $G=E\cup F$, we obtain $S_f\subset G$ and $B_{1,p}(G)=0$.

In the next theorem we deal with mappings f belonging to $W^{2,p}(\Omega;\mathsf{R}^n)$ with $p>n-1$. We recall that, by the Sobolev inequalities, such mappings have a Hölder continuous representative in their equivalence class; we shall always assume that this representative of f has been choosen.

Theorem 4.3. *Let* $p>n-1$ *and* $f\in W^{2,p}(\Omega;\mathsf{R}^n)$ *such that*
(a) $\det Df>0$ *almost everywhere in* Ω,
(b) $B_{1,p}(Z_f)=0$.

Then $B_f\subset Z_f\cup S_f, \dim_H(B_f)\le n-p$ *and* $\dim_H(f(B_f))\le \frac{p(n-p)}{2p-n}$.

Proof. Let $x_0\notin Z_f\cup S_f$; we shall prove that $x_0\notin B_f$. First we observe that, by the Sobolev inequalities, $f\in W^{1,n}(\Omega;\mathsf{R}^n)$ and satisfies the condition (N) ([9], Corollary 3.13); as in the proof of Theorem 3.1, we can show that there exist $r_1,r_2>0$ and a set $N\subset B(f(x_0),r_2)$ such that $\mathscr{L}^n(N)=0$ and $N(y,f,Q(x_0,r_1))=1$ for every $y\in B(f(x_0),r_2)\backslash N$.

Let $A_{x_0}=f^{-1}(B(f(x_0),r_2))\cap Q(x_0,r_1)$ and let's prove that $N(y,f,Q(x_0,r_1))=1$ for every $y\in f(A_{x_0})$. For this purpose we show that $f^{-1}(y)\cap Q(x_0,r_1)\subset(Z_f\cup S_f)\cap A_{x_0}$ for every $y\in N\cap f(A_{x_0})$.

Let $y\in N\cap f(A_{x_0})$ and $x\in f^{-1}(y)\cap Q(x_0,r_1)$.

First we prove that $x\in(\Omega\backslash A_{x_0})\cup(Z_f\cup S_f)$. Let's suppose by contradiction that $x \in A_{x_0} \backslash (Z_f \cup S_f)$. By the Hölder continuity of f, there exists a sequence of positive real numbers $\{\alpha_n\}_{n \in \mathsf{N}}$ such that $\{Q(x; \alpha_n)\}_{n \in \mathsf{N}}$ is a determining sequence for $(y, f, Q(x_0, r_1))$ ([10], p.329). This implies that $\{x\}$ is a component of $f^{-1}(y) \cap Q(x_0, r_1)$ ([10], II.3.1., Lemma 6); besides $y \in B(f(x_0), r_2)$ and therefore $f^{-1}(y) \cap Q(x_0, r_1)$ is a continuum by Lemma 4.1; consequently $\{x\} = f^{-1}(y) \cap Q(x_0, r_1)$ and $y \notin N$, which is a contradiction. Then $x \in (\Omega \backslash A_{x_0}) \cup (Z_f \cup S_f)$.

Finally we show that $x \in A_{x_0}$. Again by contradiction, let's suppose $x \notin A_{x_0}$. Since $x \in Q(x_0, r_1)$, we have $x \notin f^{-1}(B(f(x_0), r_2))$ and this implies that $x \notin f^{-1}(y)$, otherwise $f(x) = y \in f(A_{x_0}) \subset B(f(x_0), r_2)$, and this is a contradiction. Then $x \in A_{x_0} \cap [(\Omega \backslash A_{x_0}) \cup (Z_f \cup S_f)] = A_{x_0} \cap (Z_f \cup S_f)$ and consequently $f^{-1}(y) \cap Q(x_0, r_1) \subset (Z_f \cup S_f) \cap A_{x_0}$ for every $y \in N \cap f(A_{x_0})$.

Now, by Lemma 4.2 and (b), we obtain

$$B_{1,p}(f^{-1}(y) \cap Q(x_0, r_1)) \leq B_{1,p}(Z_f \cup S_f) = 0$$

for every $y \in N \cap f(A_{x_0})$. Since $p > n - 1$, we have

$$\dim_H(f^{-1}(y) \cap Q(x_0, r_1)) < 1$$

([15], Th. 2.6.16), and, since $f^{-1}(y) \cap Q(x_0, r_1)$ is a continuum by Lemma 4.1, then $\operatorname{diam}(f^{-1}(y) \cap Q(x_0, r_1)) = 0$, that is $N(y, f, Q(x_0, r_1)) = 1$. Therefore we have proved that $N(y, f, Q(x_0, r_1)) = 1$ for every $y \in N \cap f(A_{x_0})$ and then for every $y \in f(A_{x_0})$. This implies that the restriction of f to A_{x_0} is one – to – one; by the invariance domain theorem it follows that such restriction is open and then f is a homeomorfism from A_{x_0} to $f(A_{x_0})$, that is $x_0 \notin B_f$.

In order to show that $\dim_H(B_f) \leq n - p$, it is enough to recall that, by (b) and Lemma 4.2, $B_{1,p}(B_f) \leq B_{1,p}(Z_f) + B_{1,p}(S_f) = 0$. Hence $H^{n-p+\epsilon}(B_f) = 0$ for every $\epsilon > 0$ ([15], Theorem 2.6.16) and $\dim_H(B_f) \leq n - p$.

Finally we observe that, by the Sobolev inequalities, $f \in C^{o,(2p-n)/p}(\Omega; \mathsf{R}^n)$ and therefore $H^{p(n-p+\epsilon)/(2p-n)}(f(B_f)) \leq H^{n-p+\epsilon}(B_f) = 0$ for every $\epsilon > 0$ ([12], Theorem 29); consequently $\dim_H(f(B_f)) \leq \frac{p(n-p)}{2p-n}$.

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DIPARTIMENTO DI MATEMATICA
VIA DODECANESO, 35
16146 GENOVA
ITALIA