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MAPPINGS WITHOUT FIXED OR ANTIPODAL POINTS. SOME GEOMETRIC APPLICATIONS

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Abstract

For T a topological space and X a real normed space $S(T,X)$ denotes the set of continuous mappings from T into $S(X)=\{x\in X : \|x\|=1\}$. Given f in $S(T,X)$ we study the existence of functions e in $S(T,X)$ such that $f(t)\neq e(t)\neq -f(t)$, $\forall t\in T$. When this holds for every f, we say that $S(T,X)$ is plentiful. If $\dim X$ is an even integer or infinite this last property is automatic for any T. We show that it also verifies if T is a contractible compact space and X is an arbitrary normed space with $\dim X\geq 2$. From this we deduce that if T is completely regular and $\dim T<\dim X-1$, then $S(T,X)$ is plentiful, where $\dim T$ stands for the covering dimension of T. If $C(T,X)$ denotes the space of continuous and bounded functions from T into X endowed with the sup norm, we study the geometry of the unit ball of $C(T,X)$ for X strictly convex and $S(T,X)$ plentiful. For T completely regular and $\dim X<\infty$, we prove the following:

The necessary and sufficient condition for every f in the unit ball of $C(T,X)$ to be the mean of 3 extreme points is that $\dim T<\dim X$.

Moreover, if X is infinite-dimensional, then the previously mentioned representation remains true without any restriction about T.

1. Introduction

Let X be a real normed space. The closed unit ball and the unit sphere of X will be denoted, respectively, by $B(X)$ and $S(X)$. Moreover, $E(X)$ will stand for the set of extreme points of $B(X)$ and $co(E(X))$ for the convex hull of $E(X)$.

If T is a topological space we will denote by $C(T,X)$ the space of continuous and bounded mappings from T into X with its usual uniform norm. To simplify the notation we will frequently write Y instead of $C(T,X)$. Furthermore $S(T,X)$ will be the set of continuous functions from T into $S(X)$. Let us observe that if X is strictly convex, then $S(T,X)=E(Y)$.

Most of the known results about the extremal structure of the unit ball of $C(T,X)$ depend on the existence of continuous functions $v : S(X)\to S(X)$ verifying

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$$v(x) \neq x, \quad v(x) \neq -x, \quad \forall x \in S(X).$$

The existence of such functions was proved in [3, Proposition 12] for X an infinite-dimensional Banach space. On the other hand, if X has finite dimension, such a v exists if, and only if, the dimension of X is even.

In Section 2 we consider a more general situation. Namely we study, among other things, when every continuous function f from T into $S(X)$ admits another continuous mapping $e : T \to S(X)$ such that

$$e(t) \neq f(t), \quad e(t) \neq -f(t), \quad \forall t \in T.$$

When this occurs, we say that the set $S(T, X)$ is *plentiful*.

This last property is automatic if there exists a continuous mapping v from $S(X)$ into itself without fixed or antipodal points. We will show that there exists a wide class of pairs (T, X) such that $S(T, X)$ is plentiful but X has odd dimension. We will also prove that, when X is a normed space with infinite dimension, $S(T, X)$ is plentiful for every topological space T. As an immediate consequence the existence of continuous mappings v from $S(X)$ into $S(X)$ satisfying $x \neq v(x) \neq -x,\ \forall x \in S(X)$ is obtained, but now without assuming the hypothesis of completeness.

Section 3 is devoted to the study of the geometry of the unit ball of $C(T, X)$ for X strictly convex and $S(T, X)$ plentiful. First we show that, when X is strictly convex, a topological property ($S(T, X)$ is plentiful) is equivalent to a geometric property (every element in the unit ball of $C(T, X)$, omitting the origin, is a convenient convex combination of two extreme points). This fact makes possible to extend a technique introduced in [4] for C^*-algebras to the $C(T, X)$ spaces and so, we can prove that every convex combination of extreme points of $B(Y)$ is a mean of the same number of extreme points.

For each f in Y we define $\alpha(f) = \text{dist}\ (f, Y^{-1})$ where Y^{-1} denotes the set of the functions in Y which omit the origin.

Theorem 14 shows that every f in $B(Y)$ with $\alpha(f) < 1$ can be expressed as a convex combination of extreme points. In fact, for any $\lambda_1, ..., \lambda_n \in]0, 1]$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_k < \frac{1}{2}(1 - \alpha(f))$ for all k, there are extreme points $e_1, ..., e_n$ in $B(Y)$ such that

$$f = \lambda_1 e_1 + \cdots + \lambda_n e_n.$$

The remainder of the section explores the consequences of this theorem. In Corollary 15 we show that each element of the open unit ball of $C(T, X)$ is a mean of n extreme points for some $n \geq 2$. Corollary 17 determines the set of points in $B(Y)$ which are expressible as a convex combination of elements of $E(Y)$. Namely,

$$B(Y) \backslash \text{co}\,(E(Y)) = \{f \in B(Y) : \alpha(f) = 1\}.$$

Theorem 18 provides various equivalent assertions to the possibility of expressing each point of $B(Y)$ as a convex combination of extreme points.

As we have already said most of the known results on the extremal structure of the unit ball of $C(T, X)$ with X strictly convex (see [12], [3] and [11]) only consider the cases $\dim X$ even or infinite. In these papers they get to express every point in the unit ball of $C(T, X)$ as an average of three ([11]) or four ([12], [3]) extreme points by assuming that T is (at least) a completely regular space and $\dim T < \dim X$ (where $\dim T$ denotes the covering dimension of T, see [7] for definitions). Nevertheless in [5] and [10] the general case ($\dim X \geq 2$ arbitrary) is studied, but now every element in $B(Y)$ is expressed as a mean of eight extreme points (with the same condition on the dimensions of T and X). Cantwell conjectured that this number can be improved.

Theorem 18 gives an optimal representation of the points in $B(Y)$ as convex combination (and mean) of three extreme points when $S(T, X)$ is plentiful. This hypothesis includes the cases $\dim X$ even or infinite. Moreover, we give examples of pairs (T, X) with such property, but with $\dim X$ odd. In fact, we have obtained results on a wide class of $C(T, X)$ spaces with $\dim X$ odd (Corollaries 22 and 23). On the other hand, when X is infinite-dimensional, our results do not require the completeness of X (in [3], [10] and [11] X is complete) or the compactness of T (in [12] T is compact).

So, it is clear that our new point of view permits to generalize all the known results on the geometry of the unit ball in $C(T, X)$ spaces with X strictly convex. However, the aforementioned problem of minimal decompositions remains open when $S(T, X)$ is nonplentiful.

2. Sufficient conditions for $S(T, X)$ to be plentiful

Let T be a topological space and X a normed space. For every $f \in S(T, X)$, let us denote

$$E_f = \{e \in S(T, X) : f(t) \neq e(t) \neq -f(t),\ \forall t \in T\}.$$

Observe that if $S(T, X)$ is plentiful, then $E_f \neq \emptyset,\ \forall f \in S(T, X)$.

It is obvious that $f \notin E_f$. However, if $E_f \neq \emptyset$ we have the following result.

LEMMA 1. *Let T be a topological space, X a normed space and $f \in S(T, X)$ such that $E_f \neq \emptyset$. Then $f \in \overline{E_f}$.*

PROOF. Given $\epsilon > 0$, let us consider $\lambda \in]\frac{1}{2}, 1[$ such that $\frac{2(1-\lambda)}{2\lambda - 1} < \epsilon$ and let u be in E_f. Define v on T by

$$v(t) = \frac{\lambda f(t) + (1-\lambda)u(t)}{\|\lambda f(t) + (1-\lambda)u(t)\|}.$$

Clearly v is a continuous function from T into $S(X)$. Now, taking into account that $\|f(t)\| = \|u(t)\| = 1$ for each t in T, we have

$$2\lambda - 1 \leq \|\lambda f(t) + (1-\lambda)u(t)\| \leq 1, \quad \forall t \in T$$

and therefore

$$|\lambda - \|\lambda f(t) + (1-\lambda)u(t)\|| \leq 1 - \lambda, \quad \forall t \in T.$$

Consequently, if t is in T, then

$$\|v(t) - f(t)\| = \left\| \frac{\lambda f(t) + (1-\lambda)u(t)}{\|\lambda f(t) + (1-\lambda)u(t)\|} - f(t) \right\| =$$

$$= \left\| \frac{(\lambda - \|\lambda f(t) + (1-\lambda)u(t)\|)f(t) + (1-\lambda)u(t)}{\|\lambda f(t) + (1-\lambda)u(t)\|} \right\| \leq$$

$$\leq \frac{|\lambda - \|\lambda f(t) + (1-\lambda)u(t)\|| + (1-\lambda)}{\|\lambda f(t) + (1-\lambda)u(t)\|} \leq$$

$$\leq \frac{2(1-\lambda)}{\|\lambda f(t) + (1-\lambda)u(t)\|} \leq \frac{2(1-\lambda)}{2\lambda - 1}.$$

Hence $\|v - f\| \leq \frac{2(1-\lambda)}{2\lambda-1} < \epsilon$. Finally, to see that $v \in E_f$, let us assume, to obtain a contradiction, that there is a $t \in T$ such that $v(t) = f(t)$. Then

$$\|\lambda f(t) + (1-\lambda)u(t)\| f(t) = \lambda f(t) + (1-\lambda)u(t),$$

that is,

$$(-\lambda + \|\lambda f(t) + (1-\lambda)u(t)\|)f(t) = (1-\lambda)u(t) \quad (*).$$

Taking norms it follows that

$$|-\lambda + \|\lambda f(t) + (1-\lambda)u(t)\|| = 1 - \lambda,$$

which implies that

$$-\lambda + \|\lambda f(t) + (1-\lambda)u(t)\| = 1 - \lambda$$

or

$$-\lambda + \|\lambda f(t) + (1-\lambda)u(t)\| = -(1-\lambda).$$

From $(*)$ we get $f(t) = u(t)$ or $-f(t) = u(t)$ which is impossible since $u \in E_f$. So, $v(t) \neq f(t)$ for every t in T.
In the same way it is proved that $v(t) \neq -f(t)$ for each t in T. This completes the proof.

LEMMA 2. *Let T be a topological space, X a normed space and $f \in S(T, X)$ such that $E_f \neq \emptyset$. Then $E_g \neq \emptyset$ for every $g \in S(T, X)$ with $\|g - f\| < 1$.*

PROOF. Let u be in E_f and $g \in S(T, X)$ with $\|g - f\| < 1$. By Lemma 1, there is no loss of generality in assuming that

$$\|u - f\| < 1 - \|g - f\|.$$

Let $e : T \to S(X)$ be the function defined by

$$e(t) = \frac{g(t) + u(t) - f(t)}{\|g(t) + u(t) - f(t)\|}\ , \quad \forall t \in T.$$

Note that if $g(t) + u(t) - f(t) = 0$ for some t in T, then $g(t) - f(t) = -u(t)$ and so $\|g(t) - f(t)\| = 1$ but this can not be. Clearly e is continuous and the proof will be completed if we prove that $g(t) \neq e(t) \neq -g(t)$ for every $t \in T$. For it, let t be in T such that $e(t) = \pm g(t)$.
Taking $\alpha = \|g(t) + u(t) - f(t)\|$, we have $\pm\alpha g(t) = g(t) + u(t) - f(t)$. From here, $(\pm\alpha - 1)g(t) = u(t) - f(t)$ and hence $|\pm\alpha - 1| \in]0, 1[$. Now, for $\lambda = \frac{-1}{\pm\alpha - 1}$, we obtain

$$\|f(t) - g(t)\| = \|f(t) + \lambda(u(t) - f(t))\| \geq ||1 - \lambda| - |\lambda|| = 1$$

and this contradicts our assumption.

It is now clear that the set $\Omega = \{f \in S(T, X) : E_f \neq \emptyset\}$ is open and closed in $S(T, X)$. Therefore it is interesting to clarify when $S(T, X)$ is connected. First an elementary result is given without proof.

LEMMA 3. *Let T be a topological space and X a normed space such that $\dim X \geq 2$. The following statements are equivalent:*

1. *Any two functions f, g in $S(T, X)$ are uniformly homotopic, that is, there is a continuous function $\Phi : [0, 1] \times T \to S(X)$ satisfying*
   (a) $\Phi(0, t) = f(t), \quad \Phi(1, t) = g(t), \quad \forall t \in T.$
   (b) *For every $\epsilon > 0$ there exists $\delta > 0$ such that*
   $$s, s' \in [0, 1], |s - s'| < \delta \quad \Rightarrow \quad \|\Phi(s, t) - \Phi(s', t)\| < \epsilon,\ \forall t \in T.$$
2. *Every function f in $S(T, X)$ is uniformly nullhomotopic.*
3. *$S(T, X)$ is path-connected.*

The following known concept is useful for our aforementioned purpose.

DEFINITION 4. Let E be a metric space and $\epsilon > 0$. E is said to be ϵ-enchained if for any $f, g \in E$ there is a finite sequence $f_0, ..., f_n$ in E with $f_0 = f$ and $f_n = g$ such that $d(f_k, f_{k+1}) < \epsilon$ for all $k \in \{0, ..., n - 1\}$. We will say that E is enchained if E is ϵ-enchained for every $\epsilon > 0$.

THEOREM 5. *Let T and X be as in 3. The following six properties are equivalent:*

1. *Any two functions $f, g \in S(T,X)$ are uniformly homotopic.*
2. *Every function f in $S(T,X)$ is uniformly nullhomotopic.*
3. *$S(T,X)$ is path-connected.*
4. *$S(T,X)$ is connected.*
5. *$S(T,X)$ is enchained.*
6. *$S(T,X)$ is 2-enchained.*

Moreover, any of the above assertions implies that $S(T,X)$ is plentiful.

PROOF. $1 \Leftrightarrow 2 \Leftrightarrow 3$ is the above lemma and $3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6$ hold in every metric space. To prove $6 \Rightarrow 3$ let f, g be in $S(T,X)$. By hypothesis, there exists a finite sequence $f_0, \ldots, f_n$ in $S(T,X)$ with $f_0 = f$ and $f_n = g$ such that $\|f_k - f_{k+1}\| < 2,\ \forall k \in \{0, \ldots, n-1\}$. Then

$$f_k(t) \neq -f_{k+1}(t), \quad \forall t \in T, \quad \forall k \in \{0, \ldots, n-1\}.$$

Let us define $\gamma : [0,1] \to S(T,X)$ by

$$\gamma(s)(t) = \frac{(ns-k)f_{k+1}(t) + (1+k-ns)f_k(t)}{\|(ns-k)f_{k+1}(t) + (1+k-ns)f_k(t)\|}\ , \quad \forall t \in T,$$

$$\forall s \in \left[\frac{k}{n}, \frac{k+1}{n}\right], \quad \forall k \in \{0, \ldots, n-1\}.$$

γ is a path in $S(T,X)$ running from f to g and so we have 3.

Finally, since $\dim X \geq 2$, $\Omega := \{f \in S(T,X) : E_f \neq \emptyset\}$ is nonempty (it contains the constant mappings), and by Lemma 2, Ω is open and closed in $S(T,X)$. If one of the above conditions holds, then $S(T,X)$ is connected. Therefore, $\Omega = S(T,X)$ and so $S(T,X)$ is plentiful.

$S(T,X)$ may be plentiful and not path-connected. For example, if we take $T = S(\mathsf{R}^{2n})$ and $X = \mathsf{R}^{2n}$, then $S(T,X)$ is plentiful (there exists a continuous mapping v from $S(X)$ into itself without fixed or antipodal points) and, however, it is not path-connected by [6, Chap. XVII, Corollary 2.2].

The next results show that there is an extensive range of pairs (T,X) such that $S(T,X)$ is plentiful.

PROPOSITION 6. *Let T be a compact topological space and X a normed space with $\dim X \geq 2$. Assume that one of the following properties holds:*

1. *T is contractible.*
2. *Every $f \in S(T,X)$ is nonsurjective.*

   Then $S(T,X)$ is plentiful.

PROOF. Let f be in $S(T, X)$. If T is contractible there exist t_0 in T and a continuous mapping $\varphi : [0,1] \times T \to T$ such that

$$\varphi(0, t) = t,\ \varphi(1, t) = t_0,\ \forall t \in T.$$

In this case we can consider $x_0 = f(t_0)$ and $\Phi = f \circ \varphi$. On the other hand, if 2 holds then there is $z_0 \in S(X) \backslash f(T)$ and now Φ is defined by

$$\Phi(s,t) = \frac{(1-s)f(t) + sx_0}{\|(1-s)f(t) + sx_0\|}, \quad \forall (s,t) \in [0,1] \times T \quad (x_0 = -z_0).$$

In both cases $\Phi : [0,1] \times T \to S(X)$ is continuous and satisfies that

$$\Phi(0,t) = f(t), \quad \Phi(1,t) = x_0, \quad \forall t \in T.$$

By the compactness of $[0,1] \times T$, Φ is an uniform homotopy and so f is uniformly nullhomotopic. By the previous theorem, $S(T, X)$ is plentiful.

If X is an infinite-dimensional normed space, Y. Benyamini and Y. Sternfeld proved in [2] that the unit sphere of X is Lipschitz contractible. This permits us to obtain the following result.

PROPOSITION 7. *Let X be a normed space with infinite dimension. Then $S(T, X)$ is plentiful for any topological space T.*

PROOF. By [2], there are x_0 in $S(X)$ and a Lipschitz function Γ from $[0,1] \times S(X)$ into $S(X)$ satisfying $\Gamma(0,x) = x,\ \Gamma(1,x) = x_0,\ \forall x \in S(X)$. Hence, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$(s,x),(s',x') \in [0,1] \times S(X),\ |s - s'| + \|x - x'\| < \delta \Rightarrow$$
$$\Rightarrow \|\Gamma(s,x) - \Gamma(s',x')\| < \epsilon.$$

Let T be an arbitrary topological space. Given $f : T \to S(X)$ continuous, consider $\Phi : [0,1] \times T \to S(X)$ defined by

$$\Phi(s,t) = \Gamma(s, f(t)),\ \forall (s,t) \in [0,1] \times T.$$

Evidently Φ is continuous and satisfies:

1. $\Phi(0,t) = \Gamma(0, f(t)) = f(t), \quad \Phi(1,t) = \Gamma(1, f(t)) = x_0, \quad \forall t \in T.$
2. Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$s, s' \in [0,1], \quad |s - s'| < \delta \quad \Rightarrow \quad \|\Phi(s,t) - \Phi(s',t)\| < \epsilon, \quad \forall t \in T.$$

So, every function $f \in S(T, X)$ is uniformly nullhomotopic and $S(T, X)$ is plentiful by Theorem 5.

Our above result permits to prove the following fact which is known for infinite-dimensional Banach spaces [3, Proposition 12] (now the completeness of X is not required).

PROPOSITION 8. *Let X be an infinite-dimensional normed space. Then there is a continuous mapping $v : S(X) \to S(X)$ such that*

$$v(x) \neq x, \quad v(x) \neq -x, \quad \forall x \in S(X).$$

PROOF. It is sufficient to take $T = S(X), f$ the identity function onto $S(X)$ and to apply the preceding proposition.

Let T be a topological space and X a normed space with dim $X \geq 2$. If X is infinite-dimensional $S(A, X)$ is plentiful for any subset A of T by Proposition 7. If X has finite dimension we have the following easy lemma.

LEMMA 9. *Let T be a compact Hausdorff topological space, let X be a finite-dimensional normed space with $\dim X \geq 2$ and assume that $S(T, X)$ is plentiful and $\dim T < \dim X$. Then $S(A, X)$ is plentiful for any closed subset A of T.*

We now need the following topological concept.

For any topological space T, the cone CT over T is the quotient space $(T \times I)/R$, where $I = [0, 1]$ and R is the equivalence relation defined on $T \times I$ by

$$(t, s)R(t', s') \quad \Leftrightarrow \quad (t, s) = (t', s') \quad \text{or} \quad s = s' = 1.$$

Intuitively, CT is obtained from $T \times I$ by pinching $T \times 1$ to a single point. The elements of CT are denoted by $\langle t, s \rangle$. It is trivial to verify that the map $t \mapsto \langle t, 0 \rangle$ is a homeomorphism, so we can identify T with the subspace $\{\langle t, 0 \rangle : t \in T\}$ in CT. Also it is easy to check that if T is compact Hausdorff, then CT is it too. Moreover, CT is always contractible and it is known that if the covering dimension of T is finite, then $\dim CT = \dim T + 1$.

PROPOSITION 10. *Let T be a completely regular topological space and X a finite-dimensional normed space with $\dim X \geq 2$. Each assertion implies the following one:*

1. $\dim T < \dim X - 1$.
2. *$S(\beta(T), X)$ is plentiful where $\beta(T)$ is the Stone-Cech compactification of T.*
3. *$S(T, X)$ is plentiful.*

PROOF. $1 \Rightarrow 2$: Let T and X satisfy 1. By the above remark, $C\beta(T)$ is compact (Hausdorff) and contractible. By Proposition 6, $S(C\beta(T), X)$ is plentiful. Since $\dim C\beta(T) < \dim X$ $(\dim \beta(T) = \dim T$ by [7, Theorem 7.1.17]) and $\beta(T)$ is closed in $C\beta(T)$, $S(\beta(T), X)$ is plentiful by the above lemma.

$2 \Rightarrow 3$: Let f be in $S(T, X)$. Since $S(X)$ is compact there exists a unique continuous mapping $F : \beta(T) \to S(X)$ such that $F(t) = f(t)$, $\forall t \in T$. If 2 holds, there is a $\overline{e}$ in E_F. Then it is clear that the restriction of $\overline{e}$ to T belongs to E_f. Hence $S(T, X)$ is plentiful.

3. The main results

Let Y be a normed space. In [1] Aron and Lohman introduced the λ-function on elements f of $B(Y)$ to be the supremum, $\lambda(f)$, of numbers λ in $[0, 1]$, for which there is a pair (e, g) in $E(Y) \times B(Y)$, such that

$$f = \lambda e + (1 - \lambda)g.$$

The space Y is said to have the λ-property if $\lambda(y) > 0$ for all y in $B(Y)$, and Y has the uniform λ-property if Y verifies the λ-property and, in addition, satisfies

$$\inf\{\lambda(y) : y \in B(Y)\} > 0.$$

A complete study of the λ-property in functions spaces $C(T, X)$ with T a topological space and X a strictly convex normed space was carried out in [8]. Among other things, they got a general expression of the λ-function in these spaces. Namely,

$$\lambda(f) = \frac{1}{2}(1 + m(f) - \alpha(f)), \quad \forall f \in B(Y)$$

where $m(f) = \inf\{\|f(t)\| : t \in T\}$ and $\alpha(f) = \text{dist } (f, Y^{-1})$.

Let T be a topological space and X a normed space. In this section we assume Y denotes the space $C(T, X)$. Moreover, we suppose, unless otherwise stated, that X is strictly convex and $S(T, X)$ is plentiful. First we show that this property on $S(T, X)$ is equivalent to the fact that every function in $Y^{-1} \cap B(Y)$ is a mean of two extreme points of $B(Y)$.

The proof of the "if" half of our next result is similar to the proof of Theorem 4 in [11].

However, for the sake of completeness, we include it.

PROPOSITION 11. *Let T be a topological space and X a strictly convex normed space. The following conditions are equivalent:*

1. *$S(T, X)$ is plentiful.*
2. *For every continuous function h from T into $B(X)$ which omits the origin and, for any λ in $[\frac{1}{2}, \lambda(h)]$, there are extreme points e_1 and e_2 of $B(Y)$ such that*

$$h = \lambda e_1 + (1 - \lambda)e_2.$$

PROOF. $1 \Rightarrow 2$: Let h and λ satisfy the hypotheses of 2. Then it is obvious that $m(h) \geq 2\lambda - 1$ and therefore $\|h(t)\| \geq 2\lambda - 1 = |2\lambda - 1|$, $\forall t \in T$.

If $\lambda = 1$, then $h \in E(Y)$ and we can take $e_1 = e_2 = h$.

Let us suppose $\lambda < 1$. Let f be in $S(T, X)$ defined by $f(t) = \frac{h(t)}{\|h(t)\|}$ for every t in T. By 1, there is an element e in E_f.

Let us define $g : [0, 2] \times T \to X$ by

$$g(s,t) = \begin{cases} (1-s)f(t) + se(t) & \text{if } 0 \leq s \leq 1 \\ (2-s)e(t) - (s-1)f(t) & \text{if } 1 \leq s \leq 2 \end{cases}$$

Then g is continuous and $g(s,t) \neq 0$, $\forall (s,t) \in [0,2] \times T$. We define Γ on $[0,2] \times T$ in the following way

$$\Gamma(s,t) = \frac{g(s,t)}{\|g(s,t)\|}\ , \quad \forall (s,t) \in [0,2] \times T.$$

Evidently Γ is continuous and if we fix t in T, it follows that

$$\left\| \frac{h(t)}{1-\lambda} - \frac{\lambda}{1-\lambda}\Gamma(0,t) \right\| = \frac{\|h(t) - \lambda h(t) \|h(t)\|\|}{1-\lambda} = \frac{|\|h(t)\| - \lambda|}{1-\lambda} \leq 1$$

and

$$\left\| \frac{h(t)}{1-\lambda} - \frac{\lambda}{1-\lambda}\Gamma(2,t) \right\| = \frac{\|h(t) + \lambda h(t) \|h(t)\|\|}{1-\lambda} = \frac{\|h(t)\| + \lambda}{1-\lambda} \geq 1$$

so there is some s in $[0, 2]$ such that

$$\left\| \frac{h(t)}{1-\lambda} - \frac{\lambda}{1-\lambda}\Gamma(s,t) \right\| = 1, \quad (*)$$

that is,

$$\left\| \frac{h(t)}{\lambda} - \Gamma(s,t) \right\| = \frac{1-\lambda}{\lambda}.$$

Now, by [11, Lemma 1], there is only one s for which the above equality $(*)$ holds; if we denote it by $s(t)$, we now claim that the mapping $t \to s(t)$ from T into $[0, 2]$ is continuous. If not, there is a point $t \in T$ and a net $\{t_\nu\}$ converging to t such that $\{s(t_\nu)\} \to s \neq s(t)$. Using the continuity of Γ we find that

$$\left\{ \left\| \frac{h(t_\nu) - \lambda\Gamma(s(t_\nu), t_\nu)}{1-\lambda} \right\| \right\} \to \left\| \frac{h(t) - \lambda\Gamma(s,t)}{1-\lambda} \right\|.$$

So $\left\| \frac{h(t) - \lambda\Gamma(s,t)}{1-\lambda} \right\| = 1$, this contradicts the uniqueness of $s(t)$ and the continuity of $t \to s(t)$ is established.

It is now clear how e_1 and e_2 are to be defined on T

$$e_1(t) = \Gamma(s(t), t), \quad e_2(t) = \frac{h(t) - \lambda\Gamma(s(t), t)}{1-\lambda}\ , \quad \forall t \in T.$$

This completes the proof of the implication $1 \Rightarrow 2$.

$2 \Rightarrow 1$: Let f be in $S(T,X)$. Clearly the function $h = \frac{1}{2}f$ is an element in $B(Y) \cap Y^{-1}$. Applying 2, for $\lambda = \frac{1}{2}$ there exist e_1 and e_2 in $E(Y)$ such that $h = \frac{1}{2}(e_1 + e_2)$. Let us take $e = e_1$ or $e = e_2$. An easy verification shows that e is in E_f.

Our next result is a generalization of Proposition 5 in [11] and can be proved similarly.

Proposition 12. *Let be $u \in E(Y)$, $g \in B(Y)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ such that $\alpha > \beta$, $\alpha + \beta = \gamma + \delta$ and $\gamma, \delta \in [\beta, \alpha]$. Then there exist $e_1, e_2 \in E(Y)$ verifying that*

$$\alpha u + \beta g = \gamma e_1 + \delta e_2.$$

In [9] Kadison and Pedersen proved, by using a very laborious method, that every convex combination of extreme points of the unit ball of a C^*-algebra can be expressed as a mean of the same number of extreme points.

The above proposition permits us to obtain this same conclusion in any $C(T,X)$ space such that X is strictly convex and $S(T,X)$ is plentiful.

Corollary 13. *Each convex combination of extreme points of $B(Y)$ is a mean of the same number of extreme points.*

Proof. The proof is by induction on n. If $n = 2$, let $\alpha e_1 + \beta e_2$ be a convex combination with $e_1, e_2 \in E(Y)$. If $\alpha = \beta = \frac{1}{2}$, then we have the desired conclusion. In other case, we can suppose, without loss of generality, that $\alpha > \beta$. Let be $\gamma = \delta = \frac{1}{2}$. By the above result, there are $u_1, u_2 \in E(Y)$ such that

$$\alpha e_1 + \beta e_2 = \frac{1}{2}(u_1 + u_2).$$

Assume that the property holds for n, we will prove it for $n+1$. Let us consider
$f = \lambda_1 e_1 + \cdots + \lambda_{n+1} e_{n+1}$ $(*)$ with $\lambda_1, ..., \lambda_{n+1}$ in $[0,1]$, $\lambda_1 + \cdots + \lambda_{n+1} = 1$ and $e_1, ..., e_{n+1}$ in $E(Y)$. First let us suppose that some of the λ_i is $\frac{1}{n+1}$. For example, $\lambda_{n+1} = \frac{1}{n+1}$. Then

$$f = (1 - \lambda_{n+1})(\frac{\lambda_1}{1-\lambda_{n+1}} e_1 + \cdots + \frac{\lambda_n}{1-\lambda_{n+1}} e_n) + \lambda_{n+1} e_{n+1}.$$

By the hypotheses of induction, we have

$$\frac{\lambda_1}{1-\lambda_{n+1}} e_1 + \cdots + \frac{\lambda_n}{1-\lambda_{n+1}} e_n = \frac{1}{n}(u_1 + \cdots + u_n)$$

for some $u_1, ..., u_n \in E(Y)$.
It follows that

$$f = (1 - \frac{1}{n+1})\frac{1}{n}(u_1 + \cdots + u_n) + \frac{1}{n+1}e_{n+1},$$

that is,

$$f = \frac{1}{n+1}(u_1 + \cdots + u_n + e_{n+1})$$

which is our assertion.

Let us consider now that λ_i is not $\frac{1}{n+1}$ for $i = 1, ..., n+1$. We can always find λ_i and λ_j such that $\lambda_i < \frac{1}{n+1} < \lambda_j$. For example, $\lambda_n < \frac{1}{n+1} < \lambda_{n+1}$. If we take $\lambda'_{n+1} = \frac{1}{n+1}$ and $\lambda'_n = \lambda_n + \lambda_{n+1} - \frac{1}{n+1}$, it is immediate that

$$\lambda'_{n+1} + \lambda'_n = \lambda_n + \lambda_{n+1} \quad , \quad \lambda'_{n+1}, \lambda'_n \in [\lambda_n, \lambda_{n+1}].$$

By the above proposition, there are $u_n, u_{n+1} \in E(Y)$ such that

$$\lambda_n e_n + \lambda_{n+1} e_{n+1} = \lambda'_n u_n + \lambda'_{n+1} u_{n+1}.$$

By substituting in $(*)$ we have

$$f = \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1} + \lambda'_n u_n + \lambda'_{n+1} u_{n+1}$$

and since $\lambda'_{n+1} = \frac{1}{n+1}$ we can apply the previous argument and the proof is complete.

Let us observe that the Corollary 13 provides, in particular, the aforementioned result by Kadison and Pedersen for commutative C^* -algebras. Now we are ready to prove our main result in this section.

THEOREM 14. *For every $f \in B(Y)$ with $\alpha(f) < 1$ and any $\lambda_1, ..., \lambda_n \in]0, 1]$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_k < \frac{1}{2}(1 - \alpha(f))$ for all k, there are extreme points $e_1, ..., e_n$ in $B(Y)$ such that*

$$f = \lambda_1 e_1 + \cdots + \lambda_n e_n.$$

PROOF. Of course we can suppose that $\lambda_1 \geq \lambda_k$ for every k in $\{1, ..., n\}$. Let be $\lambda'_1 = \lambda_1 + \epsilon$ and $\lambda'_2 = \lambda_2 - \epsilon$ with $\epsilon > 0$ sufficiently small, so that $0 < \lambda'_2$ and $\lambda'_1 < \frac{1}{2}(1 - \alpha(f))$. Evidently $\lambda'_1 < \lambda(f)$ ([8]). By [1, Proposition 1.2.c)], there are $e \in E(Y)$ and $g \in B(Y)$ such that

$$f = \lambda'_1 e + (1 - \lambda'_1)g = \lambda'_1 e + (\lambda'_2 + \lambda_3 + \cdots + \lambda_n)g.$$

Since $\lambda'_1 > \lambda'_2$, we have $\lambda'_1 e + \lambda'_2 g = \lambda'_1 u_2 + \lambda'_2 e'_2$ for some u_2, e'_2 in $E(Y)$ by Proposition 12.
Repeating the argument we find u_3, e_3 in $E(Y)$ such that

$$\lambda'_1 u_2 + \lambda_3 g = \lambda'_1 u_3 + \lambda_3 e_3$$

and after $n-1$ steps we have found extreme points $u_n, e'_2, e_3, ..., e_n$ in $B(Y)$ such that

$$f = \lambda_1' u_n + \lambda_2' e_2' + \lambda_3 e_3 + \cdots + \lambda_n e_n.$$

Now use Proposition 12 on the element $\lambda_1' u_n + \lambda_2' e_2'$ to obtain extreme points e_1, e_2 in $E(Y)$ such that

$$\lambda_1' u_n + \lambda_2' e_2' = (\lambda_1 + \epsilon) u_n + (\lambda_2 - \epsilon) e_2' = \lambda_1 e_1 + \lambda_2 e_2.$$

Inserting this in the above decomposition we have the desired expression.

The above theorem was proved in [4, Theorem 3.3] in case Y is a C^*-algebra.

Taking into account that $\alpha(f) \leq \|f\|$ for each f in Y, from the above theorem we see at once that each element of the open unit ball of Y is a mean of extreme points of $B(Y)$.

Corollary 15. *If f is a element of Y such that $\|f\| < 1$, then there are n extreme points $e_1, ..., e_n$ in $B(Y)$ such that $f = \frac{1}{n}(e_1 + \cdots + e_n)$ for some integer n greater than $\frac{2}{1-\|f\|}$. So, $B(Y)$ is the closed convex hull of $E(Y)$ and Y is the linear expansion of $E(Y)$.*

Let us observe that if one considers $X = \mathbf{C}$ in the above corollary, then we obtain the result by R.R. Phelps [13, Th. 1].

Let f be in $B(Y)$. Let $u(f)$ denote the least integer n such that f is a convex combination of n extreme points in $B(Y)$, $u(f)$ will be called by extremal rank of f. Set $u(f) = \infty$ if f is not expressible as such a convex combination.

In the next result, we relate the extremal rank, $u(f)$, of a element f in the unit ball of Y to the distance, $\alpha(f)$, from f to the set Y^{-1}.

Corollary 16. *For each f in $B(Y)$ and $n \geq 2$, $u(f) \leq n$ implies $\alpha(f) \leq 1 - \frac{2}{n}$ and $\alpha(f) < 1 - \frac{2}{n}$ implies $u(f) \leq n$.*

Proof. Suppose $u(f) \leq n$ with $n \geq 2$. There exist $\lambda_1, ..., \lambda_n$ in $[0,1]$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and $e_1, ..., e_n$ in $E(Y)$ such that $f = \lambda_1 e_1 + \cdots + \lambda_n e_n$.
If $\alpha(f) = 0$ evidently $\alpha(f) \leq 1 - \frac{2}{n}\cdot$
If $\alpha(f) > 0$, then $f \notin Y^{-1}$ and, by applying [8], we have that

$$n \geq \frac{1}{\lambda(f)} = \frac{2}{1 - \alpha(f)}$$

and therefore $\alpha(f) \leq 1 - \frac{2}{n}\cdot$

Conversely if $\alpha(f) < 1 - \frac{2}{n}$, then $\frac{1}{n} < \frac{1}{2}(1 - \alpha(f))$. By Theorem 14, taking $\lambda_k = \frac{1}{n}$ for $k = 1, ..., n$, we see that f is a mean of n elements of $E(Y)$. Thus $u(f) \leq n$.

In Corollary 15 we proved that every point in the open unit ball of Y belongs to $\mathrm{co}(E(Y))$. Now, we see which points in $S(Y)$ are not expressible as a convex combination in $E(Y)$.

COROLLARY 17. $B(Y)\backslash \mathrm{co}(E(Y)) = \{f \in B(Y) : \alpha(f) = 1\}$

PROOF. If $f \in B(Y)\backslash \mathrm{co}(E(Y))$, we have that $\alpha(f) = 1$ by Theorem 14. Conversely, if $f \in \mathrm{co}(E(Y))$ then $u(f) \leq n$ for some n, and, by Corollary 16, it follows that $\alpha(f) < 1$.

In the following theorem we collect all the information about the extremal structure of the unit ball of $C(T, X)$ in case X is strictly convex and $S(T, X)$ is plentiful.

THEOREM 18. *The following conditions are equivalent:*

1. *For every f in $B(Y)$ and for any $\lambda_1, ..., \lambda_n \in]0, \frac{1}{2}[$ with $\lambda_1 + \cdots + \lambda_n = 1$, there are extreme points $e_1, ..., e_n$ in $B(Y)$ such that*

$$f = \lambda_1 e_1 + \cdots + \lambda_n e_n.$$

2. $B(Y) = \frac{E(Y) + \overset{n}{\cdots} + E(Y)}{n}$ *for every $n \geq 3$.*
3. $B(Y) = \mathrm{co}(E(Y))$.
4. $\lambda(f) = \frac{1}{2}(1 + m(f))$ *for every f in $B(Y)$.*
5. *Y has the uniform λ-property.*
6. *Y has the λ-property.*
7. $\alpha(f) < 1$ *for every f in $B(Y)$.*
8. *Y^{-1} is dense in Y.*
9. *(T, X) has the extension property.*

Moreover if we suppose that T is completely regular and X is finite-dimensional with $\dim X \geq 2$, then the conditions $1 - 9$ are equivalent to

10. $\dim T < \dim X$.

The equivalence between the conditions 4 to 10 was established in [8]. On the other hand $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 6$ is obvious and $7 \Rightarrow 1$ follows from Theorem 14.

The equivalence between $3, 5, 6, 9$ and 10 was proved in [10, Corollaries 7 and 9] without assuming that $S(T, X)$ is plentiful, but in this more general case, 2 was only obtained for $n = 8$. On the other hand, in [11] the equivalence between the conditions $1, 2, 3$ and 10 was proved when X is a Banach space and $\dim X$ is an even integer or infinite.

Let us suppose that $\overline{Y^{-1}} \neq Y$. Then, $B(Y) \neq co(E(Y))$ so that $u(f) = \infty$ for some f in $B(Y)$. Moreover, for g in $B(Y)$ with $\alpha(g) = 1$ and $n \geq 3$, set $f = \beta g$ where $1 - \frac{2}{n-1} < \beta < 1 - \frac{2}{n}$. Then f is in $co(E(Y))$ and $\alpha(f) = \beta$, so that $u(f) = n$ by Corollary 16. Clearly $u(0) = 2$ and $u(e) = 1$ for every e in $E(Y)$; so this establishes that

$$\{u(f) : f \in B(Y)\} = \mathbb{N} \cup \{\infty\}.$$

Conversely, if $\overline{Y^{-1}} = Y$ then $u(f) \leq 3$ for every f in $B(Y)$ by Theorem 18.

Since $u(f) = 1$ only for extreme points f in $B(Y)$, $\max\{u(f) : f \in B(Y)\}$ is 2 or 3. In [14], for $C(T, \mathsf{C}$ it was proved that $u(f) \leq 2$, $\forall f \in B(Y)$ if, and only if, T is an F-space and $\dim T \leq 1$. So we have

COROLLARY 19. $\max\{u(f) : f \in B(Y)\}(= \max\{u(f) : f \in \mathrm{co}(E(Y))\})$, *is* 2, 3 *or* ∞.

Taking into account that, when X is an infinite-dimensional normed space, (T, X) has the extension property and $S(T, X)$ is plentiful for every topological space T, we obtain

COROLLARY 20. *Let T be a topological space and X an infinite-dimensional strictly convex normed space. Then*

$$B(Y) = \lambda_1 E(Y) + \cdots + \lambda_n E(Y)$$

for every natural $n \geq 3$ and $\lambda_1, ..., \lambda_n \in]0, \frac{1}{2}[$ with $\lambda_1 + \cdots + \lambda_n = 1$.

Our corollary allows us to get the following interesting result.

COROLLARY 21. *Let X be as in Corollary 20. Then, for each x in $B(X)$ and every $n \geq 3$, there exist $e_1, ..., e_n$ retractions of the unit ball of X onto the unit sphere of X such that*

$$x = \frac{1}{n}(e_1(x) + \cdots + e_n(x)).$$

Corollaries 20 and 21 were obtained in [11] in case X is complete. Moreover Proposition 9 in [11] states that it is not possible to improve on the number three in the last corollary.

If T is contractible and compact, $S(T, X)$ is plentiful for every normed space X with $\dim X \geq 2$ by Proposition 6. So, when X is finite-dimensional, we have the following result.

COROLLARY 22. *Let T be a contractible and compact topological space and X a finite-dimensional strictly convex normed space with $\dim X \geq 2$. Suppose that $\dim T < \dim X$. Then*

$$B(Y) = \lambda_1 E(Y) + \cdots + \lambda_n E(Y)$$

for every natural $n \geq 3$ and $\lambda_1, ..., \lambda_n \in]0, \frac{1}{2}[$ with $\lambda_1 + \cdots + \lambda_n = 1$.

In [11] the same conclusion is obtained by assuming T completely regular and X strictly convex with even dimension. On the other hand, when X has odd dimension it is known (see [10, Corollary 7]) that every element in $B(Y)$ can be expressed as an average of eight extreme points if, and only if, $\dim T < \dim X$.

By using Proposition 10 we improve this result in a particular case.

COROLLARY 23. *Let T be a completely regular space and X a strictly convex normed space with* $\dim X \geq 2$ *(odd or even). If* $\dim T < \dim X - 1$, *then*

$$B(Y) = \lambda_1 E(Y) + \cdots + \lambda_n E(Y)$$

for every natural $n \geq 3$ *and* $\lambda_1, ..., \lambda_n \in]0, \frac{1}{2}[$ *with* $\lambda_1 + \cdots + \lambda_n = 1$.

REFERENCES

1. R.M. Aron and R.H. Lohman, *A geometric function determined by extreme points of the unit ball of a normed space*, Pacific J. Math. 127 (1987), 209–231.
2. Y. Benyamini and Y. Sternfeld, *Spheres in infinite-dimensional normed spaces are Lipschitz contractible*, Proc. Amer. Math. Soc. (88) 3 (1983), 439–445.
3. V.I. Bogachev, J.F. Mena-Jurado and J.C. Navarro-Pascual, *Extreme points in spaces of continuous functions*, Proc. Amer. Math. Soc. 123 (1995), 1061–1067.
4. L.G. Brown and G.K. Pedersen, *Approximation and convex decomposition by extremals in a C^*-algebra*, Math. Scand. 81 (1997), 69–85.
5. J. Cantwell, *A topological approach to extreme points in function spaces*, Proc. Amer. Math. Soc. 19 (1968), 821–825.
6. J. Dugundji, *Topology*, Allyn and Bacon, 1966.
7. R. Engelking, *General Topology*, Heldermann, Berlin (1989).
8. A. Jiménez-Vargas, J.F. Mena-Jurado and J.C. Navarro-Pascual, *Approximation by extreme functions*, Preprint.
9. R.V. Kadison and G.K. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. 57 (1985), 249–266.
10. J.F. Mena-Jurado and J.C. Navarro-Pascual, *The convex hull of extremal vector-valued continuous functions*, Bull. London Math. Soc. 27 (1995), 473–478.
11. J.C. Navarro-Pascual, *Extreme points and retractions in Banach spaces*, Israel J. Math. (1997).
12. N.T. Peck, *Extreme points and dimension theory*, Pacific J. Math. 25 (1968), 341–351.
13. R.R. Phelps, *Extreme points in function algebras*, Duke Math. J. 32 (1965), 267–277.
14. A.G. Robertson, *Averages of extreme points in complex functions spaces*, J. London Math. Soc. (2) 19 (1979), 345–347.

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