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SHARP $L^p - L^q$ ESTIMATES FOR SINGULAR FRACTIONAL INTEGRAL OPERATORS*

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1. Introduction

Let $Q = [-1,1] \times [-1,1]$, let $\varphi\colon Q \to R$ be a measurable function and let $\gamma_1, \gamma_2 > 0$; suppose μ is the measure on R^3 given by

$$\mu(E) = \int_Q \chi_E(x_1, x_2, \varphi(x_1, x_2))|x_1|^{\gamma_1 - 1}|x_2|^{\gamma_2 - 1} dx_1 dx_2,$$

where $dx_1 dx_2$ denotes the Lebesgue measure on R^2. Let T_μ be the convolution operator defined by $T_\mu f(x) = (\mu * f)(x)$ and let

$$E_\mu = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) : \left\|T_\mu\right\|_{L^p, L^q} < \infty, 1 \le p, q \le \infty \right\}$$

where the L^p-spaces are taken with respect to the Lebesgue measure on R^3. The set E_μ is known in several cases. For $\gamma_1 = \gamma_2 = 1$, if the graph of φ has non zero Gaussian curvature at each point, a theorem of Littman implies that E_μ is the closed triangle with vertices $(0,0)$, (1,1) and $\left(\frac{3}{4}, \frac{1}{4}\right)$ (see [O]). Now, if the curvature vanishes in some point, E_μ can be strictly contained in the above triangle. Related examples in a more general context can be found in [O], [C] and [R-S].

In this paper we study the set E_μ in the case $\varphi(x_1, x_2) = |x_1|^{\alpha_1} + |x_2|^{\alpha_2}$, $\alpha_1, \alpha_2 > 1$ and $0 < \gamma_1, \gamma_2 \le 1$. In [F-G-U] we obtain this characterization for $\gamma_1 = \gamma_2 = 1$.

Throughout this work, c will denote a positive constant not necessarily the same at each occurrence and, without loss of generality we will assume that $\frac{\alpha_1 + 2}{\gamma_1} \le \frac{\alpha_2 + 2}{\gamma_2}$.

In section 2 we find a convex closed polygonal region Σ such that $E_\mu \subset \Sigma$ and we obtain some estimates for the Fourier transform $\widehat{\mu}$. In section 3 we study $L^p - L^{p'}$ estimates for this kind of operators. In section 4 we prove,

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following a suitable extension of the ideas developed by M. Christ in [C], that, if $\frac{1}{3} \leq \gamma_1, \gamma_2 \leq 1$ then $E_\mu = \Sigma$. Also we prove that, if $0 < \gamma_1, \gamma_2 \leq 1$ then the interiors of E_μ and Σ agree.

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2. Auxiliary results

Let Q, φ, μ and E_μ be as in the introduction. The Riesz Thorin theorem implies that E_μ is a convex subset of the square $[0,1] \times [0,1]$. It is well known that if $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\mu$ then $p \leq q$. (See [S-W] p. 33). The above mentioned result due to Oberlin ([O]) implies that E_μ is contained in the closed triangular region with vertices $(0,0)$, $(1,1)$ and $(3/4, 1/4)$. In our particular case we can obtain a more precise statement.

Lemma 2.1. *If* $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\mu$, *then the following inequalities hold*

$$\frac{1}{q} \geq \frac{3}{p} - 2, \quad \frac{1}{q} \geq \frac{2\alpha_1 + 1}{\alpha_1 + 1}\frac{1}{p} - \frac{\alpha_1 + \gamma_1}{\alpha_1 + 1}, \quad \frac{1}{q} \geq \frac{2\alpha_2 + 1}{\alpha_2 + 1}\frac{1}{p} - \frac{\alpha_2 + \gamma_2}{\alpha_2 + 1},$$

$$\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma_1\alpha_2 + \gamma_2\alpha_1}{\alpha_1\alpha_2 + \alpha_1 + \alpha_2}, \quad \frac{1}{q} \geq \frac{1}{p} - \gamma_1, \quad \frac{1}{q} \geq \frac{1}{p} - \gamma_2.$$

Proof. The assertion $\frac{1}{q} \geq \frac{3}{p} - 2$ follows from Theorem 1 in [O]. To see $\frac{1}{q} \geq \frac{2\alpha_1+1}{\alpha_1+1}\frac{1}{p} - \frac{\alpha_1+\gamma_1}{\alpha_1+1}$ we take, for $0 < \delta < 1$, $f = \chi_{Q_\delta}$ where Q_δ is given by $Q_\delta = (-\delta^{\frac{1}{\alpha_1}}, \delta^{\frac{1}{\alpha_1}}) \times (-\delta, \delta) \times (-k\delta, k\delta)$ with $k = 2^{\alpha_1 - 1}\alpha_1 + 2^{\alpha_2 - 1}\alpha_2 + 1$ and we set $A_\delta = \left\{(x_1, x_2, x_3) : |x_1| < \delta^{\frac{1}{\alpha_1}}, \frac{1}{2} < |x_2| < 1, |x_3 - \varphi(x_1, x_2)| < \delta\right\}$. It is easy to see that $x \in A_\delta$ implies $\mu * f(x) \geq c\delta^{1+\frac{\gamma_1}{\alpha_1}}$. Then

$$\|\mu * f\|_q \geq \left(\int_{A_\delta} |\mu * f|^q\right)^{\frac{1}{q}} \geq c\delta^{1+\frac{\gamma_1}{\alpha_1}} |A_\delta|^{\frac{1}{q}} = c\delta^{1+\frac{\gamma_1}{\alpha_1} + \left(1+\frac{1}{\alpha_1}\right)\frac{1}{q}}.$$

Now, $\|\mu * f\|_q \leq c\|f\|_p = c\delta^{\left(2+\frac{1}{\alpha_1}\right)\frac{1}{p}}$ Since these inequalities hold for all small enough δ, the second assertion of the lemma follows. The proof of the third is analogous. To prove the fourth let $Q_\delta = \left(-\delta^{\frac{1}{\alpha_1}}, \delta^{\frac{1}{\alpha_1}}\right) \times \left(-\delta^{\frac{1}{\alpha_2}}, \delta^{\frac{1}{\alpha_2}}\right) \times (-k_1\delta, k_1\delta)$ and let

$$A_\delta = \left\{(x_1, x_2, x_3) : |x_1| < \delta^{\frac{1}{\alpha_1}}, |x_2| < \delta^{\frac{1}{\alpha_2}}, |x_3 - \varphi(x_1, x_2)| < \delta\right\}.$$

It is easy to see that if $k_1 = 1 + 2^{\alpha_1} + 2^{\alpha_2}$, then $\mu * f(x) \geq c\delta^{\frac{\gamma_1}{\alpha_1} + \frac{\gamma_2}{\alpha_2}}$ for $x \in A_\delta$. So, reasoning as above, we obtain the expected inequality. Finally, to see that $\frac{1}{q} \geq \frac{1}{p} - \gamma_1$ we choose $Q_\delta = (-\delta, \delta) \times (-1, 1) \times (-3, 3)$, and

$$A_\delta = \{(x_1, x_2, x_3) : |x_1| < \delta, |x_2| < 1, |x_3 - \varphi(x_1, x_2)| < 1\}.$$

We obtain $\mu * f(x) \geq c\delta^{\gamma_1}$ for $x \in A_\delta$. So as above the result follows. The proof of the last inequality is similar.

We denote by L, L_0, L_{α_k,γ_k}, L_{γ_k}, $(k = 1, 2)$, the lines (in the $(1/p, 1/q)$ plane) given by $\frac{1}{q} = 3\frac{1}{p} - 2$, $\frac{1}{q} = \frac{1}{p} - \frac{\gamma_1\alpha_2+\gamma_2\alpha_1}{\alpha_1\alpha_2+\alpha_1+\alpha_2}$, $\frac{1}{q} = \frac{2\alpha_k+1}{\alpha_k+1}\frac{1}{p} - \frac{\alpha_k+\gamma_k}{\alpha_k+1}$, $\frac{1}{q} = \frac{1}{p} - \gamma_k$ respectively. Also we denote by B_{α_k,γ_k}, $B^{\gamma_j}_{\alpha_k,\gamma_k}$, $B^0_{\alpha_k,\gamma_k}, k = 1, 2$, the intersection of L_{α_k,γ_k} with L, L_{γ_j} and L_0 respectively. We also set A, A_{α_k,γ_k}, A_{γ_k} and A_0 the intersection of the non principal diagonal with L, L_{α_k,γ_k}, L_{γ_k} and L_0 respectively.

A computation shows that $A = (3/4, 1/4)$ and that, for $k = 1, 2$,

$$A_{\alpha_k,\gamma_k} = \left(\frac{2\alpha_k + 1 + \gamma_k}{3\alpha_k + 2}, \frac{\alpha_k + 1 - \gamma_k}{3\alpha_k + 2}\right), \quad A_{\gamma_k} = \left(\frac{1 + \gamma_k}{2}, \frac{1 - \gamma_k}{2}\right)$$

and

$$A_0 = \left(\frac{1}{2} + \frac{\gamma_2\alpha_1 + \gamma_1\alpha_2}{2(\alpha_1 + \alpha_2 + \alpha_1\alpha_2)}, \frac{1}{2} - \frac{\gamma_2\alpha_1 + \gamma_1\alpha_2}{2(\alpha_1 + \alpha_2 + \alpha_1\alpha_2)}\right).$$

Also

$$B_{\alpha_k,\gamma_k} = \left(1 - \frac{\gamma_k}{\alpha_k + 2}, 1 - \frac{3\gamma_k}{\alpha_k + 2}\right),$$

$$B^{\gamma_j}_{\alpha_k,\gamma_k} = \left(1 - \gamma_j + \frac{\gamma_k - \gamma_j}{\alpha_k}, 1 - 2\gamma_j + \frac{\gamma_k - \gamma_j}{\alpha_k}\right),$$

$$B^0_{\alpha_1,\gamma_1} = \left(1 - \frac{\alpha_1\gamma_2 + \gamma_2 - \gamma_1}{\alpha_1 + \alpha_2 + \alpha_1\alpha_2}, 1 - \frac{\alpha_2\gamma_1 + 2\alpha_1\gamma_2 - \gamma_1 + \gamma_2}{\alpha_1 + \alpha_2 + \alpha_1\alpha_2}\right),$$

and

$$B^0_{\alpha_2,\gamma_2} = \left(1 - \frac{\alpha_2\gamma_1 + \gamma_1 - \gamma_2}{\alpha_1 + \alpha_2 + \alpha_1\alpha_2}, 1 - \frac{\alpha_1\gamma_2 + 2\alpha_2\gamma_1 - \gamma_2 + \gamma_1}{\alpha_1 + \alpha_2 + \alpha_1\alpha_2}\right).$$

REMARK 2.2. Lemma 2.1 holds for T^*_μ, taking in the proof -φ instead of φ.

Let $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$ be the closed convex polygonal region contained in Q, given by the intersection of the lower half space determined by the principal diagonal with all the upper half spaces determined by the lines L, L_0, L_{α_k,γ_k}, L_{γ_k}, $(k = 1, 2)$, and all the upper half spaces determined by their symmetric lines with respect to the non principal diagonal. Lemma 2.1, Remark 2.2 and a duality argument say that $E_\mu \subset \Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$. Now we give a more precise description of $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$. Since $\frac{\alpha_1+2}{\gamma_1} \leq \frac{\alpha_2+2}{\gamma_2}$, $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$ is determined only by L, L_0, L_{α_2,γ_2}, L_{γ_2}. Indeed, B_{α_2,γ_2} is closer to $(1, 1)$ than B_{α_1,γ_1} and if the intersection of L_{α_1,γ_1} with L_{α_2,γ_2} belongs to Q then it is discarded either by L or by L_0. Moreover L_0 lies below L_{γ_1} if and only if $\gamma_1(\alpha_2+1) < \gamma_2$, in this case, from $\frac{\alpha_1+2}{\gamma_1} \le \frac{\alpha_2+2}{\gamma_2}$, we also obtain $\gamma_2(\alpha_1+1) < \gamma_1$; adding both inequalities we get a contradiction.

Let us consider the points A, A_{γ_2}, A_{α_2,γ_2}, A_0, on the non principal diagonal. We distinguish the following cases

Case I. A is the highest of these points. This occurs if and only if $\frac{\alpha_2+2}{\gamma_2} \le 4$. In this case $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$ is the triangle with vertices $(0,0),(1,1)$ and A.

Case II. A_{α_2,γ_2} is the highest of these points and $A_{\alpha_2,\gamma_2} \ne A$. This occurs if and only if $\gamma_2 \ge \frac{1}{3}$, $\frac{\alpha_2+2}{\gamma_2} > 4$ and $\alpha_1\alpha_2 + \alpha_1 + \alpha_2 \le \gamma_1(3\alpha_2+2) + \gamma_2(\alpha_1-2)$. Here $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$ is the pentagon with vertices $(1,1)$, $B_{\alpha_2,\gamma_2}, A_{\alpha_2,\gamma_2}$ and their symmetric points with respect to the non principal diagonal.

Case III. A_{γ_2} is the highest of these points and $A_{\gamma_2} \ne A, A_{\gamma_2} \ne A_{\alpha_2,\gamma_2}$. This occurs if and only if $\gamma_2 < \frac{1}{3}$, and $\gamma_2(\alpha_1+1) \le \gamma_1$. Here $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$ is the hexagon with vertices $(1,1)$, $B_{\alpha_2,\gamma_2}, B^{\gamma_2}_{\alpha_2,\gamma_2}$ and their symmetric points with respect to the non principal diagonal.

Case IV. A_0 is the highest of these points, A_0 different from the others and $B_{\alpha_2,\gamma_2} = B^0_{\alpha_2\gamma_2}$. This happens if and only if $\frac{\alpha_1+2}{\gamma_1} = \frac{\alpha_2+2}{\gamma_2} > 4$. Here $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$ is the trapezoid with vertices $(1,1)$, B_{α_2,γ_2} and their symmetric points with respect to the non principal diagonal.

Case V. A_0 is the highest of these points, A_0 different from the others and $B_{\alpha_2,\gamma_2} \ne B^0_{\alpha_2\gamma_2}$. This happens if and only if $\frac{\alpha_2+2}{\gamma_2} > 4$, $\gamma_2(\alpha_1+1) > \gamma_1$ and $\alpha_1\alpha_2 + \alpha_1 + \alpha_2 > \gamma_1(3\alpha_2+2) + \gamma_2(\alpha_1-2)$. Now $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$ is the hexagon with vertices $(1,1)$, B_{α_2,γ_2}, $B^0_{\alpha_2,\gamma_2}$ and their symmetric points with respect to the non principal diagonal.

In order to obtain some estimate for $\widehat{\mu}$, we will need the following lemma, similar to Lemma 2.2 in [R-S].

LEMMA 2.3. *Suppose* $\alpha > 1$, $0 < \mathrm{Re}(\gamma)$, $\xi, \eta \in \Re$.

(i) *If* $\mathrm{Re}(\gamma)/\alpha \le 1/2$ *then*

$$\left| \int_0^1 e^{-i(x\xi + x^\alpha \eta)} x^{\gamma-1} dx \right| \le \frac{c_\alpha(1+|\mathrm{Im}(\gamma)|)}{\mathrm{Re}(\gamma)(1+|\eta|)^{\mathrm{Re}(\gamma)/\alpha}}$$

where c_α *is independent of* ξ, η, γ.

(ii) *If* $\mathrm{Re}(\gamma) < 1/2$ *then*

$$\left| \gamma \int_0^1 e^{-i(x\xi + x^\alpha \eta)} x^{\gamma-1} dx \right| \le \frac{d_\alpha(1+|\mathrm{Im}(\gamma)|)^2}{(1+|\eta|)^{\mathrm{Re}(\gamma)/\alpha}}$$

where d_α *is independent of* ξ, η, γ.

(iii) *If* $\mathrm{Re}(\gamma)/\alpha > 1/2$ *then*

$$\left|\int_0^1 e^{-i(x\xi+x^\alpha\eta)}x^{\gamma-1}dx\right| \leq \frac{e_{\gamma,\alpha}(1+|\mathrm{Im}(\gamma)|)}{(1+|\eta|)^{1/2}}$$

where $e_{\gamma,\alpha}$ depends only on α and $\mathrm{Re}(\gamma)$.
Proof. We can assume that $\eta > 0$. To prove (i) we note that the change of variable $x = \eta^{-\frac{1}{\alpha}}t^{\frac{1}{\mathrm{Re}(\gamma)}}$ gives

$$\int_0^1 e^{-i(x\xi+x^\alpha\eta)}x^{\gamma-1}dx = \frac{\eta^{-i\frac{\mathrm{Im}(\gamma)}{\alpha}}}{\mathrm{Re}(\gamma)\eta^{\frac{\mathrm{Re}(\gamma)}{\alpha}}}\int_0^{\eta^{\frac{\mathrm{Re}(\gamma)}{\alpha}}} e^{-i\left(t^{\frac{\alpha}{\mathrm{Re}(\gamma)}}+t^{\frac{1}{\mathrm{Re}(\gamma)}}\xi\eta^{-\frac{1}{\alpha}}-\frac{\mathrm{Im}(\gamma)}{\mathrm{Re}(\gamma)}\ln(t)\right)}dt.$$

It is enough to prove that, for $a, b \in \Re$, $a > 1$

$$\left|\int_1^a e^{-i\left(t^{\frac{\alpha}{\mathrm{Re}(\gamma)}}+bt^{\frac{1}{\mathrm{Re}(\gamma)}}-\frac{\mathrm{Im}(\gamma)}{\mathrm{Re}(\gamma)}\ln(t)\right)}dt\right| \leq c_\alpha(1+|\mathrm{Im}(\gamma)|)$$

with c_α independent of a, b and γ. Let $s_0 = \max\left\{1, \left(\frac{2|\mathrm{Im}(\gamma)||2\mathrm{Re}(\gamma)-1|}{\alpha(\alpha-2\mathrm{Re}(\gamma)+1)}\right)^{\frac{\mathrm{Re}(\gamma)}{\alpha}}\right\}$. If $a \leq s_0$, then the integral on $[1, a]$ is bounded by $\left(\frac{2|\mathrm{Im}(\gamma)||2\mathrm{Re}(\gamma)-1|}{\alpha(\alpha-2\mathrm{Re}(\gamma)+1)}\right)^{\frac{\mathrm{Re}(\gamma)}{\alpha}}$. If $s_0 \leq a$ the integral on $[1, s_0]$ has the same bound, so it only remains to study

$$\left|\int_{s_0}^a e^{-i\left(t^{\frac{\alpha}{\mathrm{Re}(\gamma)}}+bt^{\frac{1}{\mathrm{Re}(\gamma)}}-\frac{\mathrm{Im}(\gamma)}{\mathrm{Re}(\gamma)}\ln(t)\right)}dt\right|.$$

We define $\Phi : \Re \times (1, +\infty) \to \Re$ by $\Phi(b, t) = t^{\frac{\alpha}{\mathrm{Re}(\gamma)}} + bt^{\frac{1}{\mathrm{Re}(\gamma)}} - \frac{\mathrm{Im}(\gamma)}{\mathrm{Re}(\gamma)}\ln(t)$. Also we set $g_1, g_2 : (1, +\infty) \to \Re$ given by $g_1(t) = t^{\frac{1}{\mathrm{Re}(\gamma)}}$ and $g_2(t) = t^{\frac{\alpha}{\mathrm{Re}(\gamma)}} - \frac{\mathrm{Im}(\gamma)}{\mathrm{Re}(\gamma)}\ln(t)$, then $\Phi(b, t) = bg_1(t) + g_2(t)$. We note that

$$\left[\frac{\partial}{\partial t}\Phi(b,t)\right]^2 + \left[\frac{\partial^2}{\partial t^2}\Phi(b,t)\right]^2 + \left[\frac{\partial^3}{\partial t^3}\Phi(b,t)\right]^2 \neq 0 \tag{2.4}$$

for all $b \in \Re, t > 1$. Otherwise there exist $t_0 > 1$ and $b \in \Re$ such that $\frac{\partial}{\partial t}\Phi(b, t_0) = \frac{\partial^2}{\partial t^2}\Phi(b, t_0) = \frac{\partial^3}{\partial t^3}\Phi(b, t_0) = 0$. Thus $\frac{\partial}{\partial t}_{|t=t_0}\left[t\frac{\partial}{\partial t}\Phi(b, t)\right] = \frac{\partial}{\partial t}_{|t=t_0}\left[t\frac{\partial}{\partial t}\left[t\frac{\partial}{\partial t}\Phi(b, t)\right]\right] = 0$, then

$$\left(\frac{\alpha}{\mathrm{Re}(\gamma)}\right)^2 t_0^{\frac{\alpha}{\mathrm{Re}(\gamma)}} + b\left(\frac{1}{\mathrm{Re}(\gamma)}\right)^2 t_0^{\frac{1}{\mathrm{Re}(\gamma)}} = 0,$$

$$\left(\frac{\alpha}{\mathrm{Re}(\gamma)}\right)^3 t_0^{\frac{\alpha}{\mathrm{Re}(\gamma)}} + b\left(\frac{1}{\mathrm{Re}(\gamma)}\right)^3 t_0^{\frac{1}{\mathrm{Re}(\gamma)}} = 0,$$

Since the only solution of this homogeneous linear system in $t_0^{\frac{\alpha}{\mathrm{Re}(\gamma)}}$, $bt_0^{\frac{1}{\mathrm{Re}(\gamma)}}$ is the trivial one, we obtain (2.4). For a fixed $t>1$, $\left[\frac{\partial}{\partial t}\Phi(b,t)\right]^2 + \left[\frac{\partial^2}{\partial t^2}\Phi(b,t)\right]^2 + \left[\frac{\partial^3}{\partial t^3}\Phi(b,t)\right]^2$ is a quadratic expression on b with a minimum m_t. By (2.4), $m_t \neq 0$. A computation shows that

$$m_t = \left[\frac{\left(g_1'g_2''-g_1''g_2'\right)^2+\left(g_1'g_2'''-g_1'''g_2'\right)^2+\left(g_1''g_2'''-g_1'''g_2''\right)^2}{\left(g_1'\right)^2+\left(g_1''\right)^2+\left(g_1'''\right)^2}\right](t).$$

We note that

$$\left(g_1'(t)\right)^2+\left(g_1''(t)\right)^2+\left(g_1'''(t)\right)^2=$$

$$=\frac{t^{\frac{2-2\mathrm{Re}(\gamma)}{\mathrm{Re}(\gamma)}}P_1(\mathrm{Re}(\gamma))+t^{\frac{2-4\mathrm{Re}(\gamma)}{\mathrm{Re}(\gamma)}}P_2(\mathrm{Re}(\gamma))+t^{\frac{2-6\mathrm{Re}(\gamma)}{\mathrm{Re}(\gamma)}}P_3(\mathrm{Re}(\gamma))}{\mathrm{Re}(\gamma)^6}$$

where $P_j(\mathrm{Re}(\gamma))$, $j=1,2,3$ are polynomials in $\mathrm{Re}(\gamma)$ with $\deg P_j = 4$. Thus there exists $c>0$, c independent of γ, such that the last expression is bounded, for all $t>s_0$, by $ct^{\frac{2-2\mathrm{Re}(\gamma)}{\mathrm{Re}(\gamma)}}\frac{(1+\mathrm{Re}(\gamma))^4}{\mathrm{Re}(\gamma)^6}$. On the other hand

$$\left(g_1'(t)g_2''(t)-g_1''(t)g_2'(t)\right)^2=$$

$$=t^{\frac{2}{\mathrm{Re}(\gamma)}-6}\frac{\left(t^{\frac{\alpha}{\mathrm{Re}(\gamma)}}\alpha(\alpha-2\mathrm{Re}(\gamma)+1)+\mathrm{Im}(\gamma)(2\mathrm{Re}(\gamma)-1)\right)^2}{[\mathrm{Re}(\gamma)]^6}\geq$$

$$\geq t^{\frac{2}{\mathrm{Re}(\gamma)}-6}\frac{\left[t^{\frac{\alpha}{\mathrm{Re}(\gamma)}}\alpha(\alpha-2\mathrm{Re}(\gamma)+1)\right]^2}{4[\mathrm{Re}(\gamma)]6}$$

So, if $t\geq s_0$, then $m_t\geq A_{\gamma,\alpha}$ where $A_{\gamma,\alpha}=\frac{1}{4}\alpha^2\frac{(\alpha-2\mathrm{Re}(\gamma)+1)^2}{(1+\mathrm{Re}(\gamma))^4}$. We note that $\frac{4\alpha^2}{(2+\alpha)^4}\leq A_{\gamma,\alpha}\leq\frac{\alpha^2(\alpha+1)^2}{4}$. Now, let $U_{j,b}=\left\{t>s_0:\left|\frac{\partial^j\Phi}{\partial t^j}(b,t)\right|^2>\frac{A_{\gamma,\alpha}}{4}\right\}$, $j=1,2,3$. Then $U_{j,b}=\bigcup_{k\in K_j}I_{j,b,k}$ for some family $\left\{I_{j,b,k}\right\}_{k\in K_j}$ of disjoint open intervals. Moreover $\frac{\partial^j\Phi}{\partial t^j}(b,t)=\pm\frac{\sqrt{A_{\gamma,\alpha}}}{2}$ if $t\in\partial\left(I_{j,b,k}\right)$. Suppose that the equation $\frac{\partial\Phi}{\partial t}(b,t)=\frac{\sqrt{A_{\gamma,\alpha}}}{2}$ has N solutions $t_1,...,t_N$ in $(1,+\infty)$, then the equation

$$\frac{\alpha}{\mathrm{Re}(\gamma)}\left(\frac{\alpha}{\mathrm{Re}(\gamma)}-1\right)t^{\frac{\alpha}{\mathrm{Re}(\gamma)}}+b\frac{1}{\mathrm{Re}(\gamma)}\left(\frac{1}{\mathrm{Re}(\gamma)}-1\right)t^{\frac{1}{\mathrm{Re}(\gamma)}}+\frac{\mathrm{Im}(\gamma)}{\mathrm{Re}(\gamma)}=0$$

has at least $N-1$ solutions in $(1,+\infty)$. Indeed, since the left side agrees with $t^2\frac{\partial^2\Phi}{\partial t^2}(b,t)$, this assertion follows from Rolle Theorem. So

$$\frac{\alpha}{\mathrm{Re}(\gamma)}\left(\frac{\alpha}{\mathrm{Re}(\gamma)}-1\right)s^{\alpha}+b\frac{1}{\mathrm{Re}(\gamma)}\left(\frac{1}{\mathrm{Re}(\gamma)}-1\right)s+\frac{\mathrm{Im}(\gamma)}{\mathrm{Re}(\gamma)}=0$$

has at least $N-1$ solutions $s_1, ..., s_{N-1}$. Then

$$\frac{\alpha^2}{\mathrm{Re}(\gamma)}\left(\frac{\alpha}{\mathrm{Re}(\gamma)}-1\right)s^{\alpha-1}+b\frac{1}{\mathrm{Re}(\gamma)}\left(\frac{1}{\mathrm{Re}(\gamma)}-1\right)=0$$

has at least $N-2$ solutions. Thus $N \leq 3$. Similarly the equations $\frac{\partial^j \Phi}{\partial t^j}(b,t) = -\frac{\sqrt{A_{\gamma,\alpha}}}{2}$, $j=1,2,3$; have at most 3 solutions on $(1,+\infty)$. Then each $U_{j,b}$ is a union of at most 4 open intervals. Assertion (i) follows from the Van der Corput lemma applied to each $I_{j,b,k}$.

To prove (ii) we first show that

$$\left|\mathrm{Im}(\gamma)\int_0^1 e^{-i(x\xi+x^{\alpha}\eta)}x^{\gamma-1}dx\right| \leq \frac{C'_{\alpha}(1+|\mathrm{Im}(\gamma)|)}{|\eta|^{\mathrm{Re}(\gamma)/\alpha}}, \tag{2.5}$$

where C'_{α} is independent of ξ, η and γ. We can assume that $\mathrm{Im}(\gamma) \neq 0$. Now

$$\int_0^1 e^{-i(x\xi+x^{\alpha}\eta)}x^{\gamma-1}dx = \frac{1}{\eta^{\gamma/\alpha}}\int_0^{\eta^{1/\alpha}} e^{-i\left(x\frac{\xi}{\eta^{1/\alpha}}+x^{\alpha}\right)}x^{\gamma-1}dx.$$

If $\eta \geq 1$, we decompose this last integral as $\int_0^1 + \int_1^{\eta^{1/\alpha}}$. Now $\left|\int_0^1 e^{-i\left(x\frac{\xi}{\eta^{1/\alpha}}+x^{\alpha}\right)}x^{\gamma-1}dx\right| =$

$$= \left|\int_{-\infty}^0 e^{-i\left(et\frac{\xi}{\eta^{1/\alpha}}-\mathrm{Im}(\gamma)\mathrm{t}\right)}e^{\mathrm{Re}(\gamma)\mathrm{t}-\mathrm{i}e^{\alpha\mathrm{t}}}dt\right| = \left|\int_{-\infty}^0 e^{-i\phi(t)}\psi(t)dt\right|,$$

where $\phi(t) = e^t\frac{\xi}{\eta^{1/\alpha}} - \mathrm{Im}(\gamma)t$, $\psi(t) = e^{\mathrm{Re}(\gamma)\mathrm{t}-\mathrm{i}e^{\alpha\mathrm{t}}}$. We use corollary of proposition 2 in ([St], p. 334]) obtaining that if $\frac{|\mathrm{Im}(\gamma)|\eta^{1/\alpha}}{2|\xi|} \geq 1$ then $\left|\int_{-\infty}^0 e^{-i\phi(t)}\psi(t)dt\right| \leq \frac{c}{|\mathrm{Im}(\gamma)|}$, for some positive constant independent of ξ, η, γ. If $\frac{|\mathrm{Im}(\gamma)|\eta^{1/\alpha}}{2|\xi|} < 1$, we decompose the integral over $(-\infty, 0)$ in the sum of the integrals over $(-\infty, M)$ and $(M, 0)$ where $M = \log\left(\frac{|\mathrm{Im}(\gamma)|\eta^{1/\alpha}}{2|\xi|}\right)$. The same corollary gives us now $\left|\int_{-\infty}^M e^{-i\phi(t)}\psi(t)dt\right| \leq \frac{c}{|\mathrm{Im}(\gamma)|}$ and $\left|\int_M^0 e^{-i\phi(t)}\psi(t)dt\right| \leq \frac{c}{|\mathrm{Im}(\gamma)|^{1/2}} \leq \frac{c(1+|\mathrm{Im}(\gamma)|)}{|\mathrm{Im}(\gamma)|}$. The same considerations yields to (ii) in the case $\eta < 1$.

It remains to study $\int_1^{\eta^{1/\alpha}} e^{-i\left(x\frac{\xi}{\eta^{1/\alpha}}+x^\alpha\right)} x^{\mathrm{Re}(\gamma)-1} x^{i\mathrm{Im}(\gamma)} dx$ in the case $\eta > 1$. We write this integral as $\int_1^{\eta^{1/\alpha}} e^{-i\phi(x)}\psi(x)dx$ where $\phi(x) = x^\alpha + \frac{\xi}{\eta^{1/\alpha}}x$ and $\psi(x) = x^{\gamma-1}$. If $\alpha \geq 2$, we apply corollary p.334 in [St] with the second derivative to obtain

$$\left|\int_1^{\eta^{1/\alpha}} e^{-i\phi(x)}\psi(x)dx\right| \leq c(1+|\mathrm{Im}(\gamma)|). \tag{2.6}$$

If $\alpha < 2$, and $\left|\frac{\xi}{\eta^{1/\alpha}}\right| < \frac{\alpha}{2}$, the same corollary, applied with the first derivative, gives us the same bound. If $\left|\frac{\xi}{\eta^{1/\alpha}}\right| \geq \frac{\alpha}{2}$, let $J_1 = \left(-\infty, \left|\frac{\xi}{2\alpha\eta^{1/\alpha}}\right|^{\frac{1}{\alpha-1}}\right)$, $J_2 = \left(\left|\frac{\xi}{2\alpha\eta^{1/\alpha}}\right|^{\frac{1}{\alpha-1}}, \left|\frac{2\xi}{\alpha\eta^{1/\alpha}}\right|^{\frac{1}{\alpha-1}}\right)$ and $J_3 = \left(\left|\frac{2\xi}{\alpha\eta^{1/\alpha}}\right|^{\frac{1}{\alpha-1}}, +\infty\right)$ and let $I_j = J_j \cap \left[1, \eta^{\frac{1}{\alpha}}\right]$, $j = 1, 2, 3$. We decompose

$$\int_1^{\eta^{1/\alpha}} e^{-i\phi(x)}\psi(x)dx = \int_{I_1} + \int_{I_2} + \int_{I_3}.$$

To estimate these integrals we note that $|\phi'(x)| = \left|\alpha x^{\alpha-1} + \frac{\xi}{\eta^{1/\alpha}}\right| \geq \frac{\alpha}{4}$ for $x \in J_1$, $|\phi'(x)| \geq \left|\frac{\xi}{\eta^{1/\alpha}}\right| \geq \frac{\alpha}{2}$ for $x \in J_3$ and $|\phi''(x)| \geq \left|\alpha(\alpha-1)x^{\alpha-2}\right| \geq \alpha(\alpha-1)\left|\frac{2\xi}{\alpha\eta^{1/\alpha}}\right|^{\frac{\alpha-2}{\alpha-1}}$ for $x \in J_2$. We also have

$$\psi\left(\left|\frac{2\xi}{\alpha\eta^{1/\alpha}}\right|^{\frac{1}{\alpha-1}}\right) + \int_{J_2} |\psi'(x)|dx \leq$$

$$\leq c(1+|\mathrm{Im}(\gamma)|)\left|\frac{\xi}{\eta^{1/\alpha}}\right|^{\frac{\mathrm{Re}(\gamma)-1}{\alpha-1}} \leq c'(1+|\mathrm{Im}(\gamma)|)\left|\frac{\xi}{\eta^{1/\alpha}}\right|^{\frac{\alpha-2}{2(\alpha-1)}}.$$

Now we apply the corollary in [St], p. 334, to obtain (2.6) in the case $\alpha < 2$. So (2.5) holds. From (2.5) and (i) we obtain (ii). To prove (iii), we first assume that $\mathrm{Re}(\gamma) \neq 1$. We have

$$\left|\int_0^1 e^{-i(x\xi+x^\alpha\eta)}x^{\gamma-1}dx\right| \leq \sum_{r=0}^{\infty}\left|\int_{2^{-r-1}}^{2^{-r}} e^{-i(x\xi+x^\alpha\eta)}x^{\gamma-1}dx\right|.$$

We apply again the same corollary in [St] to write

$$\left|\int_{2^{-r-1}}^{2^{-r}} e^{-i(x\xi+x^\alpha\eta)}x^{\gamma-1}dx\right| \leq c_\alpha \frac{2^{-r\left(\mathrm{Re}(\gamma)-\frac{\alpha}{2}\right)}}{\eta^{\frac{1}{2}}}\left(1+\frac{|\gamma-1|\left(1-2^{1-\mathrm{Re}(\gamma)}\right)}{\mathrm{Re}(\gamma)-1}\right).$$

So

$$\left|\int_0^1 e^{-i(x\xi+x^\alpha\eta)}x^{\gamma-1}dx\right| \leq c_\alpha \frac{\eta^{-\frac{1}{2}}}{1-2^{\frac{\alpha}{2}-\mathrm{Re}(\gamma)}}\left(1+\frac{|\gamma-1|\left(1-2^{1-\mathrm{Re}(\gamma)}\right)}{\mathrm{Re}(\gamma)-1}\right).$$

Now, since $\frac{1-2^{1-\mathrm{Re}(\gamma)}}{\mathrm{Re}(\gamma)-1}$ tends to $\ln(2)$ as $\mathrm{Re}(\gamma)$ tends to 1, we obtain the case $\mathrm{Re}(\gamma)=1$ from the above, by a limit argument.

3. $L^p - L^{p'}$ estimates

THEOREM 3.1. *The following statements hold*

(i) *If the case* I *occurs, then* $A \in E_\mu$.

(ii) *If the case* II *occurs, then* $A_{\alpha_2,\gamma_2} \in E_\mu$.

(iii) *If either the case* IV *or the case* V *occurs, then* $A_0 \in E_\mu$.

(iv) *If the case* III *occurs, then* $A_{\gamma_2} \in E_\mu$.

PROOF. Let $\gamma_j(z) = 1-\left(1-\gamma_j\right)(1-z)$, $j=1,2$. We set, for $\mathrm{Re}\left(\gamma_j(z)\right)>0$,

$$\mu_z(E) = \int_Q \chi_E(x_1,x_2,\varphi(x_1,x_2))|x_1|^{\gamma_1(z)-1}|x_2|^{\gamma_2(z)-1}dx_1dx_2.$$

For $z \in C$, we consider the analytic family of distibutions I_z, that, for $\mathrm{Re}(z)>0$, are given by $I_z(t) = \frac{2^{\frac{-z}{2}}}{\Gamma\left(\frac{z}{2}\right)}|t|^{z-1}$. We set $J_z = \delta\otimes\delta\otimes I_z$, hence $(J_z)^\wedge = 1\otimes 1\otimes I_{1-z}$. We define the analytic family of operators given by $T_zf = \mu_z * J_z * f$, $f \in S\left(\Re^3\right)$.

To prove (i), we note that, since $\gamma_j > 1/2$, $\mathrm{Re}\gamma_j(z)>0$ for $\mathrm{Re}(z)\in[-1,1]$. It is easy to show that, if $\mathrm{Re}(z)=1$ then $\|T_z\|_{1,\infty} = \|\mu_z * J_z\|_\infty \leq c$. We also observe that if $\mathrm{Re}(z)=-1$, then $\frac{\mathrm{Re}\gamma_j(z)}{\alpha_j}\geq\frac{1}{2}$, $j=1,2$. Then Lemma 2.3, (iii), implies $\left|(\mu_z)^\wedge(y_1,y_2,y_3)\right| \leq c(z)(1+|y_3|)^{-1}$. Then $\|T_z\|_{2,2} \leq \left\|(\mu_z)^\wedge(J_z)^\wedge\right\|_\infty \leq c(z)\frac{2^{\frac{z-1}{2}}}{\Gamma\left(\frac{1-z}{2}\right)}$. It is easy to see that $\{T_z : -1\leq\mathrm{Re}(z)\leq 1\}$ satisfies the hypothesis of the complex interpolation theorem as stated in [S-W], p. 205. Since $T_0 = cT_\mu$, (i) follows.

To prove (ii) we study first the case $\gamma_2 > \frac{1}{3}$. Let $\tau = \frac{1}{2}\frac{\alpha_2+2\gamma_2}{\alpha_2+1-\gamma_2}$, so $\tau>0$. For $-\tau\leq\mathrm{Re}(z)\leq 1$ we have $0<\mathrm{Re}\gamma_j(z)$, $j=1,2$. As in (i), if $\mathrm{Re}(z)=1$ then $\|T_z\|_{1,\infty}\leq c$. If $\mathrm{Re}(z)=-\tau$, then $\mathrm{Re}(\gamma_1(z))>0$ and $\mathrm{Re}(\gamma_2(z))>0$, moreover $\frac{\mathrm{Re}\gamma_1(z)}{\alpha_1}\geq\frac{1}{2}$ and $\frac{\mathrm{Re}\gamma_2(z)}{\alpha_2}<\frac{1}{2}$, so Lemma 2.3 implies

$$\left|(\mu_z)^\wedge(y_1,y_2,y_3)\right| \le c(z)(1+|y_3|)^{-\tau}$$

where $c(z)$ has at most a polynomial growth in $|\mathrm{Im}(z)|$ along the line $\mathrm{Re}(z)=-\tau$. As before, by complex interpolation, (ii) follows in this case.

If $\gamma_2=\frac{1}{3}$ and $\alpha_1\alpha_2+\alpha_1+\alpha_2\le\gamma_1(3\alpha_2+2)+\gamma_2(\alpha_1-2)$, then for $\epsilon>0$ small enough, we think in the pair $(\gamma_{1,\epsilon},\gamma_{2,\epsilon})=(\gamma_1+c\epsilon,\gamma_2+\epsilon)$ instead of (γ_1,γ_2) with $\dot{c}\ge 0$ such that $\alpha_1\alpha_2+\alpha_1+\alpha_2\le\gamma_{1,\epsilon}(3\alpha_2+2)+\gamma_{2,\epsilon}(\alpha_1-2)$. We define as above the corresponding $\gamma_{j,\epsilon}(z)$, $\mu_{z,\epsilon}$ and $T_{z,\epsilon}$. We take now $\tau_\epsilon=\frac{1}{2}\frac{\alpha_2+2\gamma_{2,\epsilon}}{\alpha_2+1-\gamma_{2,\epsilon}}$ and we consider the analytic family of operators $\gamma_{2,\epsilon}(z)T_{z,\epsilon}$ on the strip $-\tau_\epsilon\le\mathrm{Re}(z)\le 1$. As above, but now taking account of (ii) and (iii) of Lemma 2.3, we have that $\left\|\gamma_{2,\epsilon}(z)T_{z,\epsilon}\right\|_{2,2}\le a(z)$ for $\mathrm{Re}(z)=-\tau_\epsilon$ and for all positive and small enough ϵ. Now (ii) follows by complex interpolation and a limit argument.

(iii) Let $\tau=\frac{\gamma_1\alpha_2+\gamma_2\alpha_1}{\alpha_1\alpha_2+(1-\gamma_1)\alpha_2+(1-\gamma_2)\alpha_1}$. Let $\gamma_j(z), j=1,2$, μ_z and T_z be defined as above, but now on the strip $-\tau\le\mathrm{Re}(z)\le 1$. We can check that $\mathrm{Re}\gamma_j(z)>0$, $j=1,2$ on this strip and that $\mathrm{Re}(\gamma_j(z))/\alpha_j<1/2, j=1,2$ if $\mathrm{Re}(z)=-\tau$. For these z, Lemma 2.3 gives us

$$\left|(\mu_z)^\wedge(y_1,y_2,y_3)\right| \le c(z)(1+|y_3|)^{-\tau}$$

for some positive constant $c(z)$. Then $\|T_z\|_{2,2}\le c(z)\frac{2^{\frac{z-1}{2}}}{\Gamma\left(\frac{1-z}{2}\right)}$. Since $\|T_z\|_{1,\infty}\le c$ for $\mathrm{Re}(z)=1$, (iii) follows by complex interpolation.

(iv) To see that $A_{\gamma_2}\in E_\mu$, we set $\gamma_j(z)$, $j=1,2$, μ_z and T_z be defined as above for $-\tau\le\mathrm{Re}(z)\le 1$, where $\tau=\frac{\gamma_2}{1-\gamma_2}$. We note that $\gamma_1(-\tau)>0$ and $\gamma_2(-\tau)=0$. For $\epsilon>0$ small enough, we set $\tau_\epsilon=\frac{\gamma_2-\epsilon}{1-\gamma_2+\epsilon}$, so for $\mathrm{Re}(z)=-\tau_\epsilon$, $\mathrm{Re}(\gamma_1(z))$ and $\mathrm{Re}(\gamma_2(z))$ are positive. We consider the analytic family of operators $\gamma_2(z)T_z$ on the strip $-\tau_\epsilon\le\mathrm{Re}(z)\le 1$. As before, $\|\gamma_2(z)T_z\|_{1,\infty}\le c|\gamma_2(z)|$ if $\mathrm{Re}(z)=1$.

We now consider $\mathrm{Re}(z)=-\tau_\epsilon$. We write $(\mu_z)^\wedge(y_1,y_2,y_3)=$

$$=\int_{-1}^{1} e^{-i(x_1y_1+|x_1|^{\alpha_1}y_3)}|x_1|^{\gamma_1(z)-1}dx_1\int_{-1}^{1} e^{-i(x_2y_2+|x_2|^{\alpha_2}y_3)}|x_2|^{\gamma_2(z)-1}dx_2=\mathscr{I}_1\mathscr{I}_2,$$

Lemma 2.3, (ii) imply that $|\gamma_2(z)\mathscr{I}_2|\le c(1+|\mathrm{Im}(z)|)(1+|y_3|)^{-\frac{\gamma_2(-\tau_\epsilon)}{\alpha_2}}$. If $\frac{\gamma_1(-\tau)}{\alpha_1}\ge\frac{1}{2}$, then $\frac{\gamma_1(-\tau_\epsilon)}{\alpha_1}>\frac{1}{2}$, so Lemma 2.3, (iii) imply that $|\mathscr{I}_1|\le c(1+|\mathrm{Im}(z)|)^2(1+|y_3|)^{-\frac{1}{2}}$. If $\frac{\gamma_1(-\tau)}{\alpha_1}<\frac{1}{2}$, then $\frac{\gamma_1(-\tau_\epsilon)}{\alpha_1}<\frac{1}{2}$, so by Lemma 2.3 (i), $|\mathscr{I}_1|\le c(1+|\mathrm{Im}(z)|)(1+|y_3|)^{-\frac{\gamma_1(-\tau_\epsilon)}{\alpha_1}}$. Moreover in these estimates we can choose c independent of ϵ.

Since case III occurs, we have for ϵ small enough, $\frac{\gamma_1(-\tau_\epsilon)}{\alpha_1}>\tau_\epsilon$ and $\frac{1}{2}>\tau_\epsilon$. Now, $|\mathscr{I}_1|\le c$ with c independent of ϵ, so $|\mathscr{I}_1|\le c(1+|\mathrm{Im}(z)|)^2$

$(1+|y_3|)^{-\tau_\epsilon+\frac{\gamma_2(-\tau_\epsilon)}{\alpha_2}}$, from this we obtain $\left|\gamma_2(z)(\mu_z)^\wedge(y_1,y_2,y_3)\right| \le c(1+|\mathrm{Im}(z)|)^2(1+|y_3|)^{-\tau_\epsilon}$ with c independent of ϵ. Now (iv) follows by complex interpolation and a limit argument.

For $j=1,2$ we consider an even function $\Phi_j \in C_c^\infty(\Re)$, such that $\operatorname{supp}\Phi_j \subset \left\{t\in\Re : 2^{\frac{1}{\alpha_j}} \le |t| \le 2^{\frac{4}{\alpha_j}}\right\}$, $0\le \Phi_j \le 1$ and $\sum_{r\in Z}\Phi_j\left(2^{\frac{r}{\alpha_j}}t\right)=1$ if $t\ne 0$. For $r_1,r_2\in N$, and a Borel set E, we set $\nu_{r_1,r_2}(E)=$

$$= \int \chi_E(x_1,x_2,\varphi(x_1,x_2))\Phi_1\left(2^{\frac{r_1}{\alpha_1}}x_1\right)\Phi_2\left(2^{\frac{r_2}{\alpha_2}}x_2\right)|x_1|^{\gamma_1-1}|x_2|^{\gamma_2-1}dx_1dx_2.$$

For $f\in S(\Re^3)$, let $T_{\nu_{r_1,r_2}}f=\nu_{r_1,r_2}*f$. We observe that $\mu\le\nu=\sum\limits_{r_1,r_2\in N}\nu_{r_1,r_2}$.

LEMMA 3.2. *There exists a positive constant c such that*

$$\left\|T_{\nu_{r_1,r_2}}\right\|_A \le c2^{\frac{r_1}{4\alpha_1}(\alpha_1-4\gamma_1+2)+\frac{r_2}{4\gamma_2}(\alpha_2-4\gamma_2+2)}, r_1,r_2\in N.$$

PROOF. We observe that $\left(\nu_{r_1,r_2}\right)^\wedge(y_1,y_2,y_3)=\mathscr{I}_{1,r_1}(y_1,y_3)\mathscr{I}_{2,r_2}(y_2,y_3)$, where

$$\mathscr{I}_{j,r_j}(y_j,y_3)=\int e^{-i\left(x_jy_j+|x_j|^{\alpha_j}y_3\right)}\Phi_j\left(2^{\frac{r_j}{\alpha_j}}x_j\right)|x_j|^{\gamma_j-1}dx_j,$$

Corollary of the proposition 2 [St. p. 334] gives us

$$\left|\mathscr{I}_{j,r_j}(y_j,y_3)\right| \le c2^{\frac{r_j}{\alpha_j}\left(\frac{\alpha_j-2}{2}+1-\gamma_j\right)}|y_3|^{-\frac{1}{2}}$$

and so

$$\left|\left(\nu_{r_1,r_2}\right)^\wedge(y_1,y_2,y_3)\right| \le c2^{\frac{r_1}{\alpha_1}\left(\frac{\alpha_1-2}{2}+1-\gamma_1\right)+\frac{r_2}{\alpha_2}\left(\frac{\alpha_2-2}{2}+1-\gamma_2\right)}|y_3|^{-1}. \tag{3.3}$$

In a similar way as in theorem 3.1, we define, for $\mathrm{Re}(z)\in[-1,1]$, the analytic family of operators $\{T_z\}$ given by $T_zf=e^{z^2}f*\nu_{r_1,r_2}*J_z$, For $\mathrm{Re}(z)=1$, $\|T_z\|_{1,\infty}\le c2^{\frac{r_1}{\alpha_1}(1-\gamma_1)+\frac{r_2}{\alpha_2}(1-\gamma_2)}$. On the other hand, (3.3) implies that if $\mathrm{Re}(z)=-1$, then $\|T_z\|_{2,2}\le c2^{\frac{r_1}{\alpha_1}\left(\frac{\alpha_1-2}{2}+1-\gamma_1\right)+\frac{r_2}{\alpha_2}\left(\frac{\alpha_2-2}{2}+1-\gamma_2\right)}$. The lemma follows by complex interpolation.

We denote with $\nu_{r_2}^{(1)}=r_1\sum\nu_{r_1,r_2}$ and with $\nu_{r_1}^{(2)}=r_2\sum\nu_{r_1,r_2}$.

LEMMA 3.4. (i) *If* $\gamma_2\ge 1/3$ *and* $\frac{\alpha_2+2}{\gamma_2}>4$, *then*

$$\left\|T_{\nu_{r_1}^{(2)}}\right\|_{A_{\alpha_2,\gamma_2}} \le c2^{\frac{r_1}{\alpha_1}\frac{(\alpha_1-2)(\alpha_2+1-\gamma_2)+(1-\gamma_1)(3\alpha_2+2)}{3\alpha_2+2}}.$$

(ii) *If* $\gamma_1\ge 1/3$ *and* $\frac{\alpha_1+2}{\gamma_1}>4$, *then*

$$\left\| T_{\nu^{(1)}_{r_2}} \right\|_{A_{\alpha_1,\gamma_1}} \leq c2^{\frac{r_2}{\alpha_2}\frac{(\alpha_2-2)(\alpha_1+1-\gamma_1)+(1-\gamma_2)(3\alpha_1+2)}{3\alpha_1+2}}.$$

Proof. To see (i), we define an analytic family of operators, on the strip $-\frac{1}{2}\frac{\alpha_2+2\gamma_2}{\alpha_2+1-\gamma_2} \leq \mathrm{Re}(z) \leq 1$, in the following way. We set $\nu^{(2)}_{r_1,z}(E) =$

$$= \int_{[-1,1]} \int \chi_E(x_1, x_2, \varphi(x_1, x_2)) \Phi_1\left(2^{\frac{r_1}{\alpha_1}} x_1\right) |x_1|^{\gamma_1(z)-1} |x_2|^{\gamma_2(z)-1} dx_1 dx_2$$

with $\gamma_j(z)$ as in theorem 3.1 and $T_z f = e^{z^2} f * \nu_{r_1,z} * J_z$. Now it is easy to show that, if $\mathrm{Re}(z) = 1$ then $\|T_z\|_{1,\infty} \leq c$. To study $\|T_z\|_{2,2}$, for $\mathrm{Re}(z) = -\frac{1}{2}\frac{\alpha_2+2\gamma_2}{\alpha_2+1-\gamma_2}$, we observe that

$$(\nu_{r_1,z})^{\wedge}(y_1, y_2, y_3) = \int e^{-i(x_1 y_1 + |x_1|^{\alpha_1} y_3)} \Phi_1\left(2^{\frac{r_1}{\alpha_1}} x_1\right) |x_1|^{\gamma_1(z)-1} dx_1 \times$$

$$\times \int_{[-1,1]} e^{-i(x_2 y_2 + |x_2|^{\alpha_2} y_3)} |x_2|^{\gamma_2(z)-1} dx_2.$$

Since

$$\left| \int e^{-i(x_1 y_1 + |x_1|^{\alpha_1} y_3)} \Phi_1\left(2^{\frac{r_1}{\alpha_1}} x_1\right) |x_1|^{\gamma_1(z)-1} dx_1 \right| \leq c 2^{\frac{r_1}{\alpha_1}\left(\frac{\alpha_1-2}{2}+1-Re(\gamma_1(z))\right)} |y_3|^{-\frac{1}{2}}$$

the assertion (i) of the lemma follows as in (ii), Theorem 3.1. Part (ii) follows in a similar way.

4. Endpoint bounds

In this section we will characterize E_μ in the case $\frac{1}{3} \leq \gamma_1, \gamma_2 \leq 1$. The use of the Littlewood Paley theory at this point, goes back to [C]. We will also describe the interior of E_μ in the case $0 < \gamma_1, \gamma_2 \leq 1$

For $U \in S'(\Re^2)$ and a test function g we set $U^\vee(g) = U(g^\vee)$, where $g^\vee(y) = g(-y)$. For $g_1 : \Re^2 \to C$ and $g_2 : \Re \to C$ we define

$$(g_1 \otimes_1 g_2)(\xi_1, \xi_2, \xi_3) = g_1(\xi_1, \xi_3) g_2(\xi_2)$$

and

$$(g_1 \otimes_2 g_2)(\xi_1, \xi_2, \xi_3) = g_1(\xi_2, \xi_3) g_2(\xi_1).$$

Also for $U \in S'(\Re^2)$ and $V \in S'(\Re)$ and $k = 1, 2$, we define

$$(U \otimes_k V)(g_1 \otimes_k g_2) = U(g_1) V(g_2).$$

For $1 \leq j \leq 2$, we introduce a C^∞ partition of unity $\{m_{j,r}\}_{r \in Z}$ in $\Re^2$ minus the coordinate axes, with $m_{j,r}$ homogeneous of degree zero (with respect to the Euclidean dilations on $\Re^2$) such that $m_{j,r}(t_1,t_2) = m_{j,0}\big(2^{-\frac{r}{\alpha_j}}t_1, 2^{-r}t_2\big)$ and supp $m_{j,r} \subset \big\{(t_1,t_2) : 2^{\frac{-r}{\alpha_j}-1}|t_1| \le 2^{-r}|t_2| \le 2^{\frac{-r}{\alpha_j}+2}|t_1|\big\}$. We also define $M_{j,r}(\xi_1,\xi_2,\xi_3) = m_{j,r}(\xi_j,\xi_3)$. We put, for $s>0$ and $\xi = (\xi_1,\xi_2,\xi_3) \in \Re^3$, $s \bullet \xi = \big(s^{\frac{1}{\alpha_1}}\xi_1, s^{\frac{1}{\alpha_2}}\xi_{2,} s\xi_3\big)$ and for $(t_1,t_2) \in \Re^2$, $s \bullet_j (t_1,t_2) = \big(s^{\frac{1}{\alpha_j}}t_1, st_2\big)$. For $g : \Re^2 \to C$, $s>0$, we set $\big(s \bullet_j g\big)(t_1,t_2) = g\big(s \bullet_j (t_1,t_2)\big)$, so we have $M_{j,r} = 2^{-r} \bullet M_{j,0}$ and $m_{j,r} = 2^{-r} \bullet_j m_{j,0}$

Let $Q_{j,r}$ be the operator with multiplier $M_{j,r}$, let C_0 be a large constant and define $\widetilde{Q}_{j,r} = \sum_{|i-r|\le C_0} Q_{j,i}$. So $\widetilde{Q}_{j,r}$ is the operator with multiplier $\widetilde{M}_{j,r} = \sum_{|i-r|\le C_0} M_{j,i}$. Let $\widetilde{m}_{j,r} = \sum_{|i-r|\le C_0} m_{j,i}$, so $\widetilde{m}_{j,r} = 2^{-r} \bullet_j \widetilde{m}_{j,0}$. We choose C_0 in such a way that $\widetilde{m}_{j,r} \equiv 1$ on supp $m_{j,r}$.

For $\epsilon_{r_k} = \pm 1$, $\big\{\widetilde{Q}_{k,r_k}\big\}_{r_k\in N}$ satisfies $\big\| \sum_{r_k\in N} \epsilon_{r_k} \widetilde{Q}_{k,r_k}\big\|_{p,p} \le c$, with c independent of $\{\epsilon_{r_k}\}$. Indeed, this follows from the Marcinkiewicz multiplier theorem (see [S], p. 109). As in [S], p. 105, we get the Littlewood Paley inequality

$$\left\| \left(\sum_{r_k\in N} \left| \widetilde{Q}_{k,r_k} f \right|^2 \right)^{1/2} \right\|_p \le c\|f\|_p.$$

By replacing C_0 by a larger constant we may define operators Q'_{k,r_k} with the same properties of the operators $\widetilde{Q}_{k,r_k}$, and such that $Q'_{k,r_k} \circ \widetilde{Q}_{k,r_k} = \widetilde{Q}_{k,r_k}$.

Let $h \in C_0^\infty\big(\Re^2\big)$ be identically one in a neighborhood of the origin, let $H_{j,r}(\xi_1,\xi_2,\xi_3) = h\big(2^{-\frac{r}{\alpha_j}}\xi_j, 2^{-r}\xi_3\big)$ and let $P_{j,r}$ be the Fourier multiplier operator with symbol $H_{j,r.}$ As in [F-G-U], Lemmas 2.4 and 2.5, there exists $\dot{c}>0$ such that, for $R \in N$, $k = 1,2$

$$\left\| \sum_{1\le r_k\le R} T_{\nu_{r_1,r_2}} P_{k,r_k} \right\|_{p,q} \le c \left\| \sum_{1\le r_k\le R} T_{\nu_{r_1,r_2}} \right\|_{p,q} \quad 1<p,q<\infty; \tag{4.1}$$

and

$$\left\| \sum_{1\le r_k\le R} T_{\nu_{r_1,r_2}} \big(I - P_{k,r_k}\big)\Big(I - \widetilde{Q}_{k,r_k}\Big) \right\|_{p,q} \le \tag{4.2}$$

$$\le c \left\| \sum_{1\le r_k\le R} T_{\nu_{r_1,r_2}} \right\|_{p,q} \quad 1<p,q<\infty.$$

Let $\mathscr{I}_{j,r}$ be defined as in the proof of Lemma 2.3. Taking account of proposition 1 in ([St], p. 331), we note that, if C_0 is large enough, then

$$\mathscr{I}_{j,0}(1-h)\big(1-\widetilde{m}_{j,0}\big) \in S\big(\Re^2\big). \tag{4.3}$$

We also have

$$\mathscr{I}_{j,r}\big(1-\widetilde{m}_{j,r}\big)(t_1,t_2) = 2^{-\frac{r}{\alpha_j}\gamma_j}\mathscr{I}_{j,0}\big(1-\widetilde{m}_{j,0}\big)\big(2^{-r}\bullet_j (t_1,t_2)\big).$$

For $R \in N$ we decompose

$$\sum_{1\le r_k\le R} T_{\nu_{r_1,r_2}} = \sum_{1\le r_k\le R} T_{\nu_{r_1,r_2}}P_{k,r_k}+$$

$$+\sum_{1\le r_k\le R} T_{\nu_{r_1,r_2}}\big(I-P_{k,r_k}\big)\Big(I-\widetilde{Q}_{k,r_k}\Big)+\sum_{1\le r_k\le R} T_{\nu_{r_1,r_2}}\big(I-P_{k,r_k}\big)\widetilde{Q}_{k,r_k}.$$

Lemma 4.4. *If* $0<\gamma_1,\gamma_2<1$ *then the kernel of the convolution operator*

$$\sum_{1\le r_2\le R} T_{\nu_{r_1,r_2}}\big(I-P_{2,r_2}\big)\Big(I-\widetilde{Q}_{2,r_2}\Big)$$

belongs to weak-$L^{\frac{\alpha_2+1}{\alpha_2+1-\gamma_2}}$ *with weak constant less than* $c2^{-\frac{r_1\gamma_1}{\alpha_1}}$. *Also*

$$\sum_{1\le r_1\le R} T_{\nu_{r_1,r_2}}\big(I-P_{1,r_1}\big)\Big(I-\widetilde{Q}_{1,r_1}\Big)$$

belongs to weak-$L^{\frac{\alpha_1+1}{\alpha_1+1-\gamma_1}}$ *with weak constant less than* $c2^{-\frac{r_2\gamma_2}{\alpha_2}}$.

Proof. We follow the proof of Lemma 2.6 in [F-G-U] to obtain that the kernel K_{r_1,r_2} of the convolution operator $T_{\nu_{r_1,r_2}}\big(I-P_{2,r_2}\big)\big(I-\widetilde{Q}_{2,r_2}\big)$ satisfies

$$K^{\vee}_{r_1,r_2} = 2^{\frac{r_2}{\alpha_2}(1-\gamma_2)+r_2}(\eta_1\otimes_1\delta)*(2^{r_2}\bullet_2 G_2\otimes_2\delta)$$

where η_1 is the measure defined by $\eta_1(E)=\int\Phi_1\big(2^{\frac{r_1}{\alpha_1}}s\big)\chi_E\big(s,-|s|^{1-\alpha_1}\big)|s|^{\gamma_1-1}ds$ and $G_2=\big(\mathscr{I}_{2,0}(1-h)\big(1-\widetilde{m}_{2,0}\big)\big)^{\wedge}$. We compute this convolution for $f\in S\big(\Re^3\big)$. We get $K^{\vee}_{r_1,r_2}(x_1,x_2,x_3)=$

$$=2^{\frac{r_2}{\alpha_2}(1-\gamma_2)+r_2}(2^{r_2}\bullet_2 G_2)(x_2,x_3+|x_1|^{\alpha_1})\Phi_1\Big(2^{\frac{r_1}{\alpha_1}}x_1\Big)|x_1|^{\gamma_1-1}.$$

So $\sum_{r_2}\big|K^{\vee}_{r_1,r_2}(x_1,x_2,x_3)\big|\le$

$$\le 2^{\frac{r_1}{\alpha_1}(1-\gamma_1)}\chi_{V^2_{r_1}}(x_1,x_2)r_2\sum 2^{\frac{r_2}{\alpha_2}(1-\gamma_2)+r_2}|2^{r_2}\bullet_2 G_2(x_2,x_3+|x_1|^{\alpha_1})|$$

where $V^2_{r_1}=\big\{(x_1,x_2)\in Q : 2^{-\frac{r_1-1}{\alpha_1}}\le|x_1|\le 2^{-\frac{r_1-4}{\alpha_1}}\big\}$. So we obtain

$$\sum_{r_2}\big|K^{\vee}_{r_1,r_2}(x_1,x_2,x_3)\big|\le 2^{\frac{r_1}{\alpha_1}(1-\gamma_1)}\chi_{V^2_{r_1}}(x_1,x_2)(|x_2|^{\alpha_2}+|x_3+|x_1|^{\alpha_1}|)^{\frac{\gamma_2-1}{\alpha_2}-1}.$$

From this we get the first statement of the lemma. The second one is analogous.

In a similar way we obtain

LEMMA 4.5. *If* $0 < \gamma_1, \gamma_2 < 1$, *then the kernel of the convolution operator*

$$\sum_{1 \le r_2 \le R} T_{\nu_{r_1,r_2}} P_{2,r_2}$$

belongs to weak-$L^{\frac{\alpha_2+1}{\alpha_2+1-\gamma_2}}$ *with weak constant less than* $c2^{-\frac{r_1\gamma_1}{\alpha_1}}$. *Also*

$$\sum_{1 \le r_1 \le R} T_{\nu_{r_1,r_2}} P_{1,r_1}$$

belongs to weak-$L^{\frac{\alpha_1+1}{\alpha_1+1-\gamma_1}}$ *and its weak constant is less than* $c2^{-\frac{r_2\gamma_2}{\alpha_2}}$.

REMARK 4.6 To prove the main result we will need Lemma 2.2 in [F-G-U] which we now state

Let $\{\sigma_r\}_{r\in N}$ be a sequence of positive measures on $\Re^{n+1}$, and let $T_r f = \sigma_r * f$, $f \in S(\Re^{n+1})$. Suppose $1 \le k \le n$, $1 < p \le 2$ and $p \le q < \infty$. If there exists $A > 0$ such that $\sup_{r\in N} \|T_r\|_{p,q} \le A$, $\left\| \sum_{1\le r\le R} T_r P_{k,r} \right\|_{p,q} \le A$ and $\left\| \sum_{1\le r\le R} T_r (I - P_{k,r})(I - \widetilde{Q}_{k,r}) \right\|_{p,q} \le A$ for all $R \in N$, then there exists $c > 0$, c independent of A, R and $\{\sigma_r\}_{r\in N}$, such that $\left\|\sum_{1\le r\le R} T_r\right\|_{p,q} \le cA$.

THEOREM 4.7. *If* $\frac{1}{3} \le \gamma_1,\ \gamma_2 \le 1$ *and* $\frac{\alpha_1+2}{\gamma_1} \le \frac{\alpha_2+2}{\gamma_2}$ *then* E_μ *is the closed convex polygonal region* $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$.

PROOF. We will prove the theorem for each one of the cases described in paragraph 2. In the case I the theorem follows from (i), Theorem 3.1. In the case II, taking account of (ii), Theorem 3.1, it is enough to check that $B_{\alpha_2,\gamma_2} \in E_\mu$. In the case IV we must only show that $B_{\alpha_2,\gamma_2} \in E_\mu$. In the case V we must prove that B_{α_2,γ_2} and $B^0_{\alpha_2,\gamma_2} \in E_\mu$.

Case II: Since $\mu \le \nu = \sum_{r_1,r_2\in N} \nu_{r_1,r_2}$, we will prove that B_{α_2,γ_2} belong to E_ν.

Lemma 3.2, the estimate $\left\|T_{\nu_{r_1,r_2}}\right\|_{1,1} \le c2^{-\frac{\gamma_1}{\alpha_1}r_1 - \frac{\gamma_2}{\alpha_2}r_2}$ and Riesz-Thorin interpolation theorem yield us to $\left\|T_{\nu_{r_1,r_2}}\right\|_{B_{\alpha_2,\gamma_2}} \le c2^{\frac{r_1}{\alpha_1}\left(\frac{\gamma_2(\alpha_1+2)-\gamma_1(\alpha_2+2)}{\alpha_2+2}\right)}$. Lemmas 4.4, 4.5 and the weak Young's inequality imply that the operators $\sum_{1\le r_2\le R} T_{\nu_{r_1,r_2}} P_{2,r_2}$ and $\sum_{1\le r_2\le R} T_{\nu_{r_1,r_2}} (I - P_{2,r_2})(I - \widetilde{Q}_{2,r_2})$ are of weak type $\left(1, \frac{\alpha_2+1}{\alpha_2+1-\gamma_2}\right)$, then (i) in Lemma 3.4, (4.1), (4.2), the Marcinkiewicz interpolation theorem (see [B-S], Remark 4.15, (d)) and a brief computation show that there exists $c > 0$, such that for $R \in N$

$$\left\| \sum_{1 \le r_2 \le R} T_{\nu_{r_1,r_2}} P_{2,r_2} \right\|_{B_{\alpha_2,\gamma_2}} \le c 2^{\frac{r_1}{\alpha_1}\left(\frac{\gamma_2(\alpha_1+2)-\gamma_1(\alpha_2+2)}{\alpha_2+2}\right)}$$

and

$$\left\| \sum_{1 \le r_2 \le R} T_{\nu_{r_1,r_2}} (I - P_{2,r_2}) \left(I - \widetilde{Q}_{2,r_2}\right) \right\|_{B_{\alpha_2,\gamma_2}} \le c 2^{\frac{r_1}{\alpha_1}\left(\frac{\gamma_2(\alpha_1+2)-\gamma_1(\alpha_2+2)}{\alpha_2+2}\right)}.$$

Remark 4.6 implies

$$\left\| \sum_{1 \le r_2 \le R} T_{\nu_{r_1,r_2}} \right\|_{B_{\alpha_2,\gamma_2}} \le c 2^{\frac{r_1}{\alpha_1}\left(\frac{\gamma_2(\alpha_1+2)-\gamma_1(\alpha_2+2)}{\alpha_2+2}\right)}. \tag{4.8}$$

Since we are in case II, we can perform the sum on r_1, to obtain the theorem, in this case.

Case V: As in case II we obtain that $B_{\alpha_2,\gamma_2} \in E_\mu$. (i) in Lemma 3.4, (4.8) and the Riesz Thorin theorem give

$$\left\| \sum_{1 \le r_2 \le R} T_{\nu_{r_1,r_2}} \right\|_{B^0_{\alpha_2,\gamma_2}} \le c, \tag{4.9}$$

with c independent of r_1 and R.

Now, Lemmas 4.4, 4.5 and the weak Young's inequality imply that the operators $\sum_{1 \le r_1 \le R} T_{\nu_{r_1,r_2}} P_{1,r_1}$ and $\sum_{1 \le r_1 \le R} T_{\nu_{r_1,r_2}} (I - P_{1,r_1})(I - \widetilde{Q}_{1,r_1})$ are of weak type $\left(1, \frac{\alpha_1+1}{\alpha_1+1-\gamma_1}\right)$, with weak constant $2^{-\frac{r_2\gamma_2}{\alpha_2}}$. Also (4.1), (4.2) and (ii) Lemma 3.4 imply that they have $\|.\|_{A_{\alpha_1,\gamma_1}}$ less than $c 2^{\frac{r_2}{\alpha_2}\frac{(\alpha_2-2)(\alpha_1+1-\gamma_1)+(1-\gamma_2)(3\alpha_1+2)}{3\alpha_1+2}}$.
We set $t \in (0, 1]$, such that

$$t \frac{(\alpha_2 - 2)(\alpha_1 + 1 - \gamma_1) + (1 - \gamma_2)(3\alpha_1 + 2)}{3\alpha_1 + 2} - (1 - t)\gamma_2 = 0$$

and we define $B = tA_{\alpha_1,\gamma_1} + (1 - t)\left(1, \frac{\alpha_1+1-\gamma_1}{\alpha_1+1}\right)$. So the operators

$$\sum_{1 \le r_2 \le R} \sum_{1 \le r_1 \le R} T_{\nu_{r_1,r_2}} P_{1,r_1}$$

and

$$\sum_{1\le r_2\le R}\sum_{1\le r_1\le R} T_{\nu_{r_1,r_2}}\left(I-P_{1,r_1}\right)\left(I-\widetilde{Q}_{1,r_1}\right)$$

are bounded on the open polygon with vertices $(1,\frac{\alpha_1+1-\gamma_1}{\alpha_1+1})$, B, A_0, $(1/2,1/2)$ and $(1,1)$ with bounds independent of R. It is easy to check that $B^0_{\alpha_2,\gamma_2}$ belongs to this polygon, so (4.9) and Remark 4.6 imply that $B^0_{\alpha_2,\gamma_2}\in E_\nu$.

Case IV: (4.8) says, in this case, that $\left\|\sum_{1\le r_2\le R} T_{\nu_{r_1,r_2}}\right\|_{B_{\alpha_2,\gamma_2}}\le c$, with c independent of r_1 and R. Also, as in case V, we obtain that

$$\left\|\sum_{1\le r_2\le R}\sum_{1\le r_1\le R} T_{\nu_{r_1,r_2}}P_{1,r_1}\right\|_{B^0_{\alpha_2,\gamma_2}}\le c$$

and

$$\left\|\sum_{1\le r_2\le R}\sum_{1\le r_1\le R} T_{\nu_{r_1,r_2}}\left(I-P_{1,r_1}\right)\left(I-\widetilde{Q}_{1,r_1}\right)\right\|_{B^0_{\alpha_2,\gamma_2}}\le c.$$

Since $B^0_{\alpha_2,\gamma_2}=B_{\alpha_2,\gamma_2}$, the theorem follows by Remark 4.6.

THEOREM 4.10. *The interior of E_μ agrees with the interior of $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$.*

PROOF. It is enough to check that the vertices of $\Sigma^{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$ belong to the boundary of E_μ, in the cases III, IV and V. We will consider analytic families of operators of the form $T_zf=\mu_z * f$ where

μ_z are complex measures defined,for $\mathrm{Re}(\gamma_1(z))$, $\mathrm{Re}(\gamma_2(z))>0$, by $\int f d\mu_z =$

$$=\frac{1}{\Gamma\left(\frac{\gamma_1(z)}{2}\right)\Gamma\left(\frac{\gamma_2(z)}{2}\right)}\int_Q f(x_1,x_2,\varphi(x_1,x_2))|x_1|^{\gamma_1(z)-1}|x_2|^{\gamma_2(z)-1}dx_1dx_2$$

with $\gamma_j(z)=k_j-\left(k_j-\gamma_j\right)(1-z)$ for a suitable choice of k_j, in each case.

To prove that $B_{\alpha_2,\gamma_2}\in\partial E_\mu$, we take, in the above construction, $k_1=\frac{\alpha_1+2}{4}$, $k_2=\frac{\alpha_2+2}{4}$ and we consider the strip $-\frac{4\gamma_2}{\alpha_2+2-4\gamma_2}+\epsilon\le\mathrm{Re}(z)\le 1$. For $\mathrm{Re}(z)=1, T_z$ is bounded by T_ν, where ν is the measure associated with α_1,α_2,k_1 and k_2. Theorem 3.1, (i) implies that $\|T_z\|_{\frac{4}{3},4}\le c$. We take $\epsilon>0$ small enough. For $\mathrm{Re}(z)=-\frac{4\gamma_2}{\alpha_2+2-4\gamma_2}+\epsilon$, it is easy to check that $\mathrm{Re}(\gamma_1(z))$, $\mathrm{Re}(\gamma_2(z))>0$ and so $\|T_z\|_{1,1}\le c$. The complex interpolation theorem implies that the interpolated point $B^\epsilon_{\alpha_2,\gamma_2}$, corresponding to $z=0$, belongs to E_μ. Since $B^\epsilon_{\alpha_2,\gamma_2}$ tends to B_{α_2,γ_2} as ϵ tends to zero, it follows that $B_{\alpha_2,\gamma_2}\in\partial E_\mu$.

Now we prove that if case V occurs, then $B^0_{\alpha_2,\gamma_2}$ belongs to ∂E_μ. Indeed, we take, in the definition of T_z,

$$k_1 = \gamma_1 \frac{\alpha_2 + \alpha_2\alpha_1 + \alpha_1}{3\alpha_2\gamma_1 - 2\gamma_2 + 2\gamma_1 + \alpha_1\gamma_2}, \quad k_2 = \gamma_2 \frac{\alpha_2 + \alpha_2\alpha_1 + \alpha_1}{3\alpha_2\gamma_1 - 2\gamma_2 + 2\gamma_1 + \alpha_1\gamma_2}$$

and we apply the complex interpolation theorem on the strip

$$\frac{3\alpha_2\gamma_1 - 2\gamma_2 + 2\gamma_1 + \alpha_1\gamma_2}{\alpha_1\alpha_2 + \alpha_1 + \alpha_2 - 3\alpha_2\gamma_1 + 2\gamma_2 - 2\gamma_1 - \alpha_1\gamma_2} + \epsilon \leq \mathrm{Re}(z) \leq 1$$

for $\epsilon > 0$ small enough., to obtain as above that $B^0_{\alpha_2,\gamma_2} \in \partial E_\mu$.

Finally, we check that, if the case III occurs then $B^{\gamma_2}_{\alpha_2,\gamma_2} \in \partial E_\mu$. We take, in the definition of T_z, $k_1 = \frac{1}{3}(\alpha_1 + 1)$, $k_2 = \frac{1}{3}$ and we apply the complex interpolation theorem on the strip $-\frac{3\gamma_2}{1-3\gamma_2} + \epsilon \leq$ Re(z)≤ 1, for $\epsilon > 0$ small enough.

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