MATH. SCAND. 83 (1998), 265–276

TWO PROBLEMS ON POTENTIAL THEORY FOR UNBOUNDED SETS

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0. Introduction. Statement of the problems.

We consider two problems which are well-known in classical potential theory when one studies *bounded* sets and measures having *compact* support. Let L denote the plurisubharmonic (psh) functions in $\mathbf{C}^N$ of logarithmic growth:

$$L = \{u \text{ psh in } \mathbf{C}^N : u(z) - \log|z| \leq c = c(u),\ |z| \to +\infty\}.$$

This class, as well as associated subclasses, have been extensively studied in relation with extremal psh functions. For an arbitrary Borel set $E \subset \mathbf{C}^N$ we define the $L-$extremal function

$$V_E(z) = \sup\{u(z) : u \in L,\ u \leq 0 \text{ on } E\}$$

and $V_E^*(z) := \limsup_{\zeta \to z} V_E(\zeta)$. If $V_E^* \not\equiv 0, +\infty$, then $V_E^* \in L$ and $\{z \in E : V_E^*(z) > 0\}$ is *pluripolar*. Recall that a set E is pluripolar (equivalently, $V_E^* \equiv +\infty$) if there exists a psh function u with $E \subset \{z : u(z) = -\infty\}$; in this case, we can even take $u \in L$. Let E_j, $j = 1, 2, \ldots$ be an increasing sequence of Borel sets in $\mathbf{C}^N$ and let $E := \cup E_j$. If E is bounded (or if E is pluripolar), then

(0.1) $\lim_{j\to+\infty} V_{E_j}^* = V_E^*$ *pointwise on* $\mathbf{C}^N$.

Here's a quick proof of (0.1) (see [5]). Clearly if E is pluripolar, so is each E_j and $V_{E_j}^* = V_E^* \equiv +\infty$. Suppose E is not pluripolar. Clearly $V := \lim_{j\to+\infty} V_{E_j}^* \geq V_E^*$ and $V \in L$. Set

$$F := \cup_j \{z \in E_j : V_{E_j}^*(z) > 0\};$$

then F is pluripolar and $V = 0$ on $E - F$. Hence $V \leq V_{E-F}^*$ and we are done provided we can show:

(0.2) *if a set F is pluripolar, then* $V_E^* = V_{E-F}^*$.

Now clearly $V_E^* \leq V_{E-F}^*$. For the reverse inequality, we can find $v \in L$ with

*Partially supported by KBN Grant No. 2 PO3A 058 09

Received July 22, 1996.

$F \subset \{z : v(z) = -\infty\}$; *since* $E - F$ *is bounded* and v is uppersemicontinuous we can assume $v \leq 0$ on $E - F$. Then for any $u \in L$ with $u \leq 0$ on $E - F$ and any $\epsilon > 0$,

$$(1 - \epsilon)u + \epsilon v \leq V_E \leq V_E^*.$$

Thus $u \leq V_E^*$ almost everywhere (a. e.) and hence everywhere.

Note how *boundedness* of E was used. If E is *unbounded*, we show that (0.1) and (0.2) are still true *if* $N = 1$ (Theorem 1.2 and Corollary 1.4) *but not if* $N > 1$. This was a problem posed by Plesniak.

The other result is a positivity and uniqueness theorem for the logarithmic energy

$$I(\mu, \mu) := \int\int \log \frac{1}{|x - y|} d\mu(x) d\mu(y)$$

of signed Borel measures in R^n which have total mass 0 but which are *not* necessarily compactly supported. We recall the classical result.

Positivity and Uniqueness Lemma ([6], Theorem 1.16)(n=2). *If* μ *is a signed Borel measure in* R^2 *with* $I(|\mu|, |\mu|) < +\infty$ *and* $|\mu|(1) < +\infty$ *having compact support and either* $\mu(1) = 0$ *or the support of* μ *is contained inside the unit disk, then* $I(\mu, \mu) \geq 0$ *and* $I(\mu, \mu) = 0$ *if and only if* $\mu = 0$.

The lemma can be extended to certain non-compactly supported measures in R^n for *any* $n \geq 2$ (Theorem 2.5). Our first attempt at solving the Plesniak problem in $\mathsf{C} = \mathsf{R}^2$ led us to this result.

In the case $N = 1$, the class L, also called the class of subharmonic functions of *minimal* growth, has been studied by many authors including Arsove, Essén, Haliste, Hayman, Huber, Lewis, Shea and Wu; we refer the reader to [3] for specific references.

We would like to thank the mathematics departments of Umeå University and the University of Auckland as well as the New Zealand Mathematical Society for their support. The third author would also like to thank V. Totik for many helpful discussions. In addition, the authors thank the referee for valuable comments.

1. The Plesniak problem.

The Plesniak problem (both (0.1) and (0.2)) has a negative answer in C^2 as the following example due to [2] shows.

Example 1.1. *Let* $E = \{(z_1, z_2) \in \mathsf{C}^2 : |z_2| \leq 1\} \cup \{(z_1, z_2) \in \mathsf{C}^2 : z_1 = 0, |z_2| \leq 2\}$ *and* $F = \{(z_1, z_2) \in \mathsf{C}^2 : z_1 = 0,\ 1 < |z_2| \leq 2\}$. *Then*

$$V^*_{E-F}(z_1,z_2) = \log^+|z_2| \neq \log^+\frac{|z_2|}{2} = V^*_E(z_1,z_2).$$

Proof. Clearly $V^*_{E-F}(z_1,z_2) = \log^+|z_2|$. To see that $V^*_E(z_1,z_2) = \log^+\frac{|z_2|}{2}$, note first that since $E \subset \{(z_1,z_2) \in \mathbf{C}^2 : |z_2| \leq 2\}$, we have $V^*_E(z_1,z_2) \geq V_E(z_1,z_2) \geq \log^+\frac{|z_2|}{2}$. Now, since $E - F \subset E$, $V_E(z_1,z_2) \leq V^*_E(z_1,z_2) \leq \log^+|z_2|$. This shows that V_E, V^*_E are independent of z_1; for if we fix z_2^0, then $f(z_1) := V^*_E(z_1,z_2^0) \leq \log^+|z_2^0|$ for *all* z_1; thus f is bounded and hence constant. Since $V_E(0,z_2) = 0$ for $|z_2| \leq 2$ we have $V_E(0,z_2) \leq \log^+\frac{|z_2|}{2}$; thus this better bound persists on all of $\mathbf{C}^2$; i.e., $V_E(z_1,z_2) = \log^+\frac{|z_2|}{2}$.

Note, in this example, for any sequence of bounded Borel sets E_j with $E_j \subset E_{j+1}$, $E = \cup E_j$, we have

$$\lim_{j\to+\infty} V^*_{E_j}(z_1,z_2) = V^*_{E-F}(z_1,z_2) = \log^+|z_2| \neq V^*_E(z_1,z_2).$$

We see here the phenomenon of *propagation of singularities* that can occur in $\mathbf{C}^N$ for $N > 1$ but not in $\mathbf{C}$; the one-variable proof sketched in the introduction of equality of V^*_{E-F} and V^*_E in the *bounded* case fails for the following reason: if $u \in L$ and $u = -\infty$ on F, then necessarily $u = -\infty$ on the whole complex line

$$F^* := \{(z_1,z_2) \in \mathbf{C}^2 : z_1 = 0,\ z_2 \in \mathbf{C}\}$$

(F^* is the *pluripolar hull* of F; see [1] for further results). For any $0 < \alpha \leq 1$, the function $v_\alpha(z) := \alpha \log|z_1|$ is $-\infty$ precisely on F^*; clearly these functions, and hence *any* $v \in L$ which is $-\infty$ on F^*, cannot remain bounded above on $E - F$.

Theorem 1.2. *Let E_j, $j = 1,2,\ldots$ be an increasing sequence of Borel sets in $\mathbf{C}$ and let $E := \cup E_j$. Then*

$$\lim_{j\to+\infty} V^*_{E_j} = V^*_E.$$

Proof. The proof is based on the following preliminary result. Below, "q.e. on E" means everywhere on E except perhaps a polar set; SH(U) denotes the subharmonic functions on U; and "$\phi \leq M$ on ∂U" means $\limsup_{z\to\zeta} \phi(z) \leq M$ for all $\zeta \in \partial U$.

Lemma 1.3. *Let $u \in L$, $u \geq 0$ (but $u \not\equiv 0$) and $E \subset \mathbf{C}$ with $u = 0$ q.e. on E. Let U be a connected component of $\{u < M\}$ for some $M > 0$. Suppose that*

$$u_{|U} = [\sup\{\phi \in \mathrm{SH}(U) : \phi \leq 0 \text{ on } E', \phi \leq M \text{ on } \partial U\}]^* \tag{1.1}$$

where $E' := \{u = 0\} \cap U$. Then given $c > 0$ with $c < M$ and given $z_0 \in U - E$, there exists $w \in L$ such that

1. *$w = u$ in the complement of U;*
2. *$w \le c$ on $E \cap U$;*
3. *$w \le u + c$ everywhere in $\mathbb{C}$; and*
4. *$w(z_0) > u(z_0) - c$.*

Furthermore, if $F' := \{w = c\} \cap U$, then

$$w_{|U} = [\sup\{\phi \in \mathrm{SH}(U) : \phi \le c \text{ on } F', \phi \le M \text{ on } \partial U\}]^*.$$

Proof of Lemma 1.3. The set U is bounded by the theorem of Wiman-Valiron (see [3], page 384 or [4], page 197). For this theorem tells us that given $t > 1$, there exist $R_j \uparrow +\infty$ such that

$$\inf_{|z|=R_j} u(z) > \frac{1}{t} \sup_{|z|=R_j} u(z).$$

Clearly $\lim_{j\to+\infty} \sup_{|z|=R_j} u(z) = +\infty$; hence, given $M > 0$, we can choose R_j with $U \cap \{z : |z| < R_j\} \ne \emptyset$ and

$$\inf_{|z|=R_j} u(z) > M.$$

Since U is connected, it is path connected; using this, an easy exercise shows that $U \subset \{z : |z| < R_j\}$.

Recall we are assuming that

$$u_{|U} = [\sup\{\phi \in \mathrm{SH}(U) : \phi \le 0 \text{ on } E', \phi \le M \text{ on } \partial U\}]^* \tag{1.1}$$

where $E' := \{u = 0\} \cap U$. Now since $E \setminus E'$ is polar, we can find g subharmonic in $\mathbb{C}$ with $g \le 0$ on U and $E \setminus E' \subset \{g = -\infty\}$. We let $U_j \uparrow U$, $j = 0, 1, 2, \ldots$ be an exhaustion sequence for U with the property that g is bounded on the boundary of every U_j. This can be achieved since by Theorem 6.3 in [3], given $z_0 \in \mathbb{C}$ and $R > 0$, one can find a circle $\{z : |z - z_0| = r\}$, $r < R$, on which g is bounded. Then any compact set contained in U may be covered by the interiors of these circles.

Fix a sequence of positive numbers $\{c_j\}$ with $c_j < c_{j+1} \uparrow c$. Set

$$h_1 := [\sup\{\phi \in \mathrm{SH}(U) : \phi \le c_1 \text{ on } E', \phi \le M \text{ on } \partial U\}]^*.$$

Then $u < h_1 \le u + c_1$ in U. In particular,

$$h_1(x) = c_1 \text{ for } x \in E'. \tag{1.2}$$

Since $h_1 > u$ in U and g is bounded on $\partial U_0 \subset\subset U$, we can choose $d_1 > 0$ such that $h_1 + d_1 g > u$ on ∂U_0 and

$$d_1 g(z_0) > -c_1. \tag{1.3}$$

Then

$$f_1 := \begin{cases} h_1 + d_1 g, & \text{on } U_0; \\ \max(h_1 + d_1 g, u), & \text{on } U \setminus U_0; \\ u, & \text{on } \mathbf{C} \setminus U \end{cases}$$

is a well defined, subharmonic function. We have $f_1(z_0) > u(z_0) - c_1$ and $f_1 \leq c_1$ on E' (see (1.2)). Furthermore, since g is negative and c_1 is positive,

$$h_1 + d_1 g \leq h_1 \leq u + c_1$$

which shows that

$$f_1 \leq u + c_1. \tag{1.4}$$

Note $E' \subset E_1' := \{f_1 \leq c_1\}$ from (1.4). Also, since $f_1 = -\infty$ on $(E - E') \cap U_0$, we have

$$(E - E') \cap U_0 \subset \text{int } E_1'. \tag{1.5}$$

Set

$$u_1 := \begin{cases} [\sup\{\phi \in \text{ SH}(U) : \phi \leq c_1 \text{ on } E_1',\ \phi \leq M \text{ on } \partial U\}]^*, & \text{on } U; \\ u, & \text{on } \mathbf{C} \setminus U. \end{cases}$$

Since $E' \subset E_1'$, if $\phi \leq c_1$ on E_1' we have $\phi \leq c_1$ on E' which shows that $u_1 \leq h_1$. Thus, for $z \in E'$ we have $u_1(z) \leq h_1(z) \leq u(z) + c_1 = c_1$ which means that $z \in E_1 := \{u_1 \leq c_1\}$; i.e., $E' \subset E_1$. We claim that, together with (1.5), this shows that

$$E' \cup [(E - E') \cap U_0] \subset E_1.$$

For if int $E_1' = \emptyset$, there is nothing to prove. If, on the other hand, int $E_1' \neq \emptyset$, then $u_1 = c_1$ on int E_1' which shows that int $E_1' \subset E_1$. Moreover, as noted above, $u_1 \leq u + c_1$; also, from (1.3), $u_1(z_0) \geq u(z_0) - c_1$. Thus we have a subharmonic function $u_1 \in L$ having the following properties:

$u_1 \leq u + c_1$;

$E' \cup [U_0 \cap (E - E')] \subset E_1 := \{u_1 \leq c_1\}$;

$u_1 = u$ outside U; and

$u_1(z_0) \geq u(z_0) - c_1$.

If we repeat this construction with u_1, U_1, E_1 and $c_2 - c_1$ in place of u, U_0, E' and c_1, we obtain $u_2 \in L$ having the following properties:

$u_2 \leq u_1 + (c_2 - c_1)$;

$E' \cup [U_1 \cap (E - E')] \subset E_1 \cup [U_1 \cap (E - E_1)] \subset E_2 := \{u_2 \leq c_2\}$;

$u_2 = u$ outside U; and

$u_2(z_0) \geq u_1(z_0) - (c_2 - c_1) \geq u(z_0) - c_1 - (c_2 - c_1) = u(z_0) - c_2$.

We continue in this fashion getting a sequence $\{u_j\}$ such that $u_j - c_j$ is decreasing; it is easy to see that the function $w := \lim u_j$ has all the required properties.

Proof of Theorem 1.2. Let E_j, $j = 1, 2, \ldots$ be an increasing sequence of Borel sets in $\mathbb{C}^N$ with $E := \cup E_j$ and let $V := \lim_{j\to\infty} V^*_{E_j}$. Then $V \geq V^*_E$. If $V \equiv 0$, then $V \geq V^*_E$ implies $V^*_E \equiv 0$. Since the case where $V \equiv +\infty$ was covered in the introduction, we need only consider the case where $V \not\equiv 0, +\infty$; i.e., $V \in L$, and it clearly suffices to show: *given $c > 0$ and $z_0 \in \mathbb{C} - E$, there exists $\psi \in L$ with $\psi \leq c$ on E and $\psi(z_0) > V(z_0) - c$.* For then $\psi - c \leq 0$ on E; hence $V^*_E(z_0) \geq \psi(z_0) - c > V(z_0) - 2c$; this inequality being valid for all $c > 0$ yields the result. To prove the italicised statement, note first that V satisfies the hypothesis (1.1) of Lemma 1.3. We now fix an exhaustion sequence for $\mathbb{C}$ of open sets U_j, each of them being a connected component of $\{V < M_j\}$ containing z_0; here, the sequence $\{M_j\}$ is chosen with $M_j \to \infty$. Fix $\{c_j\}$ with $c_0 = 0$, $c_j > 0$ for $j > 0$, and $\sum c_j < c$. We apply the lemma repeatedly, first with the data (V, c_0, U_0), obtaining u_1 as the new function; then with data (u_1, c_1, U_1) to get u_2; etc. Since $u_j - \sum_{k=0}^{j-1} c_j$ is decreasing and $\sum c_j < c$, the limit $\psi = \lim u_j$ exists; $\psi \in L$; and, by the construction, ψ satisfies the required inequalities.

Corollary 1.4. *Suppose E is a Borel set in $\mathbb{C}$ and F is a polar set. Then $V^*_{E-F} = V^*_E$.*

Proof. Take E_j bounded with $E_j \uparrow E$. Then

$$V^*_{E-F} = \lim_{j\to\infty} V^*_{E_j-F} = \lim_{j\to\infty} V^*_{E_j} = V^*_E.$$

Remark. As pointed out to the third author by S. Gardiner, for unbounded E, $V^*_E \in L$ (i.e., $V^*_E \not\equiv 0, +\infty$) if and only if E is not polar and $E^* := \{z : 1/z \in E\}$ is thin at 0. In this case, the total mass $\mu(1)$ of the Laplacian of V^*_E equals 2π; indeed, a characterization of this subclass of the class L can be found in [3], Theorem 6.32: *for a function $u \in L$, $\mu(1) = 2\pi$ if and only if*

$$\lim_{r\to+\infty} \frac{\sup_{|z|=r} u(z)}{\log r} = 1.$$

2. Logarithmic potentials in $\mathbb{R}^n$.

In this section, we discuss some general results about logarithmic potentials and energies of measures in $\mathbb{R}^n$ for *any* $n = 2, 3, \ldots$. Let μ be a positive measure on $\mathbb{R}^n$ satisfying

$$\int \log(1 + |t|)d\mu(t) < +\infty. \tag{2.1}$$

Then the logarithmic potential

$$p_\mu(x) = \int \log \frac{1}{|x-y|} d\mu(y)$$

is locally integrable and superharmonic in all of R^n. Furthermore, since

$$\log \frac{1}{|x-y|} \geq -[\log(1+|x|) + \log(1+|y|)],$$

(2.1) implies that

$$\int p_\mu(x) d\mu(x) = \int \left[\int \log \frac{1}{|x-y|} d\mu(y) \right] d\mu(x) > -\infty.$$

Definition. Let μ and ν be two positive measures satisfying (2.1). We call

$$I(\mu,\nu) = \int \left[\int \log \frac{1}{|x-y|} d\mu(x) \right] d\nu(y) := \lim_{M\to+\infty} \int \left[\int \min\left(\log \frac{1}{|x-y|}, M \right) d\mu(x) \right] d\nu(y) \tag{2.2}$$

the mutual energy of μ and ν.

Note that by Fubini's theorem we have $I(\mu,\nu) = I(\nu,\mu)$. Also, $I(\mu,\nu)$ may be equal to $+\infty$. Also note that again (2.1) implies that $I(\mu,\nu) > -\infty$. When is $I(\mu,\nu) < +\infty$?

Lemma 2.1. *Let μ, ν be two positive measures satisfying*

$$\int \log(1+|t|) d(\mu+\nu)(t) < +\infty. \tag{2.3}$$

Then $p_\mu(x) := \int \log \frac{1}{|x-y|} d\mu(y) \in L^1(d\nu)$ if and only if $I(\mu,\nu) < +\infty$. In this case,

$$\lim_{R,S\to+\infty} I(\chi_{B(0,R)}\mu, \chi_{B(0,S)}\nu) = I(\mu,\nu)$$

where $\chi_{B(0,R)}$ denotes the characteristic function of the ball $B(0,R) = \{x : |x| < R\}$.

Proof. Clearly if $p_\mu \in L^1(d\nu)$ then $I(\mu,\nu) < +\infty$. For the converse, suppose $I(\mu,\nu) < +\infty$. For $M \geq 0$, we have

$$\log \frac{1}{|x-y|} \geq \min\left(\log \frac{1}{|x-y|}, M \right) \geq -[\log(1+|x|) + \log(1+|y|)].$$

Now

$$p_\mu(x) = \int \left[\log \frac{1}{|x-y|} + \log(1+|x|) + \log(1+|y|) \right] d\mu(y) - \mu(1)[\log(1+|x|)]$$
$$- \int \log(1+|y|) d\mu(y) := f_1(x) - f_2(x)$$

where $f_2(x) := \mu(1)[\log(1+|x|)] + \int \log(1+|y|)d\mu(y) \in L^1(d\nu)$ by hypothesis (2.3). Since

$$f_1(x) := \int \left[\log\frac{1}{|x-y|} + \log(1+|x|) + \log(1+|y|)\right] d\mu(y) \geq 0,$$

it follows that $f_1 \in L^1(d\nu)$ if

$$\lim_{M\to+\infty}\int\left\{\int\left[\min\left(\log\frac{1}{|x-y|},M\right)+\log(1+|x|)+\log(1+|y|)\right]d\mu(y)\right\}d\nu(x)<+\infty.$$

This inequality follows from (2.3), the definition (2.2), and the assumption that $I(\mu,\nu) < +\infty$.

Finally for each $M \geq 0$, the equality

$$\lim_{R,S\to+\infty}\int\left[\int \min\left(\log\frac{1}{|x-y|},M\right)d[\chi_{B(0,R)}\mu](x)\right]d[\chi_{B(0,S)}\nu](y) = \int\left[\int \min\left(\log\frac{1}{|x-y|},M\right)d\mu(x)\right]d\nu(y)$$

follows from the monotone convergence theorem. Thus under the assumption $I(\mu,\nu) < +\infty$,

$$I(\mu,\nu) = \lim_{M,R,S\to+\infty}\int\left[\int \min\left(\log\frac{1}{|x-y|},M\right)d[\chi_{B(0,R)}\mu](x)\right]d[\chi_{B(0,S)}\nu](y)$$

$$= \lim_{R,S\to+\infty} I(\chi_{B(0,R)}\mu, \chi_{B(0,S)}\nu).$$

Remark. Lemma 2.1 says that under assumption (2.3),

$$p_\mu \in L^1(d\nu) \text{ if and only if } |\log|x-y|| \in L^1(\mu\times\nu).$$

We next turn to the case of *signed* Borel measures.

Definition. Suppose μ is a signed Borel measure satisfying

$$\int \log(1+|y|)d|\mu|(y) < +\infty. \tag{2.4}$$

We define the logarithmic potential

$$p_\mu(x) = \int \log\frac{1}{|x-y|}d\mu(y).$$

If ν is another signed Borel measure satisfying

$$\int \log(1+|y|)d|\nu|(y) < +\infty, \tag{2.4'}$$

and we have $I(|\mu|, |\nu|) < +\infty$, we define the mutual energy of μ, ν by

$$I(\mu, \nu) := \int\int \log \frac{1}{|x-y|} d\mu(x) d\nu(y). \tag{2.5}$$

The following result, whose proof is similar to that of Lemma 2.1, shows that this is a good definition.

COROLLARY 2.2. *Let μ, ν be two signed Borel measures satisfying* (2.4), (2.4′) *and $I(|\mu|, |\nu|) < +\infty$. Then*

1. $p_\mu \in L^1(\nu)$, $p_\nu \in L^1(\mu)$;
2. $|I(\mu, \nu)| < +\infty$;
3. $I(\mu, \nu) = I(\nu, \mu)$;
4. $\lim_{R,S\to+\infty} I(\chi_{B(0,R)}\mu, \chi_{B(0,S)}\nu) = I(\mu, \nu)$.

In particular, we get finiteness of the energy $I(\mu, \mu)$ for a signed Borel measure satisfying (2.4) and $I(|\mu|, |\mu|) < +\infty$. Under an extra hypothesis, the energy is *nonnegative*. We first deal with some preliminaries. Below, we let $dm = dm(x)$ denote Lebesgue measure on R^n; we let $D = D(\mathsf{R}^n)$ denote the space of real-valued smooth (C^∞) functions on R^n of compact support; we let S be the space of rapidly decreasing functions on R^n; and, finally, S' denotes the space of tempered distributions.

LEMMA 2.3. *Let $0 < a < n$. For $f \in D$,*

$$C_a \int\int \frac{f(x)f(y)}{|x-y|^{n-a}} dm(x) dm(y) = \int \frac{|\hat{f}(\xi)|^2}{|\xi|^a} dm(\xi)$$

where $C_a = \pi^{a-n/2} \frac{\Gamma(\frac{n-a}{2})}{\Gamma(\frac{a}{2})} > 0$ *and* $\hat{f}(\xi) := \int f(x) e^{-2\pi i x\cdot\xi} dm(x)$ *(here,* $x\cdot\xi := \sum_{i=1}^n x_i\xi_i$*).*

PROOF. Let $\phi(x) := \exp(-\pi|x|^2)$. Then $\phi = \hat{\phi}$. Also,

$$\widehat{\frac{C_a}{|x|^{n-a}}} = \frac{1}{|\xi|^a} \text{ for } 0 < a < n/2;$$

and, by the convolution identity $\frac{C_a}{|x|^{n-a}} = \frac{C_{a/2}}{|x|^{n-a/2}} * \frac{C_{a/2}}{|x|^{n-a/2}}$ for $0 < a < n/2$, the above equation holds for $0 < a < n$. For $t > 0$, let $\phi_t(x) = \phi(x/t)$. Then $\hat{\phi}_t(\xi) = t^n \hat{\phi}(t\xi) = t^n \phi(t\xi)$ and hence

$$\frac{1}{|\xi|^a} * \hat{\phi}_t(\xi) = \frac{1}{|\xi|^a} * t^n \phi(t\xi) \to \frac{1}{|\xi|^a} \tag{2.6}$$

as $t \to +\infty$. Note also that

$$\phi_t(x) \to 1 \tag{2.7}$$

as $t \to +\infty$. We show that

$$\int\left[\int \frac{\phi_t(x-y)}{|x-y|^{n-a}} f(y)dm(y)\right] f(x)dm(x) = \int \widehat{\frac{\phi_t(x)}{|x|^{n-a}}}(\xi)|\hat{f}(\xi)|^2 dm(\xi). \tag{2.8}$$

Then since

$$C_a \widehat{\frac{\phi_t(x)}{|x|^{n-a}}}(\xi) = \hat{\phi}_t(\xi) * \left[\widehat{\frac{C_a}{|x|^{n-a}}}\right](\xi) = t^n\phi(t\xi) * \frac{1}{|\xi|^a} \to \frac{1}{|\xi|^a}$$

as $t \to +\infty$ by (2.6), equations (2.7) and (2.8) prove the lemma.

To prove (2.8), we use Parseval's identity:

$$\int\left[\int \frac{\phi_t(x-y)}{|x-y|^{n-a}} f(y)dm(y)\right] f(x)dm(x) = \int\left[\int \widehat{\frac{\phi_t(x-y)}{|x-y|^{n-a}} f(y)dm(y)}\right](\xi)\overline{\hat{f}(\xi)}dm(\xi)$$

$$= \int \widehat{\frac{\phi_t(x)}{|x|^{n-a}}}(\xi)|\hat{f}(\xi)|^2 dm(\xi).$$

Lemma 2.4. *For $f \in D$ with $\int f dm = 0$,*

$$I(f,f) := \int\int \log\frac{1}{|x-y|} f(x)f(y)dm(x)dm(y) = d_n \int \frac{|\hat{f}(\xi)|^2}{|\xi|^n} dm(\xi)$$

where $d_n := 2[\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})}]$.

Proof. We have

$$C_a \int \frac{f(x)}{|y-x|^{n-a}} dm(x) = C_a \int f(x)\left[\frac{1}{|y-x|^{n-a}} - 1\right] dm(x)$$

$$= (n-a)C_a \int f(x)\left[\frac{\frac{1}{|y-x|^{n-a}} - 1}{n-a}\right] dm(x) \to d_n \int f(x) \log\frac{1}{|x-y|} dm(x) \text{ as } a \to n$$

since $(n-a)C_a \to d_n$ as $a \to n$. Now use Lemma 2.3. Note that the fact that $\hat{f}(0) = 0$ shows that $\frac{|\hat{f}(\xi)|^2}{|\xi|^n}$ is integrable near $\xi = 0$.

Theorem 2.5. *Let ν be a signed measure satisfying*
1. $\int \log(1+|t|)d|\nu|(t) < +\infty$;
2. $\nu(1) = 0$;
3. $I(|\nu|,|\nu|) < +\infty$.

Then $I(\nu,\nu) \geq 0$ with equality if and only if $\nu = 0$.

Proof. Note from 1., 3., and Corollary 2.2 we get $I(\nu,\nu) < +\infty$.

We first give the proof if ν is assumed to have compact support. Let

$\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. Given a standard smoothing kernel $\psi \in D$ ($\psi \geq 0$, $\int \psi(x)dm(x) = 1$, $\psi(x) = 0$ if $|x| \geq 1$), we let $\psi_\epsilon(x) = \frac{1}{\epsilon^n}\psi(x/\epsilon)$ and $\nu_\epsilon = \psi_\epsilon * \nu$. Then $\nu_\epsilon \in D$ and $\int \nu_\epsilon(x)dm(x) = 0$. By Lemma 2.4,

$$I(\nu_\epsilon, \nu_\epsilon) = d_n \int \frac{|\hat{\nu}_\epsilon(\xi)|^2}{|\xi|^n} dm(\xi) \geq 0. \tag{2.9}$$

We show that

$$\lim_{\epsilon \to 0} I(\nu_\epsilon, \nu_\epsilon) = I(\nu, \nu). \tag{2.10}$$

First of all, we note that $p_{\nu^+}(x) := \int \log \frac{1}{|x-y|} d\nu^+(y)$ is superharmonic so that

$$p_{\nu_\epsilon^+}(x) := \int \log \frac{1}{|x-y|} d\nu_\epsilon^+(y) \to p_{\nu^+}(x)$$

pointwise on R^n. Thus, since $\nu_\eta \to \nu$ weak-$*$ as $\eta \to 0$, for each $\epsilon > 0$,

$$\int p_{\nu_\epsilon^+}(x) d\nu_\eta^+(x) \to \int p_{\nu_\epsilon^+}(x) d\nu^+(x).$$

By 3. and dominated convergence, we have

$$\int p_{\nu_\epsilon^+}(x) d\nu_\eta^+(x) \to \int p_{\nu^+}(x) d\nu^+(x)$$

as $\epsilon, \ \eta \to 0$. Similarly,

$$\int p_{\nu_\epsilon^-}(x) d\nu_\eta^-(x) \to \int p_{\nu^-}(x) d\nu^-(x)$$

and

$$\int p_{\nu_\epsilon^+}(x) d\nu_\eta^-(x) \to \int p_{\nu^+}(x) d\nu^-(x)$$

as $\epsilon, \ \eta \to 0$. This proves (2.10). Together with (2.9), this shows that

$$I(\nu, \nu) \geq 0$$

in the case where ν has compact support. To prove the uniqueness assertion in this case, since $\nu, \ \nu_\epsilon$ have compact support, if

$$I(\nu_\epsilon, \nu_\epsilon) = d_n \int \frac{|\hat{\nu}_\epsilon(\xi)|^2}{|\xi|^n} dm(\xi) = d_n \int \frac{|\widehat{\psi_\epsilon * \nu}(\xi)|^2}{|\xi|^n} dm(\xi) \to 0$$

then $\widehat{\psi_\epsilon * \nu} \to 0$ in $L^2(\mathsf{R}^n)$. By Parseval's theorem, it follows that $\psi_\epsilon * \nu \to 0$ in $L^2(\mathsf{R}^n)$ and hence $\nu = 0$.

If ν is not assumed to have compact support, consider

$$\nu_R := \chi_{B(0,R)}\nu - \nu(B(0,R))\sigma$$

where $d\sigma = \frac{dm}{m(B(0,1))}$ denotes normalized Lebesgue measure on the unit ball. Then $\nu_R(1) = 0$ so that from the previous case we have

$$I(\nu_R, \nu_R) \geq 0.$$

Using Corollary 2.2 and hypothesis 2., as $R \nearrow +\infty$,

$$I(\chi_{B(0,R)}\nu, \chi_{B(0,R)}\nu) \to I(\nu, \nu),$$

$$I(\chi_{B(0,R)}\nu, \nu(B(0,R))\sigma) \to I(\nu, \nu(1)\sigma) = 0 \text{ and}$$

$$I(\nu(B(0,R))\sigma, \nu(B(0,R))\sigma) \to I(\nu(1)\sigma, \nu(1)\sigma) = 0;$$

it thus follows that

$$I(\nu_R, \nu_R) \to I(\nu, \nu) \geq 0.$$

Assume now that $I(\nu, \nu) = 0$. Then $I(\nu_R, \nu_R) \to 0$ so that we can find sequences $R_k \nearrow +\infty$ and $\epsilon_k \searrow 0$ with

$$I(\psi_{\epsilon_k} * \nu_{R_k}, \psi_{\epsilon_k} * \nu_{R_k}) \to 0.$$

Since

$$I(\psi_{\epsilon_k} * \nu_{R_k}, \psi_{\epsilon_k} * \nu_{R_k}) = d_n \int \frac{|\widehat{\psi_{\epsilon_k} * \nu_{R_k}}(\xi)|^2}{|\xi|^n} dm(\xi),$$

it follows that $\widehat{\psi_{\epsilon_k} * \nu_{R_k}} \to 0$ in S'. Hence $\psi_{\epsilon_k} * \nu_{R_k} \to 0$ in S' which implies $\nu = 0$.

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