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$\mathbf{L}^p$ ESTIMATES FOR MIXED SINGULAR OPERATORS

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Abstract

If we assume that a convolution kernel satisfies certain conditions of cancellation combined with a suitable decay on the Fourier transform side we obtain that the corresponding operator is bounded on $\mathbf{L}^p(\mathsf{R}^n)$ for $1 < p < \infty$. In this paper we prove that allowing slower decay could still imply that it takes $\mathbf{L}^p$ to itself for a smaller set of p's.

1. Introduction

We are interested in convolution operators, i.e. operators given by convolution with a kernel. We will use the notation T_K to denote the operator $f \mapsto f * K$. The kernels will be distributions with compact support, which by scaling and translation invariance we may assume to be the unit ball. In their paper [2] Fefferman and Stein proved that

THEOREM 1. *If K is a distribution with support in the unit ball centered at the origin, locally in $\mathbf{L}^1(\mathsf{R}^n \setminus \{0\})$ and satisfying*

$$\int_{|x| \geq 2|y|^{1-\gamma}} |K(x-y) - K(x)| dx \leq C, \tag{1}$$

where $|y| \leq 1$ and $0 \leq \gamma < 1$, and

$$|\hat{K}(\xi)| \leq |\xi|^{-\frac{n\gamma}{2}} \tag{2}$$

then T_K is bounded on $\mathbf{H}^1$.

If we take $\gamma = 1$ we get the same result without assuming condition (1), see Björk [1].

A natural way to generalize this would be to change the domain of integration in (1) and allow some more general function of ξ in (2), but still assume that they are connected. Miyachi [3] has done this by proving

THEOREM 2. *Let ψ be a positive decreasing continuous function on R, asymptotically homogeneous of some degree $\delta \leq 0$,* i.e.

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$$\lim_{x\mapsto\infty}\frac{\psi(cx)}{\psi(x)}=c^{\delta},\qquad\text{for all } c>0.$$

We also assume that $\psi(r)\le 1$, *when* $r\ge 1$ *and that* $r^{\frac{n}{2}}\psi(r)$ *is increasing. Then if we change* $|y|^{1-\gamma}$ *in (1) to* $\theta(|y|)$ *and* $|\xi|^{-\frac{n\gamma}{2}}$ *in* (2) *to* $\psi(|\xi|)$, *where* $\theta(r)=r\psi(\frac{1}{r})^{-\frac{2}{n}}$, T_K *is bounded on* $\mathbf{H}^1$.

Remark 1. *The last two conditions on* ψ *are equivalent to the assumptions that* $\theta(r)\ge r$, *when* $r\le 1$ *and that* θ *is increasing.*

Another way to generalize Theorem 1 is to change only condition (2), of course we will get a weaker conclusion but T_K could still be bounded on some $\mathbf{L}^p$. This type of generalization has been done by Sjölin [4] who proved the following

Theorem 3. *If we change* γ *in (2) to* α, *where* $\alpha<\gamma$ *and assume* $\gamma<1$ *then* T_K *is bounded on* $\mathbf{L}^p$ *for* $\left|\frac{1}{p}-\frac{1}{2}\right|\le\frac{n\alpha(1-\gamma)+2\alpha}{2n\alpha(1-\gamma)+4\gamma}$.

The next step would be to combine Theorems 2 and 3. In the general case this seems to be rather difficult because in his proof, Sjölin uses a theorem of Fefferman and Stein (see Theorem 5). This is based on the fact that T_y is bounded on $\mathbf{H}^1$ when $\hat{T}_y(\xi)=|\xi|^{iy}$. But if we restrict ourselves to Sjölin's kind of perturbation, then we don't have to worry about this problem. This case also provides some information to the general problem, see remark 4. So in this paper we will prove the following theorem

Theorem 4. *Let* α *and* θ *be two real numbers,* $0<\alpha<\theta<1$. *Set* $a=\frac{n\alpha(1-\theta)+2\theta}{n(1-\theta)+2}$, *so* $\alpha<a<\theta$. *Let* ψ *be a positive decreasing continuous function on* $[0,\infty)$, *such that*

$$\lim_{x\mapsto\infty}\frac{\psi(cx)}{\psi(x)}=1$$

for all $c>0$. *Assume further that* $1\ge\psi(r)$, *when* $r\ge 1$, *that* $r^{\frac{n(1-\theta)}{2}}\psi(r)$ *is increasing and finally that*

$$\psi\left(r^{\frac{1-a}{1-\theta}}\right)\ge A\psi(r),\quad A>0 \tag{3}$$

for $r\ge 1$. *We define a family of functions for* $0<x<1$ *by* $\sigma_x(r)=r^{1-x}\psi\left(\frac{1}{r}\right)^{-\frac{2}{n}}$. *Let* K *be a distribution with support in the unit ball and* $K\in\mathbf{L}^1_{\mathrm{loc}}(\mathsf{R}^n\setminus\{0\})$. *If* K *satisfies the following conditions*

$$\int_{|x|\ge 2\sigma_\theta(|y|)}|K(x-y)-K(x)|dx\le C,\ |y|\le 1 \tag{4}$$

and

$$|\hat{K}(\xi)| \le |\xi|^{-\frac{n\alpha}{2}}\psi(|\xi|)$$

then T_K is bounded on $\mathbf{L}^p$ when

$$\left|\frac{1}{p} - \frac{1}{2}\right| \le \frac{n\alpha(1-\theta) + 2\alpha}{2n\alpha(1-\theta) + 4\theta}.$$

REMARK 2. *The condition $1 \ge \psi(r)$ is equivalent to the condition that $\sigma_x(r) \ge r^{1-x}$. Since ψ is bounded, this condition is not so serious. The condition that $r^{\frac{n(1-\theta)}{2}}\psi(r)$ should be increasing is equivalent to the assumption that σ_θ is increasing and implies that σ_a and σ_α are increasing. The last condition on ψ, (3) is equivalent to the following condition*

$$\sigma_\theta(r^{-(\frac{1-a}{1-\theta})}) \le A^{-\frac{2}{n}}\sigma_a(r^{-1}). \tag{5}$$

Note also that A has to be ≤ 1.

REMARK 3. *The behaviour of the functions $\sigma_x(r)$, when $r > 1$ is of course not important and since we want the operator to be bounded on L^2, we might as well assume that $\hat{K}$ is bounded, which implies that the behaviour of ψ close to the origin is not important either.*

REMARK 4. *In the general case we should change the assumption, $|\hat{K}(\xi)| \le |\xi|^{-\frac{n\alpha}{2}}\psi(|\xi|)$, to the assumption that $|\hat{K}(\xi)| \le |\xi|^{-\frac{n\alpha}{2}}\chi(|\xi|)\psi(|\xi|)$ for some function χ of the same type as ψ. If $\chi(r) \le r^{-\frac{n\epsilon}{2}}$, for every $\epsilon > 0$, then we obtain from our theorem that the operator will be bounded on $\mathbf{L}^p$ where*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{n\alpha(1-\theta) + 2\alpha}{2n\alpha(1-\theta) + 4\theta}.$$

Hence the only difference is that we do not obtain boundedness at the boundary of the strip.

1.1 Examples

The conditions are for example satisfied when $\psi = 1$ which gives Theorem 3. A more interesting example is $\psi(r) = (\ln(2+r))^{-1}$ and $\sigma_\theta(r) = r^{1-\theta}\ln(2+\frac{1}{r})^{\frac{2}{n}}$.

2. Proof

The proof will follow the ideas in Sjölin [4] and consists of two parts, first we prove that convolution with $G_\beta * K$ is bounded on $\mathbf{H}^1$, where G_β is a Bessel potential, i.e. $\hat{G}_\beta(\xi) = (1 + |\xi|^2)^{-\frac{\beta}{2}}$, for $\beta = \frac{n(a-\alpha)}{2}$, then we apply Fefferman and Stein's theorem [2] below, with $\delta = \frac{n\alpha}{2}$ and $\gamma = \beta$, to remove G_β.

THEOREM 5. *Let m be a Fourier multiplier on H^1, such that $|m(\xi)| \le C|\xi|^{-\delta}$, for some $\delta \ge 0$. Then $|\xi|^\gamma m(\xi)$ is a Fourier multiplier for $\mathbf{L}^p$ where $1 < p < \infty$,*

$$\left|\frac{1}{p}-\frac{1}{2}\right| \leq \frac{1}{2}-\frac{\gamma}{2\delta}$$

and $\gamma \geq 0$.

PROOF OF THEOREM 4. We decompose G_β by choosing a function $\phi \in C_0^\infty(\mathsf{R}^n)$, such that $\phi \geq 0$, $\operatorname{supp}\phi \subset \{x; \frac{1}{4} < |x| < 1\}$ and $\sum_{k=0}^{\infty} \phi(2^k x) = 1$ when $0 < |x| < \frac{1}{2}$. Set $\phi_k(x) = \phi(2^k x)$ and $\Phi(x) = \sum_{k=0}^{\infty} \phi_k(x)$. Almost every $\phi_k(x) = 0$ hence we see that $\Phi \in C_0^\infty(\mathsf{R}^n)$. Since $(1-\Phi)G_\beta$ is a Schwartz function we get that $(1-\Phi)G_\beta * K \in \mathbf{L}^1$ so it is trivially bounded on $\mathbf{H}^1$. It is easily seen that $(\Phi G_\beta) * K$ satisfies

$$\widehat{\Phi G_\beta * K}(\xi) \leq |\xi|^{-\frac{na}{2}}\psi(|\xi|).$$

Hence to prove that it is bounded on $\mathbf{H}^1$ it suffices, by Theorem 2 (with $\theta(r) = 2A^{-\frac{2}{n}} r^{1-a}\psi(\frac{1}{r})^{-\frac{2}{n}}$ and $\psi(\mathrm{r}) = \mathrm{r}^{-\frac{na}{2}}\psi(\mathrm{r})$), to prove that

$$\int_{|x| \geq 4A^{-\frac{2}{n}}\sigma_a(|y|)} |K * (\Phi G_\beta)(x-y) - K * (\Phi G_\beta)(x)| dx \leq C, \ |y| \leq 1.$$

We begin by proving some estimates for $G_{\beta,k} = \phi_k G_\beta$. If $|\xi| \leq 2^k$ then

$$\begin{aligned} |\hat{G}_{\beta,k}(\xi)| &\leq \int_{|\eta| \leq 2^{k+1}} |\hat{\phi}_k(\eta)\hat{G}_{\beta,k}(\xi-\eta)| d\eta + \int_{|\eta| \geq 2^{k+1}} |\hat{\phi}_k(\eta)\hat{G}_{\beta,k}(\xi-\eta)| d\eta \\ &\leq 2^{-nk} \int_{|\eta| \leq 2^{k+1}} |\hat{G}_\beta(\eta)| d\eta + 2^{-k\beta} \int_{|\eta| \geq 2^{k+1}} 2^{-kn} |\hat{\phi}(2^{-k}\eta)| d\eta \\ &\leq 2^{-k\beta}. \end{aligned}$$

If $|\xi| \geq 2^k$ we get by repeated partial integration, since $D^\alpha G_\beta(y) \sim |y|^{-n+\beta+|\alpha|}$ when y goes to 0, that

$$|\hat{G}_{\beta,k}(\xi)| \leq \frac{C2^{-k\beta}}{(2^{-k}|\xi|)^N},$$

from which it follows easily that

$$\left(\int_{|\xi| \geq 2^k} |\hat{G}_{\beta,k}|^2 d\xi\right)^{\frac{1}{2}} \leq C2^{-k(\beta-\frac{n}{2})}.$$

This implies

$$\|G_{\beta,k}\|_{\mathbf{L}^2} \leq C2^{-k(\beta-\frac{n}{2})},$$

and hence

$$\|G_{\beta,k}\|_{\mathbf{L}^1} \leq C2^{-k\beta}$$

Let $\delta_k > 2^{-k} + |y|$ and

$$J_{1,k} = \int_{|x|>\delta_k} |G_{\beta,k} * K(x-y) - G_{\beta,k} * K(x)|dx$$

then

$$J_{1,k} \le \|G_{\beta,k}\|_{\mathbf{L}^1} \int_{|x|>\delta_k - 2^{-k}} |K(x-y) - K(x)|dx \tag{6}$$

$$\le C2^{-k\beta} \int_{|x|>\delta_k - 2^{-k}} |K(x-y) - K(x)|dx. \tag{7}$$

If we write

$$K(x-y) - K(x) = \sum_{i=1}^{m} \left(K\left(x - \frac{iy}{m}\right) - K\left(x - \frac{(i-1)y}{m}\right)\right)$$

then using the triangle inequality, condition (6) and choosing m to be the minimal positive integer such that $2\sigma_\theta(\frac{|y|}{m}) < \delta_k - 2^{-k} - |y|$, we obtain

$$J_{1,k} \le C2^{-k\beta} m \le \frac{C2^{-k\beta}|y|}{\sigma_\theta^{-1}\left(\frac{\delta_k - 2^{-k} - |y|}{2}\right)} + C2^{-k\beta}. \tag{8}$$

By Schwarz' inequality the second part

$$J_{2,k} = \int_{|x|\le\delta_k} |G_{\beta,k} * K(x-y) - G_{\beta,k} * K(x)|dx$$

satisfies

$$J_{2,k} \le C\delta_k^{\frac{n}{2}} \|G_{\beta,k} * K\|_{\mathbf{L}^2} = C\delta_k^{\frac{n}{2}} \|\hat{G}_{\beta,k}\hat{K}\|_{\mathbf{L}^2}.$$

We now divide the $\mathbf{L}^2$ norm into two parts

$$\|\hat{G}_{\beta,k}\hat{K}\|_{\mathbf{L}^2} \le \left(\int_{|\xi|\le 2^k} |\hat{G}_{\beta,k}(\xi)\hat{K}(\xi)|^2 d\xi\right)^{\frac{1}{2}} + \left(\int_{|\xi|\ge 2^k} |\hat{G}_{\beta,k}(\xi)\hat{K}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

In the first part we use the estimate above for $\hat{G}_{\beta,k}$

$$\left(\int_{|\xi|\le 2^k} |\hat{G}_{\beta,k}(\xi)\hat{K}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \le 2^{-k\beta}\left(\int_{|\xi|\le 2^k} |\hat{K}(\xi)|^2 d\xi\right)^{\frac{1}{2}},$$

which by our assumption on $\hat{K}$ is

$$\leq 2^{-k\beta}\left(\int_{|\xi|\leq 2^k}|\xi|^{-n\alpha}\psi(|\xi|)^2 d\xi\right)^{\frac{1}{2}}$$

$$= C2^{-k\beta}\left(\int_0^{2^k} s^{n(1-\alpha)-1}\psi(s)^2 ds\right)^{\frac{1}{2}}.$$

To be able to move ψ outside the integral we rewrite the integrand, observing that a simple calculation gives $a-\alpha \geq 1-\theta$,

$$C2^{-k\beta}\left(\int_0^{2^k} s^{n(1-a)-1}s^{n(a-\alpha)}\psi(s)^2 ds\right)^{\frac{1}{2}}$$

$$\leq C2^{-k(\beta-\frac{n(a-\alpha)}{2})}\psi(2^k)\left(\int_0^{2^k} s^{n(1-a)-1}ds\right)^{\frac{1}{2}}$$

$$= C\psi(2^k)2^{\frac{nk(1-a)}{2}}$$

$$= C\sigma_a(2^{-k})^{-\frac{n}{2}}.$$

For the second part we use our earlier obtained estimate (2)

$$\left(\int_{|\xi|\geq 2^k}|\hat{G}_{\beta,k}(\xi)\hat{K}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

$$\leq 2^{-\frac{nk\alpha}{2}}\psi(2^k)\left(\int_{|\xi|\geq 2^k}|\hat{G}_{\beta,k}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

$$\leq 2^{-\frac{nk\alpha}{2}}\psi(2^k)2^{-k\beta+\frac{kn}{2}}$$

$$= \sigma_a(2^{-k})^{-\frac{n}{2}},$$

which shows that

$$J_{2,k} \leq C\delta_k^{\frac{n}{2}}\sigma_a(2^{-k})^{-\frac{n}{2}}. \tag{9}$$

We divide the sum into three cases

2.1 *Case* 1: $2^{-k} < |y|$

Let $\delta_k = 4A^{-\frac{2}{n}}\sigma_a(|y|)$ then using $\sigma_a(|y|) \geq |y|$ we obtain $\delta_k - 2^{-k} - |y| \geq 2A^{-\frac{2}{n}}\sigma_a(|y|)$ so we can use (8)

$$J_{1,k} \leq \frac{C2^{-k\beta}|y|}{\sigma_\theta^{-1}(A^{-\frac{2}{n}}\sigma_a(|y|))} + C2^{-k\beta},$$

which by (5) implies

$$\sum_{2^{-k}<|y|} J_{1,k} \leq C,$$

because $\beta + 1 = \frac{1-a}{1-\theta}$. By our choice of δ_k we don't have to consider $J_{2,k}$.

2.2 *Case* 2: $|y| \leq 2^{-k} < \sigma_\alpha{}^{-1}\sigma_\theta(|y|)$

Set $s = \frac{2}{n(1-\theta)+2}$, $t = \frac{n(1-\theta)}{n(1-\theta)+2}$ and let $\delta_k = 7A^{-\frac{2}{n}}\sigma_\theta(|y|)^s\sigma_\alpha(2^{-k})^t$. Since we are in case 2 it is easy to see, using the upper bound on 2^{-k} and $\sigma_\alpha(2^{-k}) \geq 2^{-k}$, that $\delta_k - 2^{-k} - |y| \geq 2A^{-\frac{2}{n}}\sigma_\theta(|y|)^s\sigma_\alpha(2^{-k})^t$, which means that we can use (8). Thus

$$J_{1,k} \leq \frac{C2^{-k\beta}|y|}{\sigma_\theta{}^{-1}(A^{-\frac{2}{n}}\sigma_\theta(|y|)^s\sigma_\alpha(2^{-k})^t)} + C2^{-k\beta}$$

and since σ_α is increasing, we can estimate this by

$$\begin{aligned} &\leq \frac{C2^{-k\beta}|y|}{\sigma_\theta{}^{-1}(A^{-\frac{2}{n}}\sigma_\theta(|y|)^s\sigma_\alpha(|y|)^t)} + C2^{-k\beta} \\ &\leq \frac{C2^{-k\beta}|y|}{\sigma_\theta{}^{-1}(A^{-\frac{2}{n}}\sigma_a(|y|))} + C2^{-k\beta}. \end{aligned}$$

where we used the easily proved fact that $\sigma_\theta^s\sigma_\alpha^t = \sigma_a$. This implies for the sum over k in case 2

$$\begin{aligned} \sum_{case2} J_{1,k} \quad &\leq \frac{C|y|^{\beta+1}}{\sigma_\theta^{-1}(A^{-\frac{2}{n}}\sigma_a(|y|))} + C \\ &\leq C, \end{aligned}$$

where the last inequality follows from (5). For the second part we obtain by (9)

$$\begin{aligned} J_{2,k} \quad &\leq C\sigma_a(2^{-k})^{-\frac{n}{2}}\delta_k^{\frac{n}{2}} \\ &= C\sigma_a(2^{-k})^{-\frac{n}{2}}\sigma_\theta(|y|)^{\frac{sn}{2}}\sigma_\alpha(2^{-k})^{\frac{tn}{2}}, \end{aligned}$$

as σ_a is increasing this is

$$\leq C\sigma_a(|y|)^{-\frac{n}{2}}\sigma_\theta(|y|)^{\frac{sn}{2}}\sigma_\alpha(2^{-k})^{\frac{tn}{2}}.$$

By definition and the relation $\sigma_\alpha^{-t} = \sigma_\theta^s \cdot \sigma_a$ we obtain

$$\begin{aligned} &= C\sigma_\alpha(|y|)^{-\frac{tn}{2}}(2^{-k})^{\frac{tn(1-\alpha)}{2}}\psi(2^k)^{-t} \\ &\leq C\sigma_\alpha(|y|)^{-\frac{tn}{2}}(2^{-k})^{\frac{tn(1-\alpha)}{2}}\psi\left(\frac{1}{|y|}\right)^{-t}, \end{aligned}$$

hence

$$\sum_{case2} J_{2,k} \leq C\sigma_\alpha(|y|)^{-\frac{nt}{2}}\sigma_\alpha(|y|)^{\frac{nt}{2}} = C.$$

2.3 *Case* 3: ${\sigma_\alpha}^{-1}\sigma_\theta(|y|) \le 2^{-k}$

In this case we take $\delta_k = 4\sigma_\alpha(2^{-k})$. Since $\delta_k - 2^{-k} - |y| \ge 2\sigma_\theta(|y|)$ it follows from (8) and (6) that

$$\sum_{\sigma_\alpha^{-1}\sigma_\theta(|y|)\le 2^{-k}} J_{1,k} \le \sum C2^{-k\beta} \le C.$$

By our definitions $\frac{\sigma_\alpha(r)}{\sigma_a(r)} = r^{\frac{2\beta}{n}}$, hence by (9)

$$\sum_{\sigma_\alpha^{-1}\sigma_\theta(|y|)\le 2^{-k}} J_{2,k} \le \sum C\delta_k^{\frac{n}{2}}\sigma_a(2^{-k})^{-\frac{n}{2}} \le \sum C2^{-k\beta} \le C.$$

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