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WEIGHTED TANGENTIAL BOUNDARY LIMITS OF SUBHARMONIC FUNCTIONS ON DOMAINS IN R^n $(n \geq 2)$

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Abstract

In the paper we consider weighted non-tangential and tangential boundary limits of non-negative subharmonic functions on bounded domains in R^n, $n \geq 2$.

The main result of the paper is as follows: Let f be a non-negative subharmonic function on a bounded domain Ω with C^1 boundary satisfying

$$\int_\Omega \delta(y)^\gamma f^p(y)dy < \infty$$

for some $p > 0$, and some $\gamma > -1 - \beta(p)$, where $\beta(p) = \max\{(n-1)(1-p), 0\}$ and $\delta(y)$ denotes the distance from y to $\partial\Omega$. Suppose $\tau \geq 1$. Then for a.e. $\zeta \in \partial\Omega$,

$$f^p(y) = o\Big(\delta(y)^{(\frac{n-1}{\tau})-\gamma-n}\Big)$$

uniformly as $y \to \zeta$ in each $\Gamma_{\tau,\alpha}(\zeta)$, where for $\alpha > 0$ ($\alpha > 1$ when $\tau = 1$)

$$\Gamma_{\tau,\alpha}(\zeta) = \{y \in \Omega : |y - \zeta|^\tau < \alpha\delta(y)\}.$$

1. Introduction.

The results of this paper were motivated by the following result of F. W. Gehring [4] (see also [13, Theorem IV. 41]):

THEOREM. *Suppose $w(z)$ is a non-negative subharmonic function in the unit disc $|z| < 1$ in C satisfying*

$$\iint_{|z|<1} w^p(z)\,dx\,dy < \infty, \quad z = x + iy, \tag{1.1}$$

for some $p > 1$. Then for almost every θ,

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$$w(z) = o\Big((1-|z|)^{-1/p}\Big)$$

uniformly as $z \to e^{i\theta}$ in each non-tangential approach region $\Gamma_\alpha(e^{i\theta})$.

This last statement is equivalent to

$$\lim_{r\to 1^-} \sup_{\substack{z\in\Gamma_\alpha(e^{i\theta})\\ |z|\geq r}} (1-|z|)w^p(z) = 0$$

for almost every θ, where for $\alpha > 1$,

$$\Gamma_\alpha(e^{i\theta}) = \{z : |e^{i\theta} - z| < \alpha(1-|z|),\ |z| < 1\}. \tag{1.2}$$

The proof of the theorem used the Hardy-Littlewood theorem which accounts for the assumption that $p > 1$.

Using techniques of potential theory we extend the previous theorem in several directions. First, we remove the restriction on $p > 1$ and prove that the result of Gehring is valid for all p, $0 < p < \infty$. Second, we extend the result to subharmonic functions on bounded domains in R^n, $n \geq 2$, with C^1 boundary. Finally, in addition to non-tangential limits, we will also consider weighted boundary limits along tangential approach regions.

For a bounded domain $\Omega \subset \mathsf{R}^n$, $n \geq 2$, and $x \in \Omega$, let $\delta(x)$ denote the distance from x to $\partial\Omega$, the boundary of Ω. The boundary of Ω is said to be C^1 if there exists a C^1 function $\rho : \mathsf{R}^n \to \mathsf{R}$ such that $\Omega = \{x \in \mathsf{R}^n : \rho(x) < 0\}$, $\partial\Omega = \{x \in \mathsf{R}^n : \rho(x) = 0\}$, and $\nabla\rho(x) \neq 0$ for all $x \in \partial\Omega$. This last condition ensures that at each $\zeta \in \partial\Omega$ there is a tangent plane and an outward unit normal, denoted by $\mathbf{n}_\zeta$.

Let $\zeta \in \partial\Omega$. For $\tau \geq 1$ and $\alpha > 0$ ($\alpha > 1$ when $\tau = 1$), set

$$\Gamma_{\tau,\alpha}(\zeta) = \{y \in \Omega : |y-\zeta|^\tau < \alpha\delta(y)\}. \tag{1.3}$$

In the unit disc, when $\tau = 1$ and $\alpha > 1$, these are the non-tangential regions Γ_α defined above. As we will see below, when $\tau > 1$, the regions $\Gamma_{\tau,\alpha}(\zeta)$ have tangential contact in all directions at ζ.

Finally, as in [12], for $p > 0$, set $\beta(p) = \max\{(n-1)(1-p), 0\}$. The main result of the paper is as follows:

THEOREM 1. *Let f be a non-negative subharmonic function on a bounded domain Ω with C^1 boundary satisfying*

$$\int_\Omega \delta(y)^\gamma f^p(y)\,dy < \infty, \tag{1.4}$$

for some $p > 0$, and $\gamma > -1 - \beta(p)$. Then for each $\tau \geq 1$ and $\alpha > 0$ ($\alpha > 1$ when $\tau = 1$)

$$\lim_{\rho\to 0} \sup_{y\in\Gamma_{\tau,\alpha,\rho}(\zeta)} \delta(y)^{n+\gamma-(\frac{n-1}{\tau})} f^p(y) = 0 \quad \text{for a.e. } \zeta \in \partial\Omega,$$

where $\Gamma_{\tau,\alpha,\rho}(\zeta) = \{y \in \Gamma_{\tau,\alpha}(\zeta) : \delta(y) < \rho\}$.

The special case $n = 2$, $\gamma = 0$, and $\tau = 1$ gives the result of Gehring in the setting of the unit disc. In the hypothesis of Theorem 1 we require that $\gamma > -1 - \beta(p)$, since by Theorem 2 of [12], if $\gamma \le -1 - \beta(p)$, then the only non-negative subharmonic function f satisfying (1.4) on a bounded domain with C^2 boundary for some $p > 0$ is the zero function. The proof of Theorem 1 will be given in Section 3. In Section 4 we give two extensions of Theorem 1. The first is a restatement of Theorem 1 in terms of d-dimensional Hausdorff measure, while the second provides an extension to include unbounded domains. The analogue of Theorem 1 for functions that are subharmonic with respect to the Laplace-Beltrami operator on the unit ball in C^n was proved by the author in [11].

Tangential boundary limits of harmonic functions or Green potentials have been considered by many authors, including Y. Mizuta [7, 8], A. Nagel, W. Rudin, and J. H. Shapiro [9], and J-M. G. Wu [14], among many others. A good reference for the numerous results concerning non-tangential and tangential boundary limits of Green potentials on the upper half-space in R^n is the paper by R. D. Berman and W. S. Cohn [1]. Many of the results involving tangential boundary limits of Green potentials were motivated by the results of G. T. Cargo [2] and J. R. Kinney [6] concerning tangential boundary limits of Blaschke products in the unit disc.

Many of the above referenced results involve tangential boundary limits in the half-space $\mathscr{H}$ in R^n, where for $n \ge 2$,

$$\mathscr{H} = \{(x', x_n) : x' \in \mathsf{R}^{n-1}, x_n > 0\}.$$

For $\zeta = (\zeta', 0) \in \partial\mathscr{H}$, $\tau \ge 1$, and $\beta > 0$ ($\beta > 1$ when $\tau = 1$), set

$$\mathscr{A}_{\tau,\beta}(\zeta) = \{y \in \mathscr{H} : |y - \zeta|^\tau < \beta y_n\}.$$

When $\tau = 1$ and $\beta > 1$, $\mathscr{A}_{1,\beta}(\zeta)$ is an open cone at ζ with axis in the direction $(0', 1)$ and angle $\arccos \frac{1}{\beta}$. On the other hand, when $\tau = 2$, $\beta > 0$,

$$\mathscr{A}_{2,\beta}(\zeta) = \{y \in \mathscr{H} : |y' - \zeta'|^2 + (y_n - \tfrac{1}{2}\beta)^2 < (\tfrac{1}{2}\beta)^2\} = B_{\frac{1}{2}\beta}(\zeta', \tfrac{1}{2}\beta),$$

where $B_r(x)$ is the open ball of radius r centered at x.

As we will see, the approach regions $\Gamma_{\tau,\alpha}(\zeta)$ are very similar to the regions $\mathscr{A}_{\tau,\alpha}$. Fix $\zeta \in \partial\Omega$. By translation and rotation we can assume without loss of generality that $\zeta = 0$, and that in a neighborhood U of 0, Ω is given by

$$U \cap \Omega = \{(y', y_n) \in U : y' \in V, y_n > \varphi(y')\},$$

where V is a neighborhood of $0'$ in R^{n-1}, φ is a C^1 function defined on V with $\varphi(0') = 0$ and $\nabla\varphi(0') = 0'$. For purposes of illustration we assume that $\varphi(y') \geq 0$ for all $y' \in V$. If $\varphi(y') = 0$ for all $y' \in V$, then $\Gamma_{\tau,\alpha}(0) = \mathscr{A}_{\tau,\alpha}(0)$. Since φ is assumed to be non-negative, we have $\delta(y) \leq y_n$ for all $y \in U \cap \Omega$. Thus

$$\Gamma_{\tau,\alpha}(0) \cap U \subset \mathscr{A}_{\tau,\alpha}(0) \cap U$$

for all $\tau \geq 1$ and $\alpha > 0$ ($\alpha > 1$ when $\tau = 1$). If $\varphi(y') \leq 0$, then the reverse containment holds.

Since $\partial\Omega$ is C^1, there exists $\beta_o > 1$ such that $\overline{\mathscr{A}}_{1,\beta}(0) \cap \partial\Omega \cap U = \{0\}$ for all β, $1 < \beta \leq \beta_o$. If in addition, $\partial\Omega$ is $C^{1,\gamma}$ ($0 < \gamma \leq 1$) near 0, that is, there exists a positive constant C such that

$$|\nabla\varphi(\zeta') - \nabla\varphi(\xi')| \leq C\,|\zeta' - \xi'|^\gamma$$

for all ζ', $\xi' \in V$, then there exists $\beta_o > 0$ such that

$$\overline{\mathscr{A}}_{1+\gamma,\beta}(0) \cap \partial\Omega \cap U = \{0\}$$

for all β, $0 < \beta \leq \beta_0$. This is an immediate consequence of the fact that if φ is $C^{1,\gamma}$, then $|\varphi(\zeta')| \leq C|\zeta'|^{1+\gamma}$ for all $\zeta' \in V$.

Suppose now that $\tau \geq 1$ and $\beta_o > 0$ ($\beta_o > 1$ if $\tau = 1$) is such that

$$\overline{\mathscr{A}}_{\tau,\beta_o}(0) \cap \partial\Omega \cap U = \{0\}.$$

We will show that if this is the case, then given $\alpha > 0$, there exists $\beta_\alpha < \beta_o$ such that

$$\mathscr{A}_{\tau,\beta}(0) \cap U \subset \Gamma_{\tau,\alpha}(0) \cap U$$

for all $\beta \leq \beta_\alpha$. Let $y \in \mathscr{A}_{\tau,\beta}(0)$, $\beta < \beta_o$, and let $\zeta \in \partial\Omega \cap U$ be such that $\delta(y) = |\zeta - y|$. Then $y = \zeta - \delta(y)\mathbf{n}_\zeta$. Hence if we write $y = (y', y_n)$, we have $y_n = \varphi(\zeta') + \delta(y)/A$, where $A = \sqrt{|\nabla\varphi(\zeta')|^2 + 1}$. Since $\zeta \in \partial\Omega, |\zeta|^\tau \geq \beta_o\varphi(\zeta')$. Thus

$$\begin{aligned}
|y|^\tau < \beta y_n &< \left(\frac{\beta}{\beta_o}\right)|\zeta|^\tau + \beta\delta(y) \leq \left(\frac{\beta}{\beta_o}\right)(|y| + \delta(y))^\tau + \beta\delta(y) \\
&< \left(\frac{\beta}{\beta_o}\right)(|y|^\tau + \tau 2^{\tau-1}|y|^{\tau-1}\delta(y)) + \beta\delta(y) \\
&< \left(\frac{\beta}{\beta_o}\right)|y|^\tau + \beta c\delta(y)
\end{aligned}$$

for some positive constant c. Hence

$$|y|^{\tau} < \left(\frac{\beta\beta_o c}{\beta_o - \beta}\right)\delta(y).$$

From this it now follows that there exists $\beta_\alpha < \beta_o$ such that $\mathscr{A}_{\tau,\beta}(0) \subset \Gamma_{\tau,\alpha}(0)$ for all $\beta \le \beta_\alpha$.

2. Preliminaries.

Prior to proving Theorem 1 we first state and prove several preliminary results. As in the Introduction, for $\zeta \in \partial\Omega$, $\tau \ge 1$, and $\alpha > 0$ ($\alpha > 1$ if $\tau = 1$), set

$$\Gamma_{\tau,\alpha}(\zeta) = \{y \in \Omega : |y-\zeta|^{\tau} < \alpha\delta(y)\}.$$

Also, for $x \in \Omega$, let

$$B(x) = B(x, \tfrac{1}{3}\delta(x)) = \{y \in \Omega : |x-y| < \tfrac{1}{3}\delta(x)\}.$$

Lemma 1.

(a) *For all* $y \in B(x)$, $\frac{2}{3}\delta(x) \le \delta(y) \le \frac{4}{3}\delta(x)$.

(b) *Let* $\zeta \in \partial\Omega$. *Suppose* $\tau \ge 1$, $\alpha > 0$ *(*$\alpha > 1$ *when* $\tau = 1$*). Then there exists* $\alpha' > \alpha$ *such that* $B(x) \subset \Gamma_{\tau,\alpha'}(\zeta)$ *for all* $x \in \Gamma_{\tau,\alpha}(\zeta)$ *with* $\delta(x) \le 1$.

Proof. (a) Let $y \in B(x)$. Suppose $\zeta \in \partial\Omega$ is such that $|x-\zeta| = \delta(x)$. Then

$$\delta(y) \le |y-\zeta| \le |y-x| + |x-\zeta| < \tfrac{4}{3}\delta(x).$$

On the other hand, if $\xi \in \partial\Omega$ is such that $|y-\xi| = \delta(y)$, then

$$\delta(x) \le |x-\xi| \le |x-y| + |y-\xi| < \tfrac{1}{3}\delta(x) + \delta(y).$$

Thus $\delta(y) \ge \frac{2}{3}\delta(x)$.

(b) Suppose $x \in \Gamma_{\tau,\alpha}(\zeta)$ with $\delta(x) \le 1$. Then for $y \in B(x)$,

$$\begin{aligned} |y-\zeta|^{\tau} &\le (|y-x| + |x-\zeta|)^{\tau} \\ &\le 2^{\tau}(|y-x|^{\tau} + |x-\zeta|^{\tau}) \\ &\le (\tfrac{2}{3})^{\tau}\delta(x)^{\tau} + 2^{\tau}\alpha\delta(x), \end{aligned}$$

which by (a),

$$\le \alpha'\delta(y)$$

for some $\alpha' > \alpha$.

For $y \in \Omega$, let

$$\widetilde{\Gamma}_{\tau,\alpha}(y) = \{\zeta \in \partial\Omega : y \in \Gamma_{\tau,\alpha}(\zeta)\}.$$

Also, for $\zeta \in \partial\Omega$ and $r > 0$, let

$$S(\zeta, r) = \{\xi \in \partial\Omega : |\zeta - \xi| < r\}.$$

If $\partial\Omega$ is C^1 and σ denotes surface area measure on $\partial\Omega$, then there exists a positive constant C such that $\sigma(S(\zeta, r)) \le C r^{n-1}$ for all $\zeta \in \partial\Omega$, $r > 0$.

Lemma 2. *Let Ω be a bounded domain with C^1 boundary. If $y \in \Gamma_{\tau,\alpha}(\zeta)$, then*

$$\sigma(\widetilde{\Gamma}_{\tau,\alpha}(y)) \le C\,\delta(y)^{(n-1)/\tau},$$

where C is a positive constant depending only on τ and α.

Proof. Suppose $y \in \Gamma_{\tau,\alpha}(\zeta)$. If $\xi \in \widetilde{\Gamma}_{\tau,\alpha}(y)$, then

$$|\xi - \zeta| \le |\xi - y| + |y - \zeta| \le 2\alpha^{1/\tau}\delta(y)^{1/\tau}.$$

Therefore $\widetilde{\Gamma}_{\tau,\alpha}(y) \subset S(\zeta, c\delta(y)^{1/\tau})$, with $c = 2\alpha^{1/\tau}$. Thus

$$\sigma(\widetilde{\Gamma}_{\tau,\alpha}(y)) \le \sigma(S(\zeta, c\delta(y)^{1/\tau})) \le C\,\delta(y)^{(n-1)/\tau}.$$

The following generalization of an inequality of Fefferman and Stein will be crucial in the proof of Theorem 1.

Lemma 3. *Let Ω be a proper open subset of R^n,and let f be a non-negative subharmonic function on Ω. Then there exists a constant $C(n,p)$, depending only on n and p, such that*

$$f^p(x) \le \frac{C(n,p)}{\delta(x)^n}\int_{B(x)} f^p(y)\,dy \tag{2.1}$$

for all $p > 0$.

Remark. Inequality (2.1) has previously been stated by Riihentaus in [10] and by Susuki in [12] For $p \ge 1$, the inequality follows immediately from the mean value property of subharmonic functions. For $0 < p < 1$, inequality (2.1) was proved in [3] for $|h|$, where h is harmonic on Ω. The same proof also works for non-negative subharmonic functions, and thus is omitted.

3. Proof of Theorem 1.

For $\rho > 0$, set $\Omega_\rho = \{x \in \Omega : \delta(x) < \rho\}$, and

$$\Gamma_{\tau,\alpha,\rho}(\zeta) = \Gamma_{\tau,\alpha}(\zeta) \cap \Omega_\rho.$$

Also, for $\zeta \in \partial\Omega$, $\rho > 0$, set

$$M_\rho(\zeta) = \sup\{\delta(x)^{n+\gamma-(\frac{n-1}{\tau})} f^p(x) : x \in \Gamma_{\tau,\alpha,\rho}(\zeta)\}.$$

By Lemmas 1 and 3, if $x \in \Gamma_{\tau,\alpha,\rho}(\zeta)$,

$$\delta(x)^{n+\gamma-(\frac{n-1}{\tau})} f^p(x) \leq C \int_{B(x)} \delta(y)^{\gamma-(\frac{n-1}{\tau})} f^p(y)\, dy$$

$$\leq C \int_{\Gamma_{\tau,\alpha',\rho'}(\zeta)} \delta(y)^{\gamma-(\frac{n-1}{\tau})} f^p(y)\, dy,$$

where $\alpha' > \alpha$ and $\rho' = \frac{4}{3}\rho$. Thus

$$M_\rho(\zeta) \leq C \int_{\Omega_{\rho'}} \chi_{\Gamma_{\tau,\alpha'}(\zeta)}(y) \delta(y)^{\gamma-(\frac{n-1}{\tau})} f^p(y)\, dy,$$

where for a set E, χ_E denotes the characteristic function of E. Integrating over $\partial\Omega$ with respect to surface area measure σ gives

$$\int_{\partial\Omega} M_\rho(\zeta)\, d\sigma(\zeta) \leq C \int_{\partial\Omega} \int_{\Omega_{\rho'}} \chi_{\Gamma_{\tau,\alpha'}(\zeta)}(y) \delta(y)^{\gamma-(\frac{n-1}{\tau})} f^p(y)\, dy\, d\sigma(\zeta),$$

which by Fubini's theorem and Lemma 2,

$$\leq C \int_{\Omega_{\rho'}} \sigma(\widetilde{\Gamma}_{\tau,\alpha'}(y)) \delta(y)^{\gamma-(\frac{n-1}{\tau})} f^p(y)\, dy$$

$$\leq C \int_{\Omega_{\rho'}} \delta(y)^\gamma f^p(y) dy.$$

If f satisfies (1.4), then

$$\lim_{\rho\to 0} \int_{\Omega_{\frac{4}{3}\rho}} \delta(y)^\gamma f^p(y)\, dy = 0.$$

Thus if we let $M(\zeta) = \lim_{\rho\to 0} M_\rho(\zeta)$, by Fatou's lemma and the above,

$$\int_{\partial\Omega} M(\zeta)\, d\sigma(\zeta) \leq \lim_{\rho\to 0} C \int_{\Omega_{\rho'}} \delta(y)^\gamma f^p(y)\, dy = 0.$$

Hence $M(\zeta) = 0$ a.e. on $\partial\Omega$. Thus

$$\lim_{\rho\to 0} \sup_{y\in\Gamma_{\tau,\alpha,\rho}(\zeta)} \delta(y)^{n+\gamma-(\frac{n-1}{\tau})} f^p(y) = 0$$

for a.e. $\zeta \in \partial\Omega$.

4. Extensions of Theorem 1.

We conclude the paper by giving two extensions of Theorem 1. Since the proofs follow easily from what has already been presented, they are omitted.

Our first result is a restatement of Theorem 1 in terms of d-dimensional Hausdorff measure H_d.

THEOREM 2. *Let f be a non-negative subharmonic function on a bounded domain Ω with C^1 boundary satisfying*

$$\int_{\Omega} \delta(y)^{\gamma} f^{p}(y)\, dy < \infty,$$

for some $p > 0$, and $\gamma > -1 - \beta(p)$. Let $0 < d \leq n-1$. Then for each $\tau \geq 1$, there exists a subset E_τ of $\partial\Omega$ with $H_d(E_\tau) = 0$ such that

$$\lim_{\rho\to 0} \sup_{y\in\Gamma_{\tau,\alpha,\rho}(\zeta)} \delta(y)^{n+\gamma-(\frac{d}{\tau})} f^{p}(y) = 0 \quad \text{for all } \zeta \in \partial\Omega \setminus E_\tau.$$

The proof of Theorem 2 follows in the same way as Theorem 1, except that surface area measure is replaced by a measure ν on $\partial\Omega$ satisfying $\nu(S(\zeta, r)) \leq C\, r^d$ for all $\zeta \in \partial\Omega$.

Since the proof of Theorem 1 only involves the local boundary behavior of f, Theorem 1 is also valid for unbounded domains. Thus we have

THEOREM 3. *Let Ω be a domain with C^1 boundary, and let f be a non-negative subharmonic function on Ω satisfying*

$$\int_{\Omega\cap\{|y|\leq r\}} \delta(y)^{\gamma} f^{p}(y)\, dy < \infty, \tag{1.4}$$

for some $p > 0$, $\gamma > -1 - \beta(p)$, and every $r > 0$. Then, for each $\tau \geq 1$ and $\alpha > 0$ ($\alpha > 1$ when $\tau = 1$),

$$\lim_{\rho\to 0} \sup_{y\in\Gamma_{\tau,\alpha,\rho}(\zeta)} \delta(y)^{n+\gamma-(\frac{n-1}{\tau})} f^{p}(y) = 0 \quad \text{for a.e. } \zeta \in \partial\Omega.$$

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