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HARMONIC MEASURE AND HYPERBOLIC DISTANCE IN JOHN DISKS

KIWON KIM and NAVAH LANGMEYER

1. Introduction.

Suppose that D is a domain in the complex plane C. Let $D^* = \mathsf{C} \setminus \bar{D}$ be the exterior of D in C and let $B(z, r) = \{\zeta : |\zeta - z| < r\}$ for $z \in \mathsf{C}$ and $r > 0$.

In this paper, we find several characterizations of John disks which have analogues in the class of quasidisks. John disks can be thought of as "one-sided quasidisks". For example, a Jordan domain $D \subset \mathsf{C}$ is a quasidisk if and only if D and D^* are John disks. Also, every quasidisk is a John disk [GM3]. The results presented here are likewise one-sided versions of characterizations of quasidisks. These characterizations involve the conformal invariants harmonic measure and hyperbolic distance.

A simply-connected bounded domain $D \subset \mathsf{C}$ is said to be a *c-John disk* if there exist a point $z_0 \in D$ and a constant $c \geq 1$ such that each point $z_1 \in D$ can be joined to z_0 by an arc γ in D satisfying

$$\ell(\gamma(z_1, z)) \leq c\,(z, \partial D)$$

for each $z \in \gamma$, where $\ell(\gamma(z_1, z))$ is the euclidean length of the subarc of γ with endpoints z_1, z. We call z_0 a *John center*, c a *John constant* and γ a *c-John arc*. We say that D is *John* if it is c-John disk for some c.

A bounded domain $D \subset \mathsf{C}$ is John if and only if each pair of points $z_1, z_2 \in D$ can be joined by an arc γ which satisfies

$$\min_{j=1,2} \ell(\gamma(z_j, z)) \leq c\,(z, \partial D) \tag{1.1}$$

for all $z \in \gamma$. We call γ a *double c-cone arc*. This definition can be used to define the unbounded John disks $D \subset \mathsf{C}$ as well [NV, 2.26].

A domain $D \subset \mathsf{C}$ is said to be *c-uniform* if there is a constant $c \geq 1$ such

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that each pair of points $z_1, z_2 \in D$ can be joined by an arc $\gamma \subset D$ which satisfies (1.1) and

$$\ell(\gamma) \leq c\,|z_1 - z_2|.$$

We say that D is *uniform* if it is c-uniform for some $c \geq 1$.

We say that a domain $D \subset \mathsf{C}$ is a *K-quasidisk*, $1 \leq K < \infty$, if it is the image of the unit disk B under a K-quasiconformal self mapping of $\bar{\mathsf{C}} = \mathsf{C} \cup \infty$. A Jordan domain $D \subset \mathsf{C}$ is uniform if and only if it is a quasidisk [MS].

In section 2, we show that a bounded Jordan domain $D \subset \mathsf{C}$ satisfies a harmonic doubling condition if and only if D is a John disk. This is a one-sided analogue of a characterization for quasidisks due to Jerison and Kenig [JK]. It is also a one-sided version of a characterization for quasidisks due to Krzyż who compares the harmonic measures of adjacent arcs on the boundary when considered from inside and outside the domain [Kr].

In section 3, we characterize John disks D in terms of various properties of the hyperbolic geodesics in D; in particular, the position of the euclidean midpoint of the geodesic or the quasiextremal distance property in D with respect to the geodesic. The first of these leads to a third characterization in terms of the Hölder continuity of analytic functions in D similar to a well-known theorem of Hardy and Littlewood [HL]. Finally, we characterize unbounded Jordan John disks in terms of the hyperbolic geodesics in their exteriors.

In section 4, we characterize John disks in terms of a euclidean estimate for the hyperbolic distance between points of D. This is again a one-sided analogue of a theorem due to Gehring and Osgood [GO], who showed that a domain D is uniform if and only if it satisfies

$$k_D(z_1, z_2) \leq c\, j_D(z_1, z_2) + d$$

for all $z_1, z_2 \in D$ and some constants c and d, where k_D is the quasihyperbolic metric in D and

$$j_D(z_1, z_2) = \tfrac{1}{2}\log\left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)} + 1\right)\left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)} + 1\right). \tag{1.2}$$

Our result replaces the euclidean distance $|z_1 - z_2|$ with the inner distance between these points and yields an analogous estimate for $h_D(z_1, z_2)$, the hyperbolic distance between z_1 and z_2.

We will repeatedly use a result of Gehring and Hayman.

LEMMA 1.3 [GH, Theorem 2], [Ja]. *Suppose that D is a simply connected domain in C. If γ is a hyperbolic geodesic in D and if α is any curve which joins the endpoints of γ in D, then*

$$\ell(\gamma) \le k\,\ell(\alpha),$$

where k is an absolute constant, $4.5 \le k \le 17.5$.

2. Harmonic measure in John disks.

A bounded Jordan domain $D \subset \mathbf{C}$ is said to satisfy a *harmonic doubling condition* if for some $z_0 \in D$ and some constant $c_0 > 0$,

$$\omega(z_0, \alpha; D) \le c_0\, \omega(z_0, \beta; D) \tag{2.1}$$

for each pair of consecutive arcs α, β on ∂D with $\operatorname{dia}(\alpha) \le 2\operatorname{dia}(\beta)$, where $\omega(z_0, \gamma; D)$ is the harmonic measure of γ at the point z_0 with respect to D.

REMARK 2.2. If D satisfies (2.1) for some $z_0 \in D$, then it satisfies (2.1) for every $z_1 \in D$ with a constant c_1 which depends on c_0, z_0 and z_1.

PROOF. Fix $z_1 \in D$ and fix consecutive arcs $\alpha, \beta \subset \partial D$ with $\operatorname{dia}(\alpha) \le 2\operatorname{dia}(\beta)$. Since ω is nonnegative and harmonic,

$$\frac{\omega(z_1, \alpha; D)}{\omega(z_0, \alpha; D)} \le k \qquad \text{and} \qquad \frac{\omega(z_0, \beta; D)}{\omega(z_1, \beta; D)} \le k\,,$$

where $k = e^{h_D(z_0, z_1)}$. (See, for example, [H, Theorem 6].) Thus by hypothesis we have

$$\frac{\omega(z_1, \alpha; D)}{\omega(z_1, \beta; D)} \le \frac{\omega(z_0, \alpha; D)}{\omega(z_0, \beta; D)} k^2 \le c_0 k^2 = c_1$$

and hence (2.1) holds for every $z_1 \in D$ with $c_1 = c_0\,(e^{h_D(z_0, z_1)})^2$.

THEOREM 2.3. *A bounded Jordan domain $D \subset \mathbf{C}$ is a c-John disk if and only if it satisfies a harmonic doubling condition.*

To prove Theorem 2.3 we need a lemma.

LEMMA 2.4. *Suppose that D is a bounded Jordan domain in $\mathbf{C}$ and let $z_0 \in D$. Then the following conditions are equivalent, where the constants in each condition need not be the same but depend on each other:*

(1) *D is a c-John disk.*

(2) *There exist constants c and* $\delta > 0$ *such that*

$$\frac{\operatorname{dia}(\alpha_1)}{\operatorname{dia}(\alpha)} \le c \left(\frac{\omega(z_0, \alpha_1; D)}{\omega(z_0, \alpha; D)} \right)^{\delta} \tag{2.5}$$

for all arcs $\alpha_1 \subset \alpha \subset \partial D$.

(3) *There exists a constant* $c > 1$ *such that*

$$\omega(z_0, \alpha; D) \le c\, \omega(z_0, \alpha_1; D) \tag{2.6}$$

for all arcs $\alpha_1 \subset \alpha \subset \partial D$ *with* $\mathrm{dia}(\alpha) \le 2\,\mathrm{dia}(\alpha_1)$.

PROOF. The equivalence of (1) and (2) is proved in [P, Theorem 1]. To prove the equivalence of (2) and (3), we first assume that (2) holds and let $\alpha_1 \subset \alpha$ be arcs on ∂D with $\mathrm{dia}(\alpha) \le 2\,\mathrm{dia}(\alpha_1)$. Then

$$\frac{\omega(z_0, \alpha_1; D)}{\omega(z_0, \alpha; D)} \ge c^{-\frac{1}{\delta}} \left(\frac{\mathrm{dia}(\alpha_1)}{\mathrm{dia}(\alpha)} \right)^{\frac{1}{\delta}} \ge (2c)^{-\frac{1}{\delta}}$$

and hence we have (2.6) with a constant $(2c)^{\frac{1}{\delta}}$. Next suppose that (3) holds. Then by induction it is not difficult to show that

$$\omega(z_0, \alpha; D) \le c^n\, \omega(z_0, \alpha_1; D) \tag{2.7}$$

for all arcs $\alpha_1 \subset \alpha \subset \partial D$ with $\mathrm{dia}(\alpha) \le 2^n \mathrm{dia}(\alpha_1)$ and for each integer $n > 0$.

Now given any arcs $\alpha_1 \subset \alpha \subset \partial D$, there exists an integer $n > 0$ such that

$$2^{n-1} \mathrm{dia}(\alpha_1) \le \mathrm{dia}(\alpha) \le 2^n \mathrm{dia}(\alpha_1). \tag{2.8}$$

Then by (2.7) we have

$$\omega(z_0, \alpha; D) \le c^n \omega(z_0, \alpha_1; D). \tag{2.9}$$

Let $\delta = \frac{\log 2}{\log c}$. Then by (2.8) and (2.9) we obtain

$$\frac{\omega(z_0, \alpha; D)}{\omega(z_0, \alpha_1; D)} \le c^n = c\left(2^{\frac{1}{\delta}}\right)^{n-1} = c(2^{n-1})^{\frac{1}{\delta}} \le c\left(\frac{\mathrm{dia}(\alpha)}{\mathrm{dia}(\alpha_1)} \right)^{\frac{1}{\delta}}.$$

Hence we get (2.5) with a constant c^δ.

PROOF OF THEOREM 2.3. For the necessity suppose that a harmonic doubling condition does not hold for D. Then for $j = 1, 2, \ldots$ there are consecutive arcs α_j, β_j on ∂D such that

$$\mathrm{dia}(\alpha_j) \le 2\,\mathrm{dia}(\beta_j) \qquad \text{and} \qquad \omega(z_0, \alpha_j; D) \ge 3^j\, \omega(z_0, \beta_j; D). \tag{2.10}$$

Thus $\mathrm{dia}(\alpha_j \cup \beta_j) \le 3\,\mathrm{dia}(\beta_j)$ and hence by Lemma 2.4 (2) and by (2.10)

$$\tfrac{1}{3} \le \frac{\mathrm{dia}(\beta_j)}{\mathrm{dia}(\alpha_j \cup \beta_j)} \le c\left(\frac{\omega(z_0, \beta_j; D)}{\omega(z_0, \alpha_j \cup \beta_j; D)} \right)^\delta \le c\left(\frac{\omega(z_0, \beta_j; D)}{\omega(z_0, \alpha_j; D)} \right)^\delta \le c\,(3^{-j})^\delta$$

which yields a contradiction as $j \to \infty$.

For the sufficiency, by Lemma 2.4 it suffices to show that D satisfies (2.6). Let $\alpha_1 \subset \alpha$ be arcs of ∂D with $\mathrm{dia}(\alpha) \le 2\,\mathrm{dia}(\alpha_1)$.

Suppose first that α_1, α have a common endpoint. Then $\mathrm{dia}(\alpha \setminus \alpha_1) \le 2\,\mathrm{dia}(\alpha_1)$ and hence by (2.1), $\omega(z_0, \alpha \setminus \alpha_1; D) \le c_0\, \omega(z_0, \alpha_1; D)$ for some $z_0 \in D$. Thus

(2.11) $$\omega(z_0, \alpha; D) \le (c_0 + 1)\,\omega(z_0, \alpha_1; D).$$

Next suppose that $\alpha \setminus \alpha_1$ consists of two disjoint subarcs α_2, α_3. Then for $j = 2, 3$ $\operatorname{dia}(\alpha_1 \cup \alpha_j) \le 2\operatorname{dia}(\alpha_1)$ and hence $\omega(z_0, \alpha_1 \cup \alpha_j; D) \le (c_0 + 1)$ $\omega(z_0, \alpha_1; D)$ by what was proved above. Thus

(2.12) $$\omega(z_0, \alpha; D) \le 2(c_0 + 1)\,\omega(z_0, \alpha_1; D).$$

Therefore by (2.11) and (2.12) D satisfies (2.6) with $c = 2(c_0 + 1)$.

3. Hyperbolic geodesics in John disks.

We say that a domain $D \subset \mathbf{C}$ is a *M-quasiextremal distance or M-*QED *domain with respect to $E \subset D$*, $1 \le M < \infty$, if for each pair of disjoint continua $F_1, F_2 \subset E$

(3.1) $$\operatorname{mod}(\Gamma) \le M \operatorname{mod}(\Gamma_D),$$

where Γ and Γ_D are the families of curves joining F_1 and F_2 in $\mathbf{C}$ and in D, respectively.

THEOREM 3.2. *Suppose that D is a bounded simply connected domain in* $\mathbf{C}$. *Then the followings are equivalent:*

(1) *D is a c-John disk.*

(2) *There exists a constant $c > 0$ such that for each hyperbolic geodesic $\gamma \subset D$*

(3.3) $$\ell(\gamma) \le c\,(z_0, \partial D),$$

where z_0 is the euclidean midpoint of γ.

(3) *There exists a constant $c > 0$ such that if f is analytic with*

(3.4) $$|f'(z)| \le 1$$

in D, then for all $z_1, z_2 \in D$

(3.5) $$|f(z_1) - f(z_2)| \le c \operatorname{dist}(z_0, \partial D),$$

where z_0 is the euclidean midpoint of the hyperbolic geodesic $\gamma \subset D$ joining z_1 to z_2.

(4) *D is a M-*QED *domain with respect to all hyperbolic geodesics in D with a given point $z_0 \in D$ as an endpoint.*

Here the constants in each condition need not be the same but depend on each other. In particular, from (4) *we obtain a John constant c in* (1), *which depends on M and a given point z_0.*

PROOF OF EQUIVALENCE OF (1) AND (2). Let $D \subset \mathbf{C}$ be a bounded c-John disk and let z_0 be the euclidean midpoint of a hyperbolic geodesic γ with endpoints z_1 and z_2 in D. By [GHM, 2.16 Lemma], there exists a crosscut α of D containing z_0 which separates the components of $\gamma \setminus \{z_0\}$ in D and

$$\ell(\alpha) \le c_1 \operatorname{dist}(z_0, \partial D), \tag{3.6}$$

where c_1 is an absolute constant. Next since D is a c-John disk, there exists a John center x_0, a c-John arc β_1 from z_1 to x_0, and a c-John arc β_2 from z_2 to x_0.

If x_0 is in the component of $D \setminus \alpha$ which contains z_2, then by (3.6) there exists a point w in $\alpha \cap \beta_1$ such that

$$\ell(\beta_1(z_1, w)) \le c \operatorname{dist}(w, \partial D) \le c\,\ell(\alpha) \le cc_1 \operatorname{dist}(z_0, \partial D).$$

Since $\beta_1(z_1, w) \cup \alpha(z_0, w)$ is a curve which joins z_1 to z_0 in D and since $\gamma(z_1, z_0)$ is a hyperbolic geodesic in D with z_1 and z_0 as its end points, Lemma 1.3 and (3.6) imply that

$$\ell(\gamma) = 2\,\ell(\gamma(z_1, z_0)) \le 2k\,(\ell(\beta_1(z_1, w)) + \ell(\alpha(z_0, w))) = 2kc_1(c+1) \operatorname{dist}(z_0, \partial D)$$

where k is an absolute constant. If x_0 is in the component of $D \setminus \alpha$ which contains z_1, then the above argument applied to the arc β_2 yields the desired inequality. Finally if $x_0 \in \alpha$, then by Lemma 1.3 and (3.6),

$$\begin{aligned} \ell(\gamma) &= 2\,\ell(\gamma(z_1, z_0)) \le 2k\,\big(\ell(\beta_1) + \ell(\alpha(x_0, z_0))\big) \\ &\le 2k(c \operatorname{dist}(x_0, \partial D) + c_1 \operatorname{dist}(z_0, \partial D)). \end{aligned}$$

Since α joins x_0 to ∂D,

$$\ell(\gamma) \le 2k(c\,\ell(\alpha) + c_1 \operatorname{dist}(z_0, \partial D)) \le 2kc_1(c+1) \operatorname{dist}(z_0, \partial D)\,.$$

Conversely, suppose that (2) holds and let $L = \sup \ell(\gamma)$, where the supremum is taken over all possible hyperbolic geodesics γ with endpoints in D. Then there exist two points z_1, $z_2 \in D$ such that $\ell(\gamma) = \frac{L}{2}$, where γ is the hyperbolic geodesic joining z_1 to z_2 in D. Let z_0 be the euclidean midpoint of γ. Then by (3.3),

$$\operatorname{dia}(D) \ge \operatorname{dist}(z_0, \partial D) \ge \frac{1}{c}\,\ell(\gamma) = \frac{1}{2c}\,L. \tag{3.7}$$

Now fix a point $z \in D$ and let w_0 be the euclidean midpoint of the hyperbolic geodesic α joining z to z_0 in D. If $x \in \alpha(w_0, z)$, then we can find a point $x_1 \in \alpha(x, z_0)$ with $\ell(\alpha(z, x)) = \ell(\alpha(x, x_1))$ and by (3.3) applied to $\alpha(z, x_1)$,

$$\ell(\alpha(z, x)) = \tfrac{1}{2}\,\ell(\alpha(z, x_1)) \le \frac{c}{2} \operatorname{dist}(x, \partial D). \tag{3.8}$$

If $x \in \alpha(z_0, w_0)$, then we can find a point $x_2 \in \alpha(x, z)$ with $\ell(\alpha(x, z_0)) = \ell(\alpha(x_2, x))$. Then again by (3.3) applied to $\alpha(z_0, x_2)$ and by (3.7),

$$\ell(\alpha(z,x)) \le L \le 2c\,\mathrm{dist}(z_0,\partial D) \le 2c\left(\ell(\alpha(x_2,z_0)) + \mathrm{dist}(x,\partial D)\right) \\ \le 2c(c+1)\,\mathrm{dist}(x,\partial D). \tag{3.9}$$

Hence by (3.8) and (3.9) D is a c_1-John disk with $c_1 = 2c(c+1)$.

Proof of equivalence of (2) and (3). First suppose that D satisfies (2). Then D is a b-John disk, where b depends only on c. Let f be analytic and satisfy

$$|f'(z)| \le \mathrm{dist}(z,\partial D)^{\alpha-1} \tag{3.10}$$

for some $0 < \alpha \le 1$ in D. Fix z_1, $z_2 \in D$, and let γ be the hyperbolic geodesic joining z_1 to z_2 in D. Next let s denote arclength measured along γ from z_1, let $z(s)$ denote the corresponding representation for γ, and set $g(s) = f(z(s))$. Then

$$|g'(s)| = |f'(z(s))|$$

while

$$\min(s,\, l-s) \le b_1\,\mathrm{dist}(z(s),\partial D)\ , \qquad l = \ell(\gamma)\,,$$

where $b_1 \ge 1$ is a constant depending only on b, by [GHM, Theorem 4.1]. Thus

$$|g'(s)| \le \mathrm{dist}(z(s),\partial D)^{\alpha-1} \le \left(\frac{\min(s,\, l-s)}{b_1}\right)^{\alpha-1}$$

for $0 < s < l$, and hence

$$\begin{aligned} |f(z_1) - f(z_2)| &= |g(l) - g(0)| \\ &\le \int_0^l |g'(s)|\,ds \le 2{b_1}^{1-\alpha}\int_0^{\frac{l}{2}} s^{\alpha-1}\,ds \\ &= \frac{2{b_1}^{1-\alpha}}{\alpha}\left(\frac{l}{2}\right)^{\alpha} \le \frac{c_1}{\alpha}\mathrm{dist}(z_0,\partial D)^{\alpha}, \end{aligned} \tag{3.11}$$

where $c_1 = b_1\,c$. If f satisfies (3.4) in D, then f satisfies (3.10) with $\alpha = 1$. Hence, f satisfies (3.11) with $\alpha = 1$, i.e. f satisfies (3.5).

Now suppose that (3.5) holds for any analytic function f on D which satisfies (3.4). By [KW, Theorem 1] with $k = 1$, for z_1, $z_2 \in D$

$$\inf_{\beta}\int_{\beta} |d\zeta| \le c_1 \sup_{f} |f(z_1) - f(z_2)|,$$

where the infimum is taken over all Jordan arcs β in D joining z_1 to z_2, c_1 is an absolute constant, and the supremum is taken over all analytic functions f on D with $|f'(z)| \le 1$. Thus by Lemma 1.3,

$$\ell(\gamma) \le k \inf_{\beta} \ell(\beta) \le k\, c_1 \sup_{f} |f(z_1) - f(z_2)| \le k\, c_1\, c\, \mathrm{dist}(z_0, \partial D)$$

for an absolute constant k, where γ is the hyperbolic geodesic joining z_1 to z_2 in D and z_0 is the euclidean midpoint of γ.

Remark 3.12. Note that this proof shows that if (3.4) implies (3.5), then D satisfies (2) and hence

$$|f'(z)| \le \mathrm{dist}(z, \partial D)^{\alpha - 1}$$

in D implies

$$|f(z_1) - f(z_2)| \le \frac{c}{\alpha}\, \mathrm{dist}(z_0, \partial D)^{\alpha}$$

for any $0 < \alpha \le 1$.

In order to prove the equivalence of (1) and (4), we need a lemma which shows that each hyperbolic line in D which joins two points on ∂D lies in the middle of D. See [R, Lemma 4.13] and [PR, Theorem 3.3].

Lemma 3.13. *Suppose that D is a simply connected proper subdomain in* C *and that $\gamma \subset D$ is a hyperbolic line joining $w_1, w_2 \in \partial D$ and dividing D into disjoint subdomains D_1 and D_2. Then*

$$\frac{1}{b} \le \frac{\mathrm{dist}(z, \alpha_1)}{\mathrm{dist}(z, \alpha_2)} \le b, \qquad b = 3 + 2\sqrt{2}$$

for all $z \in \gamma$, where $\alpha_j = \partial D_j \setminus \overline{\gamma}$, $j = 1, 2$.

Proof of equivalence of (1) and (4). Suppose that $D \subset \mathsf{C}$ is a bounded c-John disk with fixed John center z_0. Fix $z_1 \in D$ and let γ be the hyperbolic geodesic joining z_0 to z_1 in D. Fix two disjoint continua F_1, F_2 of γ. Then by [K, Theorem 2.1] and the construction on [GO, pp. 67-68], there is a K-quasidisk G_1 in D such that $\gamma \in \overline{G_1}$, where K depends only on c. Thus by [GM3, Remark 2.23], G_1 is M-QED with respect to G_1 for some constant M, $1 \le M < \infty$, which depends only on K, and hence only on c. Therefore, since $\Gamma_{G_1} \subset \Gamma_D$,

$$\mathrm{mod}(\Gamma) \le M\, \mathrm{mod}(\Gamma_{G_1}) \le M\, \mathrm{mod}(\Gamma_D),$$

where Γ, Γ_{G_1}, Γ_D are the families of curves which join F_1 and F_2 in C, G_1, D, respectively.

Suppose next that z_0 is a point in D and that D is M-QED with respect to all hyperbolic geodesics in D which have z_0 as an endpoint. Fix $z_1 \in D$, $z_1 \neq z_0$ and let γ be the hyperbolic geodesic joining z_0 to z_1 in D.

We show first that for some constant $a > 1$ and for all $z \in \gamma$

$$\min(|z_0 - z|, |z - z_1|) \leq a \operatorname{dist}(z, \partial D). \tag{3.14}$$

Suppose otherwise. Then for each constant $a > 1$, there is a point $z \in \gamma$ such that $\min(|z_0 - z|, |z - z_1|) > a \operatorname{dist}(z, \partial D)$. Fix a constant $a > 1$ and let $b = 3 + 2\sqrt{2}$. Then for a constant $ab > 1$ there is a point $z \in \gamma$ such that

$$\min(|z_0 - z|, |z - z_1|) > ab \operatorname{dist}(z, \partial D).$$

Consider the hyperbolic line in D which contains γ and which has the endpoints $w_1, w_2 \in \partial D$ and let α_1, α_2 be as described in Lemma 3.13. Then $\operatorname{dist}(z, \partial D) = \min_{j=1,2} \operatorname{dist}(z, \alpha_j)$. Thus we may assume that $\operatorname{dist}(z, \partial D) = \operatorname{dist}(z, \alpha_1)$ and hence by Lemma 3.13

$$\operatorname{dist}(z, \alpha_2) \leq b \operatorname{dist}(z, \partial D). \tag{3.15}$$

Let $r = b \operatorname{dist}(z, \partial D)$. By means of a preliminary similarity mapping we may assume that $z = 0$. Then $z_0, z_1 \notin B(0, ar)$. Let $A = B(0, ar) \setminus \overline{B}(0, \sqrt{a}r)$. For $j = 0, 1$, let F_j denote a component of $A \cap \gamma(0, z_j)$ which joins the two boundary circles of A. Then by [V, Theorem 10.12],

$$\operatorname{mod}(\Gamma) \geq \operatorname{mod}(\Gamma_A) = \frac{2}{\pi} \log \sqrt{a}, \tag{3.16}$$

where Γ, Γ_A are the families of curves joining F_0 and F_1 in C and in A, respectively. Now let $B = B(0, \sqrt{a}r) \setminus \overline{B}(0, r), E = \partial B(0, r)$, and $F = \partial B(0, \sqrt{a}r)$. Then by (3.15), Γ_D is minorized by Γ_B and hence by [V, 7.5]

$$\operatorname{mod}(\Gamma_D) \leq \operatorname{mod}(\Gamma_B) = 2\pi \left(\log \frac{\sqrt{a}r}{r} \right)^{-1} = \frac{2\pi}{\log \sqrt{a}}, \tag{3.17}$$

where Γ_B is the family of curves joining E and F in B and Γ_D is the family of curves joining F_0 and F_1 in D. Then by the hypothesis, (3.16) and (3.17)

$$\frac{2}{\pi} \log \sqrt{a} \leq \operatorname{mod}(\Gamma) \leq M \operatorname{mod}(\Gamma_D) \leq \frac{2\pi M}{\log \sqrt{a}}$$

and hence

$$M \geq \left(\frac{\log \sqrt{a}}{\pi} \right)^2.$$

This holds for each constant $a > 1$ and it leads a contradiction, which establishes (3.14).

Next to show that D is a c-John disk, by [NV, Lemma 2.10] we need to prove that for some constant $c \geq 1$ and for all $z \in \gamma$

$$|z - z_1| < c\,\mathrm{dist}(z, \partial D). \tag{3.18}$$

For this let $L = \max\{|z_0 - z| : z \in \partial D\}$, $k = \frac{L}{\mathrm{dist}(z_0, \partial D)}$ and $c_1 = \max(a, k)$. If $|z - z_1| < |z - z_0|$, then by (3.14)

$$|z - z_1| < a\,\mathrm{dist}(z, \partial D). \tag{3.19}$$

If $|z - z_1| > |z - z_0|$, then $|z_0 - z_1| \leq L$ and (3.14) give

$$\begin{aligned}\frac{|z - z_1|}{c_1} &< \frac{|z - z_0|}{a} + \frac{|z_0 - z_1|}{k} < \mathrm{dist}(z, \partial D) + \mathrm{dist}(z_0, \partial D)\\ &\leq |z - z_0| + 2\,\mathrm{dist}(z, \partial D) < (a + 2)\,\mathrm{dist}(z, \partial D).\end{aligned}$$

Hence

$$|z - z_1| < c_1(a + 2)\,\mathrm{dist}(z, \partial D). \tag{3.20}$$

Therefore by (3.19) and (3.20) we obtain (3.18) with $c = c_1(a + 2)$, which depends on M and z_0.

Note that in the proof of equivalence of (1) and (4) in Theorem 3.2, what we get from (4) is the John condition on all hyperbolic geodesics with a given point z_0 as an end point and a fixed constant $c = c(z_0, M)$. If c were independent of z_0, we are in the uniform domain case as follows.

Corollary 3.21. *Suppose that D is a bounded finitely connected domain in* $\mathbf{C}$. *Then D is c-uniform if and only if D is a M-*QED *domain with respect to all hyperbolic geodesics in D. Here c and M depend only on each other.*

Proof. Suppose that D is c-uniform. Then by [GM3, Theorem 2.22], D is a M-QED domain with respect to D, $M = M(c)$, and hence with respect to all hyperbolic geodesics in D. For the sufficiency, let z_1, z_2 be two disjoint points in D and let γ be the hyperbolic geodesic in D with endpoints z_1, z_2. Then by an argument similar to that for the proof of (3.14)

$$\min(|z_1 - z|, |z - z_2|) \leq a\,\mathrm{dist}(z, \partial D) \tag{3.22}$$

for all $z \in \gamma$ and for some constant $a > 1$. Also by the same argument as the proof of [GM3, Lemma 2.7]

$$\ell(\gamma) \leq k\,|z_1 - z_2|, \tag{3.23}$$

where k, $1 < k < \infty$, is a constant depending only on M. Therefore [NV, Theorem 2.16], (3.22) and (3.23) imply that D is c-uniform with $c = \max(a, k)$, which depends only on M.

Next we characterize unbounded Jordan John disks with $\infty \in \partial D$ in terms of the hyperbolic geodesics in their exteriors.

LEMMA 3.24 [GHM], [NV], [R]. *A Jordan domain $D \subset \mathbf{C}$ is a c-John disk if and only if each pair of points z_1, $z_2 \in D^*$ can be joined by a continuum $E \subset D^*$ with*

$$\operatorname{dia}(E) \leq c_1 \, |z_1 - z_2|.$$

Here the constants c and c_1 depend only on each other.

THEOREM 3.25. *A Jordan domain $D \subset \mathbf{C}$ with $\infty \in \partial D$ is a c-John disk if and only there is a constant $c_0 \geq 1$ such that for each hyperbolic geodesic γ in D^**

$$\operatorname{dia}(\gamma) \leq c_0 |z_1 - z_2|, \tag{3.26}$$

where z_1, z_2 are the endpoints of γ. Here c and c_0 depend on each other.

To prove this we need a lemma which gives the diameter version of the Gehring-Hayman inequality in Lemma 1.3. See [R, Lemma 3.22] and [PR, Theorem 3.2].

LEMMA 3.27. *Suppose that γ is a hyperbolic geodesic in a simply connected proper subdomain $D \subset \mathbf{C}$ and that α is an arc which joins the endpoints of γ in $D \cap \bar{B}(z_0, r)$ for $z_0 \in \mathbf{C}$. Then*

$$\gamma \subset \bar{B}(z_0, br), \qquad b = 3 + 2\sqrt{2}.$$

PROOF OF THEOREM 3.25. Suppose first that a Jordan domain $D \subset \mathbf{C}$ is a c-John disk with $\infty \in \partial D$. Then by Lemma 3.24 for each pair of points z_1, $z_2 \in D^*$ there exists a continuum $E \subset D^*$ such that $\operatorname{dia}(E) \leq c_1 |z_1 - z_2|$. Thus by [NV, Lemma 4.3], E can be replaced by an arc $\alpha \subset D^*$ with $\operatorname{dia}(\alpha) \leq c_2 |z_1 - z_2|$ for any $c_2 > c_1$. Next let γ be the hyperbolic geodesic joining z_1 and z_2 in D^*. Then α is an arc which joins the endpoints of γ. Now choose a point $z_0 \in \alpha$ such that $|z_1 - z_0| = |z_2 - z_0|$ and let $r = \operatorname{dia}(\alpha)$. Thus $\alpha \subset D^* \cap \overline{B}(z_0, r)$, while $\gamma \subset \overline{B}(z_0, br)$ with $b = 3 + 2\sqrt{2}$ by Lemma 3.27. Hence

$$\operatorname{dia}(\gamma) \leq 2br = 2b \operatorname{dia}(\alpha) \leq 2bc_2 \, |z_1 - z_2|$$

and this shows (3.26) with $c_0 = 2bc_2$.

Suppose next that (3.26) holds. Then by Lemma 3.24, D is a c-John disk.

4. Hyperbolic distance in John disks.

We define a one-sided analogue of the function j_D in (1.2) as follows:

$$j'_D(z_1, z_2) = \tfrac{1}{2}\log\left(\frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)} + 1\right)\left(\frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)} + 1\right),$$

where λ_D is the inner distance on D,

$$\lambda_D(z_1, z_2) = \inf_{\gamma} \ell(\gamma),$$

and the infimum is taken over all paths $\gamma \subset D$ with z_1 and z_2 as endpoints. The main result of this section relates h_D and j'_D in John disks. As mentioned in the introduction, this is a one-sided analogue of a characterization of quasidisks due to Gehring and Osgood [GO]. Their two-sided version characterizes uniform domains, regardless of connectivity, when the hyperbolic metric is replaced by the quasihyperbolic metric.

THEOREM 4.1. *A simply connected proper subdomain $D \subset \mathbf{C}$ is a c-John disk if and only if there exists a constant $b \geq 1$ such that*

$$h_D(z_1, z_2) \leq b\, j'_D(z_1, z_2) \tag{4.2}$$

for all z_1, $z_2 \in D$. Here the constants c and b depend only on each other.

We will use the following inequality, which is easily derived.

LEMMA 4.3. *For any $c \geq 1$ and $x \geq 0$,*

$$\log(cx + 1) \leq c\log(x + 1)\,.$$

PROOF OF NECESSITY. Suppose that D is a c-John disk. Then by [GHM, Theorem 4.1] each z_1, $z_2 \in D$ can be joined by a hyperbolic geodesic γ in D such that for all $z \in \gamma$

$$\min_{j=1,2} \ell(\gamma(z_j,\, z)) \leq c_1 \operatorname{dist}(z, \partial D) \tag{4.4}$$

for some constant c_1 depending only on c. Choose $z_0 \in \gamma$ so that $\ell(\gamma(z_0, z_1)) = \ell(\gamma(z_0, z_2))$. Then by the triangle inequality it is sufficient to show that

$$h_D(z_j,\, z_0) \leq b\, \log\left(\frac{\lambda_D(z_1,\, z_2)}{\operatorname{dist}(z_j, \partial D)} + 1\right) \tag{4.5}$$

for $j = 1$, 2, where $b = (3c_1 + 2)k$ and k is an absolute constant. By symmetry we may assume that $j = 1$.

Suppose first that

$$\ell(\gamma(z_1, z_0)) \le \frac{c_1}{c_1+1}\operatorname{dist}(z_1, \partial D). \tag{4.6}$$

Then $z_0 \in B(z_1, \frac{c_1}{c_1+1}\operatorname{dist}(z_1, \partial D))$. If $z \in [z_1, z_0]$, then

$$\operatorname{dist}(z, \partial D) \ge \operatorname{dist}(z_1, \partial D) - |z_1 - z| \ge \frac{1}{c_1+1}\operatorname{dist}(z_1, \partial D)$$

and hence

$$\begin{aligned} |z_1 - z| + \operatorname{dist}(z_1, \partial D) &\le c_1 \operatorname{dist}(z, \partial D) + (c_1+1)\operatorname{dist}(z, \partial D) \\ &\le (2c_1+1)\operatorname{dist}(z, \partial D)\,. \end{aligned}$$

If $\rho_D(z)$ is the hyperbolic density in D, then the Schwarz lemma and the Koebe distortion theorem give the inequalities

$$\frac{1}{4\operatorname{dist}(z, \partial D)} \le \rho_D(z) \le \frac{1}{\operatorname{dist}(z, \partial D)}\,.$$

Thus Lemma 1.3 and Lemma 4.3 yield

$$\begin{aligned} h_D(z_1, z_0) &\le \int_{[z_1, z_0]} \frac{ds}{\operatorname{dist}(z, \partial D)} \\ &\le \int_0^{|z_1 - z_0|} \frac{(2c_1+1)\,ds}{s + \operatorname{dist}(z_1, \partial D)} \\ &\le (2c_1+1)\,\log\left(\frac{\ell(\gamma)}{\operatorname{dist}(z_1, \partial D)} + 1\right) \\ &\le (2c_1+1)k\,\log\left(\frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)} + 1\right). \end{aligned}$$

where k is an absolute constant. This implies (4.5).

Next suppose that (4.6) does not hold and choose $y_1 \in \gamma(z_1, z_0)$ so that

$$\ell(\gamma(z_1, y_1)) = \frac{c_1}{c_1+1}\operatorname{dist}(z_1, \partial D).$$

If $z \in \gamma(y_1, z_0)$, then

$$\operatorname{dist}(z, \partial D) \ge \frac{1}{c_1}\ell(\gamma(z_1, z))$$

by (4.4) and hence again

$$\begin{aligned} h_D(y_1, z_0) &\le c_1 \log\left(\frac{c_1+1}{c_1}\frac{\ell(\gamma(z_1, z_0))}{\operatorname{dist}(z_1, \partial D)}\right) \\ &\le c_1 \log\left(\frac{c_1+1}{c_1}\frac{\ell(\gamma)}{\operatorname{dist}(z_1, \partial D)} + 1\right) \\ &\le (c_1+1)k \log\left(\frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)} + 1\right). \end{aligned}$$

We also have

$$h_D(z_1, y_1) \le (2c_1+1)k \log\left(\frac{\lambda_D(z_1, y_1)}{\operatorname{dist}(z_1, \partial D)} + 1\right)$$

by what was proved above. Then (4.5) follows from the triangle inequality.

Proof of sufficiency. Suppose that (4.2) holds. Fix $z_1, z_2 \in D$ and let γ be the hyperbolic geodesic joining z_1 to z_2 in D. We may assume that $\operatorname{dist}(z_1, \partial D) \ge \operatorname{dist}(z_2, \partial D)$.

Suppose first that

$$2\lambda_D(z_1, z_2) \le \operatorname{dist}(z_1, \partial D) \tag{4.7}$$

Then $|z_1 - z_2| \le \operatorname{dist}(z_1, \partial D)/2$ and hence

$$z_2 \in B\left(z_1, \frac{\operatorname{dist}(z_1, \partial D)}{2}\right) \subset D.$$

Thus $\lambda_D(z_1, z_2) = |z_1 - z_2|$ and since euclidean disks in D are convex with respect to the hyperbolic geometry in D [Jø],

$$\gamma \subset \bar{B}\left(\frac{z_1+z_2}{2}, \frac{|z_1-z_2|}{2}\right) \subset B\left(z_1, \frac{\operatorname{dist}(z_1, \partial D)}{2}\right).$$

Then by Lemma 1.3

$$\min_{j=1,2} \ell(\gamma(z_j, z)) \le \ell(\gamma) \le k\,|z_1 - z_2| \le k \operatorname{dist}(z, \partial D) \tag{4.8}$$

for all $z \in \gamma$ and k is an absolute constant.

Next suppose that (4.7) does not hold. By compactness there exists a point $z_0 \in \gamma$ with

$$\operatorname{dist}(z_0, \partial D) = \sup_{z\in\gamma} \operatorname{dist}(z, \partial D).$$

Let m denote the largest integer for which

$$2^m \operatorname{dist}(z_1, \partial D) \le \operatorname{dist}(z_0, \partial D)$$

and let y_0 be the first point of $\gamma(z_1, z_0)$ with

$$\operatorname{dist}(y_0, \partial D) = 2^m \operatorname{dist}(z_1, \partial D)$$

as we traverse γ from z_1 towards z_0. Clearly

$$\operatorname{dist}(y_0, \partial D) \le \operatorname{dist}(z_0, \partial D) < 2 \operatorname{dist}(y_0, \partial D). \tag{4.9}$$

Let $y_1 = z_1$ and choose points $y_2, \dots, y_{m+1} \in \gamma(z_1, z_0)$ so that y_i is the first point of $\gamma(z_1, z_0)$ for which

$$\operatorname{dist}(y_i, \partial D) = 2^{i-1} \operatorname{dist}(y_1, \partial D) \tag{4.10}$$

as we traverse γ from z_1 towards z_0. Then $y_{m+1} = y_0$ and let $y_{m+2} = z_0$.

We show first that for $i = 1, , m+1$

$$\begin{cases} h_D(y_i, y_{i+1}) \le 2^4 \cdot b^2 \\ \ell(\gamma(y_i, y_{i+1})) \le 2^7 \cdot b^2 \operatorname{dist}(y_i, \partial D).. \end{cases} \tag{4.11}$$

Fix $i \in \{1, , m+1\}$ and set

$$t = \frac{\ell(\gamma(y_i, y_{i+1}))}{\operatorname{dist}(y_i, \partial D)}.$$

If $z \in \gamma(y_i, y_{i+1})$, then by (4.9), (4.10)

$$\operatorname{dist}(z, \partial D) \le \operatorname{dist}(y_{i+1}, \partial D) \le 2 \operatorname{dist}(y_i, \partial D)$$

and hence

$$t = \int_{\gamma(y_i, y_{i+1})} \frac{|dz|}{\operatorname{dist}(y_i, \partial D)} \le 8\, h_D(y_i, y_{i+1}).$$

Since

$$j_D'(y_i, y_{i+1}) \le \log\left(\frac{\lambda_D(y_i, y_{i+1})}{\operatorname{dist}(y_i, \partial D)} + 1\right) \le \log(t+1),$$

(4.2) implies that

$$h_D(y_i, y_{i+1}) \le b \log(t+1) \le b\,(t+1)^{1/2}.$$

If $t \ge 1$, then

$$t \le 8\, h_D(y_i, y_{i+1}) \le 8b\,(t+1)^{1/2} \le 8b\,(2t)^{1/2}$$

which implies

$$t \le 2^7 \cdot b^2$$

and hence

$$h_D(y_i, y_{i+1}) \le b\,(2 \cdot 2^7 \cdot b^2)^{1/2} = 2^4 \cdot b^2.$$

Thus we obtain (4.11). If $t < 1$, then $t \leq 2^7 \cdot b^2$ and again we obtain (4.11). This completes the proof of (4.11).

Now [GP, Lemma 2.1] and (4.11) imply that for $z \in \gamma(y_i, y_{i+1})$, $i = 1, \ldots, m+1$

$$\log \frac{\operatorname{dist}(y_{i+1}, \partial D)}{\operatorname{dist}(z, \partial D)} \leq 4\, h_D(z, y_{i+1}) \leq 4 h_D(y_i, y_{i+1}) < 2^6 \cdot b^2 = c_0$$

and thus

$$\operatorname{dist}(y_{i+1}, \partial D) \leq e^{c_0} \operatorname{dist}(z, \partial D). \tag{4.12}$$

If $z \in \gamma(z_1, z_0)$, then $z \in \gamma\,[y_{i_0}, y_{i_0+1}]$ for some $i_0 \in \{1, , m+1\}$ and hence by (4.10), (4.11) and (4.12)

$$\begin{aligned} \min_{j=1,2} \ell(\gamma(z_j, z)) \leq \ell(\gamma(z_1, z)) &\leq \sum_{i=1}^{i_0} \ell(\gamma\,[y_i, y_{i+1}]) \\ &\leq 2\, c_0 \sum_{i=1}^{i_0} \operatorname{dist}(y_i, \partial D) = 2\, c_0\, (2^{i_o} - 1) \operatorname{dist}(y_1, \partial D) \\ &< 2\, c_0 \operatorname{dist}(y_{i_o+1}, \partial D) \leq 2\, c_0\, e^{c_0} \operatorname{dist}(z, \partial D). \end{aligned} \tag{4.13}$$

Likewise, if $z \in \gamma(z_2, z_0)$, then we also have (4.13). Therefore by (4.8) and (4.13) D is a c-John disk with $c = 2\, c_0\, e^{c_0}$.

REMARK 4.14. Theorem 4.1 is easily translated into a result for the quasihyperbolic distance in D, k_D. If we assume that quasihyperbolic geodesics are double c-cone arcs in D, the result for k_D can be generalized to finitely connected domains in the plane, and to domains in R^n which are quasiconformal images of uniform domains. In the quasihyperbolic case, the proof of sufficiency of Theorem 4.1 shows that in a domain $D \subset \mathsf{C}$ satisfying

$$k_D(z_1, z_2) \leq b j'_D(z_1, z_2)\,,$$

quasihyperbolic geodesics are double c-cone arcs, where c depends only on b.

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DEPARTMENT OF MATHEMATICS
PUSAN WOMEN'S UNIVERSITY
PUSAN
617–736 KOREA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109
USA