MATH. SCAND. 84 (1999), 203–212

$L^p - L^q$ MULTIPLIERS ON NON-COMPACT RIEMANNIAN SYMMETRIC SPACES

A. NILSSON

Abstract

In this paper we prove a Hörmander-Michlin type theorem for $L^p - L^q$ multipliers on Riemannian symmetric spaces of the non-compact type. We use the Helgason Fourier transform, the Radon transform and the Abel transform, in particular the support properties of the Abel transform. The kernel is shown to be in L^q at ∞ and locally it satisfies cancellation and boundedness conditions of the usual type.

1. Introduction

In [Hö] Hörmander proved an L^2-version of Michlin's multiplier theorem

THEOREM 1. *If a function m in R^n satisfies for $k \in \mathsf{Z}$*

$$\sum_{|\alpha|=0}^{[\frac{n}{2}]+1} \int_{2^k \leq |\xi| \leq 2^{k+1}} |2^{k\alpha} D^\alpha m(\xi)|^2 d\xi \leq C 2^{nk}, \tag{1}$$

then the operator T_κ acting by convolution with $\kappa = \mathscr{F}^{-1} m$ has weak-type (1,1).

The proof consists in proving that the operator T_κ satisfies the hypothesis of another theorem in that paper:

THEOREM 2. *The conditions*
1) $\int_{|x| \geq 2|y|} |\kappa(x-y) - \kappa(x)| dx \leq C$,
2) $|\hat{\kappa}(\xi)| \leq C$,

implies that the operator T_κ is weak-type (1,1).

This theorem is actually stated for weak-type (1,q). We will use the notation $\mathscr{C}\mathscr{O}_r^s$ for convolution operators taking L^r to L^s.

Received January 14, 1997.

THEOREM 2′. *If*
1′) $\int_{|x|\geq 2|y|} |\kappa(x-y) - \kappa(x)|^q dx \leq C$,

2′) $T_\kappa \in \mathscr{C}\mathscr{O}_r^s(\mathsf{R}^n)$ *for some r, s such that* $\frac{1}{r} - \frac{1}{s} = \frac{1}{q'}$,
then the operator is weak-type (1,q).

This gives rise to the natural question: Is there a condition similar to (1) in Theorem 1 ensuring that the operator in question will be weak-type (1,q)? The answer is yes. (For the proof see [N] where an $\boldsymbol{H}^1 - \boldsymbol{L}^q$-version is proved, in the same way as there condition (1′) could be shown to be true and then (2′) follows by interpolation.)

The history of multipliers on Riemannian Symmetric spaces of the non-compact type goes back about 20 years to [CS] where they showed a Michlin type theorem in the complex case. Later the real case was settled by Anker [A]. His proof is close to the one in [Hö] but there are two major differences:

- The volume of balls grows exponentially.
- One cannot use Bernstein's inequality.

The second is not so serious because, as already Hörmander observed, instead of using Bernstein one could easily give a more direct proof. The first is overcome by assuming that the function m is holomorphically extendable to a certain tube. As was first noted in [CS] this is also a necessary condition. In view of the first part of the introduction it would be natural to try to prove a weak type (1,q) version. In this paper we will obtain such a theorem. The main new problem compared to [A] is that the condition corresponding to 2′ no longer follows trivially from the assumptions. This is the same in R^n and we will use a modified version of the proof in [N], after transferring the problem to R^n using the Radon transform. For more historical background see [A].

ACKNOWLEDGMENT. I would like to thank the director Kjell-Ove Widman and the secretarial staff for making my stay at the Mittag-Leffler institute such a pleasant one. I also want to thank the scientific organizers for arranging the special year of "analysis on Lie groups". Finally I would like to thank Toshiyuki Kobayashi for many valuable comments.

2. Notation

Let G be a real semi-simple, connected, non-compact Lie group with finite center and K a maximal compact subgroup. Set $X = G/K$. We shall denote the Cartan involution by θ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition and $\mathfrak{a}$ a maximal Abelian subspace in $\mathfrak{p}$. Denote the root system of $(\mathfrak{g}, \mathfrak{a})$ by Σ and the associated Weyl group by W. Choose a set Σ^+ of positive roots and set $\rho(H) = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha(H)$ where $m_\alpha = \dim \mathfrak{g}_\alpha$ and $\mathfrak{g}_\alpha$ is the α-root space. We have the following decompositions of G

CARTAN $G = K(\exp \overline{\mathfrak{a}^+})K$

IWASAWA $G = K(\exp \mathfrak{a})N$,

where N is the analytic subgroup of G with Lie algebra $\mathfrak{n} = \oplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$. Set $n = \dim X, a = \dim \mathfrak{a}$ and $b = \dim \mathfrak{n} = n - a$. We also need some function spaces. Let $\mathscr{S}(G/K)$ be the space of functions $f \in \mathscr{C}^\infty(G/K)$ such that

$$\sup_{k_1,k_2\in K, H\in\overline{\mathfrak{a}^+}} \langle H\rangle^r e^{\rho(H)} |L_{D_1} R_{D_2} f(k_1 \exp(H) k_2)| < \infty$$

for all $D_1, D_2 \in \mathfrak{U}(\mathfrak{g})$ and $r \geq 0$. (We have used the notation L_D and R_D to denote left resp. right differentiation and $\langle . \rangle = (1 + |.|^2)^{\frac{1}{2}}$). We will use the following Schwartz spaces $\mathscr{S}(K\backslash G/K) = \mathscr{S}(G/K)^K$, $\mathscr{S}(\mathfrak{a})$ and $\mathscr{S}(\mathfrak{a}^*)$. These spaces are connected by the three transforms

THE SPHERICAL FOURIER TRANSFORM:

$$\mathscr{H}f(\lambda) = \int_G \phi_\lambda(x) f(x)\, dx\,, \quad f \in \mathscr{S}(K\backslash G/K)$$

THE ABEL TRANSFORM:

$$\mathscr{A}f(H) = e^{\rho(H)} \int_N f((\exp H)n)\, dn\,, \quad f \in \mathscr{S}(K\backslash G/K)$$

THE EUCLIDEAN FOURIER TRANSFORM:

$$\mathscr{F}f(\lambda) = \int_{\mathfrak{a}} f(H) e^{i\lambda(H)}\, dH\,, \quad f \in \mathscr{S}(\mathfrak{a})^W.$$

We define some different types of balls:

$$V_r = \{H \in \mathfrak{a} | |H| \leq r\}$$

$$U_r = K(\exp V_r)K$$

$$\mathscr{V}_r = \{H \in \mathfrak{a} | (w.\rho)(H) \leq |\rho| r\,, w \in W\}$$

$$\mathscr{U}_r = K(\exp \mathscr{V}_r)K.$$

For us the fundamental result about these sets is

PROPOSITION 1 (Helgason).
1) $\mathrm{supp}(f) \subset U_r \Leftrightarrow \mathrm{supp}(\mathscr{A}f) \subset V_r$
2) $\mathrm{supp}(f) \subset \mathscr{U}_r \Leftrightarrow \mathrm{supp}(\mathscr{A}f) \subset \mathscr{V}_r$

This proposition implies that a decomposition of f correspond to a decomposition of $\mathscr{A}(f)$.

Let $\boldsymbol{H}_2^{\sigma}(\mathfrak{a}^*)$ be the usual $\boldsymbol{L}^2$- Sobolev spaces and we take $\boldsymbol{H}_{2,\infty}^{\sigma,\tau}$ to be the space of all distributions on $\mathfrak{a}^*$ satisfying

i) $h(\lambda) = \sum_{k=0}^{\infty} h_k(2^{-k}\lambda)$
ii) $h_k \in \boldsymbol{H}_2^{\sigma}$
iii) $\operatorname{supp}(h_0) \subset \{\lambda \in \mathfrak{a}^* | |\lambda| \leq 1\}$
iv) $\operatorname{supp}(h_k) \subset \{\lambda \in \mathfrak{a}^* | \frac{1}{4} \leq |\lambda| \leq 1\}$, $k \geq 1$
v) $\sup(2^{(\tau+\frac{a}{2})k}\|h_k\|_{\boldsymbol{H}_2^{\sigma}}) < \infty$.

With norm given by the infimum of the value in the left hand side of (v) taken over all decompositions of h as in (i). These spaces are treated in more detail in [A]. Let $\mathscr{T}^{\nu} = \mathfrak{a}^* + i\mathrm{Cv}(W.2\nu\rho)$, where Cv denotes convex hull. For $1 \leq q \leq 2$, we define $\nu = \frac{2-q}{2q}$, $\sigma = [n\nu] + 1$ and $\tau = -n\nu + \frac{b}{2}$. (So when $q = 1 : \nu = \frac{1}{2}, \sigma = [\frac{n}{2}] + 1, \tau = -\frac{a}{2}$)

3. Main Part

Theorem 3. *If m is a function $\mathscr{T}^{\nu} \mapsto \mathsf{C}$ satisfying*

1) *m is W-invariant and continuous,*
2) *m is holomorphic in the interior of $\mathscr{T}^{\nu}$,*
3) *m grows at most polynomially,*
4) *$m(. + i2\nu\rho) \in \boldsymbol{H}_{2,\infty}^{\sigma,\tau}$.*

then, with $\kappa = \mathscr{H}^{-1}m$, T_κ is a right convolution operator on X of weak-type (1,q) and furthermore $T_\kappa \in \mathscr{RCO}_r^s(X)$ for all pairs (r,s) such that $\frac{1}{r} - \frac{1}{s} = \frac{1}{q'}$, here $\mathscr{RCO}$ denotes right convolution operators.

Remark 1. The assumptions made on m implies that $m \in \mathscr{H}_{2,\infty}^{\sigma,\tau,\nu}$, for more information about these spaces see [A].

Remark 2. The case $q = 1$ was proved in [A].

Remark 3. Observe that, in contrast with Theorem 1, when we define the spaces $\boldsymbol{H}_{2,\infty}^{\sigma,\tau}$ we assume $k \geq 0$, this is to ensure that the kernel will be in $\boldsymbol{L}^q$ at infinity. This condition is artificial even when $q = 1$ and in the rank one case Giulini, Mauceri and Meda [GMM] have been able to remove this.

Proof. We can, by regularization, assume that $\kappa \in \mathscr{S}(K\backslash G/K)$. Let ψ be identically equal to 1 in $\{|x| \leq \frac{1}{2}\}$ and 0 outside the unit ball. Put $\kappa^0 = \psi\kappa$ and $\kappa^{\infty} = (1-\psi)\kappa$. We shall prove that $\kappa^{\infty} \in \boldsymbol{L}^q$ and then that κ^0 satisfies Hörmander's weak-type (1,q)-conditions for convolution operators on spaces of homogeneous type, i.e. κ^0 satisfies suitably modified versions of the assumptions in $2'$.

3.1. κ^∞-*case*

To prove $\kappa^\infty \in L^q$ it clearly suffices to prove

$$\int_{\frac{1}{2}<|x|<R} |\kappa(x)|^q dx \le C \tag{2}$$

$$\int_{\mathscr{U}_{j+1}\setminus\mathscr{U}_j} |\kappa(x)|^q dx \le Cj^{((a-1)\nu-\sigma)q}. \tag{3}$$

Because if $n\nu = k + r$, where $k = [n\nu]$ then since $a \le \frac{n}{2}$

$$\begin{aligned}\sigma - (a-1)\nu &\ge k + 1 - \frac{k}{2} - \frac{r}{2} + \nu \\ &= \frac{k}{2} + 1 + \nu - \frac{r}{2} \\ &> \frac{k}{2} + \frac{1}{2} + \nu \\ &\ge \frac{q+2-q}{2q} \\ &= \frac{1}{q}.\end{aligned}$$

As in [A] the estimates 2 and 3 follows by using Hölder's inequality to turn them into L^2-estimates and then use Prop 2 to get a corresponding question in R^n. The main point being that on account of the exponential growth of balls the inequality of Hölder introduces exponential factors, which in R^n will precisely give rise to the shift to the edge of the tube on the Fourier transform side.

3.2. κ^0 – *case*

For the second part we prove the following two conditions

- $\int_{|x|\ge 2|y|} |\kappa^0(y^{-1}x) - \kappa^0(x)|^q dx \le C$
- $\kappa^0 \in \mathscr{RCO}_s^r$

for some s and r such that $\frac{1}{s} - \frac{1}{r} = \frac{1}{q'}$.

3.2.1. *Cancellations*

The first condition could be proved as in [A] but the proof of Lemma 15 has to be slightly altered because as it is stated there it is necessary that, in his notation, $\sigma_2 > \max(0, \frac{\tau-1}{2})$ but in our case $\sigma_2 = \sigma$ and $\tau = \frac{b}{2}$ hence his condition might not be satisfied. However this obstacle is easily overcome because if one, in the formula at the top of page 615, moves only $2^{2\sigma_2 j}$ in under the integral sign, a simple argument shows that the proof goes through also in our case.

3.2.2. *Boundedness*
The final step is to prove that $\kappa^0 \in \mathscr{RCO}_s^r(X)$. We will first prove that $\kappa^0 \in \mathscr{RCO}(\boldsymbol{H}^1(U_1), \boldsymbol{L}^q)$. Observe that U_1 is a space of homogeneous type so that $\boldsymbol{H}^1$ and $\boldsymbol{BMO}$ are well defined by [CW2]. This will suffice by the following lemma by Calabi see [AL]

LEMMA 1. *For every $R > 0$ there exists balls $\boldsymbol{B}(x_i, R) = \{x \in G/K | d(x_i, x) \leq R\}$ which cover G/K and intersect uniformly finitely.*

Since $\kappa^\infty \in \boldsymbol{L}^q$ it will be enough, by the cancellation property to show that

$$\left(\int_{|x|\leq 2r} |a * \kappa(x)|^q dx\right)^{\frac{1}{q}} \leq C$$

where a is an atom with support in a ball of radius r contained in U_1. This reduces to

$$\|a * \kappa\|_{\boldsymbol{L}^2} \leq C r^{-\frac{n(2-q)}{2q}}. \tag{4}$$

By Plancherel's formula the square of the left hand side of (9) becomes, with $B = K/M$

$$\int_{\mathfrak{a}_+^* \times B} |\tilde{a}(\lambda, b) m(\lambda)|^2 |\boldsymbol{c}(\lambda)|^{-2} d\lambda db.$$

Here we have used $\tilde{}$ to denote Helgason's Fourier transform, for more information on this transform see [He2]. Let $m = \sum_{j=0}^\infty m_j(2^{-j}.)$ be a decomposition of m as in the definiton of the $\boldsymbol{H}_{2,\infty}^{\sigma,\tau}$ spaces and let $\varDelta_0 = \{|\lambda| \leq 1\}$ and $\varDelta_j = \{\lambda | 2^{j-2} \leq |\lambda| \leq 2^j\}$. Since the m_j's have bounded overlap we can move out the sum and, using Hölder's inequality, we obtain (leaving out the B integral)

$$C \sum \left(\int |m_j(2^{-j}\lambda)|^{q'} |c(\lambda)|^{-2} d\lambda\right)^{\frac{1}{q}} \left(\int_{\varDelta_j} |\tilde{a}(\lambda, b)|^{\frac{2q}{2-q}} |\boldsymbol{c}(\lambda)|^{-2} d\lambda\right)^{\frac{2-q}{q}}$$

$$\leq C \sum \left(\int_{\varDelta_j} |\tilde{a}(\lambda, b)|^{\frac{2q}{2-q}} |\boldsymbol{c}(\lambda)|^{-2} d\lambda\right)^{\frac{2-q}{q}},$$

where the inequality follows because $\kappa^\infty \in \boldsymbol{L}^q$ and in the course of the proof of the cancellation property one proves that the kernel corresponding to $\hat{\psi} * \tilde{m}_j$ also belongs to $\boldsymbol{L}^q$, where $\tilde{m}_j(.) = m_j(2^{-j}.)$. (of course when $q = 2$ we just have the square of the supremum over Δ_j.) When $q = 1$ we are finished because $\|a\|_{L^2} \leq r^{-\frac{n}{2}}$. We shall now prove the $q = 2$ case and then the general case will follow by interpolation. When $q = 2$ we have to prove that

$$\sum \sup_{\Delta_j} |\tilde{a}(\lambda, b)|^2 \leq C.$$

We begin by dividing the sum into two parts

$$J_1^2 = \sum_{2^j > \frac{1}{r}} \sup_{\Delta_j} |\tilde{a}(\lambda, b)|^2,$$

$$J_2^2 = \sum_{2^j \leq \frac{1}{r}} \sup_{\Delta_j} |\tilde{a}(\lambda, b)|^2.$$

To estimate these we use a lemma which allows us to move the situation to the Euclidean setting.

LEMMA 2. *The Radon transform*

$$\mathscr{R}_k f(H) = \int_N f(k \exp(H) n.o) dn$$

takes atoms on X supported in balls of radius r contained in the unit ball to atoms on A with support in balls of radius r. Moreover this is uniform in k.

PROOF. Let f be an atom supported in a ball of radius r. We want to prove the following three things:
1) $\mathscr{R}_k f$ is supported in a ball of radius r,
2) $\int_{\mathfrak{a}} \mathscr{R}_k f(H) dH = 0$,
3) $|\mathscr{R}_k f| \leq C r^{-a}$, C independent of k.

(1) is trivial if one recalls the estimate $|H| \leq c |k \exp(H) n|$ (see [He] chapter VI, Ex B2(iv)),

(2) follows from the decomposition of the integral

$$\int_X f(x) dx = \int_{\mathfrak{a}} \int_N f(k \exp H n.o) dn dH.$$

(3) is a bit more difficult because the original atom only satisfies $|f| \leq C r^{-n}$, so we have to show that the intersection of a horocycle and the support of f has the N-measure $O(r^b)$. This corresponds to the fact that the intersection of a hyperplane and a ball of radius r in R^n will be a ball in the hyperplane with radius $\leq r$.

First we assume that the ball supporting f is centered at the origin. There the situation is K-invariant so we only need to consider the standard family of horocycles, i.e. $k\exp(H)N.o$ with $k = e$. Let V be the inverse image of the set U_1 for the map

$$\mathfrak{a} \times \mathfrak{n} \mapsto X,$$
$$(H, Y) \mapsto \exp(H)\exp(Y)$$

Since $\mathfrak{a} \perp \mathfrak{n}$ we can choose orthonormal bases $\{X_1, \ldots, X_a\}$ in $\mathfrak{a}$ and $\{Y_1, \ldots, Y_b\}$ in $\mathfrak{n}$ respectively, such that the inverse image of the boundary of the ball of radius r centered at the origin is given by

$$g(X, Y) = r^2$$

where g is a function which satisfies, with $X = \sum x_j X_j$ and $Y = \sum y_j Y_j$,

$$g(X, Y) = \sum x_j^2 + \sum y_j^2 + o(\sum x_j^2 + \sum y_j^2).$$

By the implicit function theorem we get that

$$\sqrt{\sum y_j^2} = \sqrt{r^2 - \sum x_j^2} + o(r^2 + \sum x_j^2)$$

As each horosphere of the form $aN.o$ corresponds to constant X in the coordinate (X,Y) and the N-measure corresponds to the Euclidean measure dY, we obtain that the N-measure of the intersection of the horocycle with the ball is $C(r^2 - \sum x_j^2)^{\frac{b}{2}} + o(r^b)$, which is $O(r^b)$, because by (1), $\sum x_j^2 \leq Cr^2$.

In the case when the ball supporting f is not centered at the origin, we will show that we can move it to the origin without changing the result. We can assume that we are considering the standard family of horocycles, because otherwise we can rotate by an element in K which moves the ball, but since we are considering an arbitrary ball it does not matter. Next we rotate the horocycle, moving the center of the ball to the A-orbit $A.o$, this does not change the N-measure. Finally we translate it to the origin which only change the measure by a bounded factor, since we are in U_1.

We know that $\widehat{\mathscr{R}_k a}(\lambda + i\rho) = \tilde{a}(\lambda, b)$. Let ψ be a function in $\mathscr{C}_0^\infty(\mathfrak{a})$ such that ψ is 1 on the support of $\mathscr{R}_k a$ and $\|\psi\|_{L^2} \leq Cr^{\frac{a}{2}}$. Since the Euclidean Fourier transform takes products to convolutions we obtain $\widehat{\mathscr{R}_k a}(\lambda) = \widehat{\mathscr{R}_k a} * \hat{\psi}(\lambda)$. Let $\Delta_j^* = \Delta_{j-1} \cup \Delta_j \cup \Delta_{j+1}$ then $\lambda \in \Delta_j$ and $\eta \notin \Delta_j^*$ implies that $|\lambda - \eta| \geq 2^j$. Hence it is easily seen that

$$\int_{\mathfrak{a}^* \setminus \Delta_j^*} |\hat{\psi}(\lambda + i\rho - \eta)|^2 d\eta \leq Cr^{-2} 2^{-j(a+2)}.$$

By Schwarz' inequality

$$\sum \sup_{\Delta_j} \left| \int_{\mathfrak{a}^* \setminus \Delta_j^*} \widehat{\mathscr{R}_k a}(\eta) \hat{\psi}(\lambda + i\rho - \eta) d\eta \right|^2 \leq C$$

We also have

$$\sum \sup_{\Delta_j} \left| \int_{\Delta_j^*} \widehat{\mathscr{R}_k a}(\eta) \hat{\psi}(\lambda + i\rho - \eta) d\eta \right|^2 \leq C \|\mathscr{R}_k a\|_{\boldsymbol{L}^2}^2 \|\psi\|_{\boldsymbol{L}^2}^2 \leq C.$$

For J_2 we use the wellknown fact that $|\widehat{\mathscr{R}_k a}(\lambda + i\rho)| \leq Cr|\lambda + i\rho|$ (easily proved using (2).) This implies that

$$J_2^2 \leq Cr^2 \sum_{2^j \leq r^{-1}} \sup_{\Delta_j} |\lambda + i\rho|^2 \leq C.$$

For the general case let $p = \frac{2}{1-\theta}$, set $c_j = \|\widehat{\mathscr{R}_k a}\chi_j\|_{\boldsymbol{L}^\infty}$ and $d_j = \|\widehat{\mathscr{R}_k a}\chi_j\|_{\boldsymbol{L}^2}$, then by interpolation

$$\|\widehat{\mathscr{R}_k a}\chi_j\|_{\boldsymbol{L}^p} \leq c_j^\theta d_j^{1-\theta}.$$

Hence

$$\begin{aligned} \|\|\widehat{\mathscr{R}_k a}\chi_j\|_{\boldsymbol{L}^p}\|_{\boldsymbol{l}^2} &\leq \|c_j^\theta d_j^{1-\theta}\|_{\boldsymbol{l}^2} \\ &\leq \|c_j\|_{\boldsymbol{l}^2}^\theta \|d_j\|_{\boldsymbol{l}^2}^{1-\theta} \\ &\leq C(r^{-\frac{n}{2}})^{1-\theta} \\ &= Cr^{-\frac{n}{p}}. \end{aligned}$$

Finally we set $p = \frac{2q}{2-q}$.

By duality it follows that $\kappa^0 \in \mathscr{RCO}(\boldsymbol{L}^{q'}(U_2), \boldsymbol{BMO}(U_1))$. Restricting to U_1 and using the fact that $\kappa^0 \in \mathscr{RCO}_2^2$ we obtain $\kappa^0 \in \mathscr{RCO}(\boldsymbol{L}^{q'}(U_1), \boldsymbol{BMO}(U_2))$. Interpolation hence implies that κ^0 is in $\mathscr{RCO}(\boldsymbol{L}^r(U_1), \boldsymbol{L}^s(G/K))$, where $\frac{1}{r} - \frac{1}{s} = \frac{1}{q'}$. Hence by the Homogeneous type version of Theorem 2′, which can be proved using the techniques developed in [CW1] since the proof in R^n is similar to the proof of Theorem 2, we obtain that κ^0 is also of weak type (1,q). These results combine with the result for κ^∞ showing that $\kappa \in \mathscr{RCO}_r^s(G/K)$ and that it is of weak-type (1,q).

REFERENCES

[A] J-Ph. Anker, *L^p Fourier multipliers on Riemannian symmetric spaces of the noncompact type*, Ann. of Math. 132 (1990), 597–628.

[AL] J-Ph. Anker, and N. Lohoué, *Multiplicateurs sur certain espaces symétriques*, Amer. J. Math. 108 (1986), 1303–1354.

[CS] J-L. Clerc and E.M. Stein, *L^p multipliers for noncompact symmetric spaces*, Proc. Nat. Acad. Sci. USA 71 (1974), 3911–3912.

[CW1] R.R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Math. 242, 1971.

[CW2] R.R. Coifman and G. Weiss, *Extension of Hardy Spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[GMM] S. Giulini, G. Mauceri and S. Meda, *L^p multipliers on noncompact symmetric spaces*, J. Reine Angew. Math. 482 (1997), 151–175.

[He] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press (1978).

[He2] S. Helgason, *Geometric Analysis on Symmetric spaces*, Mathematical Surveys and Monographs, 39, AMS (1994).

[Hö] L. Hörmander, *Estimates for translation invariant operators in L^p-spaces*, Acta Math. 104 (1960), 93–140.

[N] A. Nilsson, *Weakly strongly singular integrals and multipliers*, Ark. Mat. 36 (1998), 379–383.

MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
106 91 STOCKHOLM
SWEDEN