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HAHN DECOMPOSITION FOR THE RIESZ CHARGE OF δ-SUBHARMONIC FUNCTIONS

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We use standard notions of the potential theory. Let w be a δ-subharmonic function: i.e. a function represented as a difference of two subharmonic functions. Such a function is well defined outside a certain polar set; i.e. quasi-everywhere. For simplicity, we consider functions in the whole complex plane.

Defining a continuous function of $t > 0$

$$t \mapsto N_w(t, z) = \frac{1}{2\pi} \int_0^{2\pi} w(z + te^{i\theta})\, d\theta\,,$$

we have quasi-everywhere

$$w(z) = \lim_{t \searrow 0} N_w(t, z)\,.$$

In fact, that relation can be regarded as a definition of the value $w(z)$ if the limit on the right-hand side exists in $[-\infty, \infty]$. Consider the set $\mathscr{E}_+$ of points z such that the function $t \mapsto N_w(t, z)$ has intervals $[r_n(z), R_n(z)]$, $0 < r_n(z) < R_n(z)$, $R_n(z) \searrow 0$ as $n \to \infty$, on which

$$N_w(r_n, z) \le N_w(R_n, z)\,.$$

By $\mu[w]$ we denote the Riesz charge of $w(z)$.

Theorem. *The set $\mathscr{E}_+$ is Borelian, and the restriction $\mu[w]|_{\mathscr{E}_+}$ is a non-negative measure.*

Corollaries.

1. *If $w(z) \ge 0$ quasi-everywhere, then*

$$\mathscr{E}_+ \supset \{z :\ \liminf_{t \searrow 0} N_w(t, z) = 0\} \supset \{z :\ w(z) = 0\}\,,$$

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and we obtain that

$$\mu[w]\big|_{\{z:\, w(z)=0\}} \geq 0\,.$$

This statement is known as Grishin's lemma. It was proved in [6], however weaker versions go back to De la Valléé Poussin [11, p.21] and Brelot [1]. It found a number of applications in the potential theory [2], in the subharmonic approach to value-distribution theory [3, 4], in the theory of orthogonal polynomials [9], in complex dynamics [10]. A different proof with further generalizations was given in [5].

2. *Let* $\mathscr{E}_{-\infty} = \{z :\ \liminf_{t\to 0} N_w(t,z) = -\infty\}$. *Then* $\mathscr{E}_+ \supset \mathscr{E}_{-\infty}$. *Hence*

$$\mu[w]|_{\mathscr{E}_{-\infty}} \geq 0\,.$$

3. *Let*

$$\mathscr{E}^* = \{z :\ \liminf_{t\to 0} N_w(t,z) < \limsup_{t\to 0} N_w(t,z)\}$$

be the polar set of undeterminancy of w. *Then* $\mathscr{E}_+ \supset \mathscr{E}^*$ *and* $\mu[w]|_{\mathscr{E}^*} \geq 0$. *Replacing* w *by* $-w$, *we conclude that*

$$|\mu[w]|(\mathscr{E}^*) = 0\,.$$

Corollaries 2 and 3 answer questions posed by Eremenko. Another proof of these two corollaries based on the fine potential theory was given by Fuglede.

4. *Define a "dual" set* $\mathscr{E}_-$ *of points* z *such that for certain intervals* $[r_n(z), R_n(z)]$, $0 < r_n(z) < R_n(z)$, $R_n(z) \searrow 0$ *as* $n \to \infty$,

$$N_w(r_n, z) \geq N_w(R_n, z)\,.$$

Then the domain of the function $w(z)$ *splits into three mutually disjoint sets* $E_0 = \mathscr{E}_+ \cap \mathscr{E}_-, E_+ = \mathscr{E}_+ \backslash \mathscr{E}_-$, $E_- = \mathscr{E}_- \backslash \mathscr{E}_+$ *and*

$$\mu[w]|_{E_+} \geq 0\,, \quad \mu[w]|_{E_-} \leq 0\,,$$

whilst

$$|\mu[w]|(E_0) = 0\,.$$

Here, E_+ is the set of points z such that the function $t \mapsto N_w(t,z)$ increases strictly for $0 < t < t_0(z)$, and E_0 is the set of points z such that the function $t \mapsto N_w(t,z)$ decreases strictly for $0 < t < t_0(z)$.

This is the version of the Hahn decomposition of $\mu[w]$ promised in the title of this paper.

Proof of the Theorem.

We repeat almost verbatim Grishin's original arguments. First, we prove that $\mathscr{E}_+$ is a Borelian set. Since the function $z \mapsto N_w(t,z)$ is continuous, the sets $\mathscr{E}(r,R) = \{z : N_w(R,z) \ge N_w(r,z)\}$ are closed. In the definition of $\mathscr{E}_+$ we may assume without loss of generality that the endpoints r_n, R_n are rational numbers. Therefore,

$$\mathscr{E}_+ = \bigcap_{N=1}^{\infty} \bigcup_{\{m,n:\frac{m}{n}<\frac{1}{N}\}} \bigcup_{\{p,q:\frac{p}{q}<\frac{m}{n}\}} \mathscr{E}\left(\frac{p}{q},\frac{m}{n}\right)$$

is a Borelian set. Now, let $z \in \mathscr{E}_+$. Then by the Jensen formula

$$\int_{r_n}^{R_n} \frac{\mu[w](D(z,t))}{t} = N_w(R_n,z) - N_w(r_n,z) \ge 0\,,$$

here, $D(z,t) = \{\zeta : |\zeta - z| < t\}$. Hence, for each $z \in \mathscr{E}_+$, there is a sequence of shrinking discs $D(z,t_n(z))$, $t_n(z) \searrow 0$, such that $\mu[w](D(z,t_n(z))) \ge 0$. It remains to apply the following

Claim. *Let E be a Borelian set, and let ν be a charge on the complex plane. Suppose that for each $z \in E$ there is a sequence $t_n(z) \searrow 0$ such that*

$$\nu(D(z,t_n(z))) \ge 0\,.$$

Then the restriction $\nu\big|_E$ is a non-negative measure.

Proof of the Claim.

Let $F \subset E$ be an arbitrary compact subset. We prove that $\nu(F) \ge 0$ which implies the statement.

First, we choose a decreasing sequence of open sets O_j such that $F = \bigcap_j O_j$ and

$$\lim_j |\nu|(O_j \backslash F) = 0\,.$$

Fixing j, we assume that all radii $t_n(z)$, $z \in F$, are less than the distance from F to O_j; i.e. all discs $D(z,t_n(z))$ are contained in O_j for $z \in F$.

Now, the generalized Vitali theorem [7, 2.8] gives us a sequence of mutually disjoint discs D_n (depending on j) from the whole collection $\{D(z,t_n(z))\}$, $z \in F$, $n \in \mathsf{N}$, such that for $G_j = \bigcup_n D_n$

$$|\nu|(F \backslash G_j) = 0\,.$$

Thus $\nu(G_j) = \sum_n \nu(D_n) \geq 0$, and

$$\begin{aligned}\nu(F) &= \lim_j \nu(O_j) \\ &= \lim_j \big[\nu(O_j \backslash (G_j \cup (F \backslash G_j))) + \nu(G_j) + \nu(F \backslash G_j)\big] \geq 0\end{aligned}$$

because $|\nu|\big(O_j \setminus (G_j \cup (F \setminus G_j))\big) \leq |\nu|(O_j \setminus F) \to 0$. We are done.

Remarks.

1. The assertion of the theorem can be slightly strengthened using the "Phragmén – Lindelöf trick". Let us fix an increasing C^1-function $h(t)$, $0 \leq t < 1$, $h(0) = 0$, such that

$$h'(t) = o(t) \quad \text{as} \quad t \to 0\,.$$

Then we define a set $\mathscr{E}_+^{(h)}$ making use of the function $t \mapsto N_w(t,z) + h(t)$ instead of $N_w(t,z)$. Evidently, $\mathscr{E}_+^{(h)} \supset \mathscr{E}_+$. And we still have

$$\mu[w]\Big|_{\mathscr{E}_+^{(h)}} \geq 0\,.$$

To prove this we replace $\mu[w]$ by $\mu_\varepsilon = \mu[w] + \varepsilon m$, where m is the planar-area measure, and $\varepsilon > 0$. Then, for $z \in \mathscr{E}_+^{(h)}$ and large enough n, we have

$$\begin{aligned}&\int_{r_n}^{R_n} \frac{\mu_\varepsilon(D(z,t))}{t}\, dt \\ &= N_w(R_n,z) - N_w(r_n,z) + \frac{\pi}{2}\varepsilon(R_n^2 - r_n^2) \\ &\geq [N_w(R_n,z) + h(R_n)] - [N_w(r_n,z) + h(r_n)] \geq 0\,.\end{aligned}$$

(in the first inequality we have used the estimate

$$h(R_n) - h(r_n) = \int_{r_n}^{R_n} h'(t)\, dt \leq \pi\varepsilon \int_{r_n}^{R_n} t\, dt = \frac{\pi\varepsilon}{2}[R_n^2 - r_n^2]$$

if n is large enough).

Now, using the Claim we conclude that $\mu_\varepsilon|_{\mathscr{E}_+^{(h)}} \geq 0$ and since ε is an arbitrary number $\mu[w]|_{\mathscr{E}_+^{(h)}} \geq 0$.

2. One may go a little bit further in this direction. Let E be a Borelian set. Suppose that for some $s > 0$ there is a non-negative measure λ_s such that for each $z \in E$ and each t, $0 < t < t_0(z)$,

$$\lambda_s(D(z,t)) \geq t^s\,.$$

The existence of such a measure is provided by the so-called "anti Frostman lemma" proved in [8, Lemma 4] for

$$s > \Delta(E) = \limsup_{\varepsilon \to 0} \frac{\log M_\epsilon(E)}{\log 1/\varepsilon},$$

where $M_\varepsilon(E)$ is the smallest number of discs of radius ε required to cover E. The value $\Delta(E)$ is called the upper Minkowski dimension or the box counting dimension. In general, it is bigger than or equal the Hausdorff dimension, however for many "reasonable" sets these two dimensions coincide. For the discussion of this concept see [7, Chapter 5] and [8]. Ifsuch λ_s exists, we can consider the intersection $\mathscr{E}_+^{(h)} \cap E$ where $\mathscr{E}_+^{(h)}$is defined with an increasing $h(t)$, $h(0) = 0$, which is C^1 and such that

$$h'(t) = o(t^{s-1}), \quad t \to 0.$$

The same argument as above with $\mu_\varepsilon = \mu[w] + \varepsilon\lambda_s$, $\varepsilon > 0$, shows that

$$\mu[w]\Big|_{\mathscr{E}_+^{(h)} \cap E} \geq 0.$$

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