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STABLE RANK OF THE C^*-ALGEBRAS OF AMENABLE LIE GROUPS OF TYPE I

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Abstract

In this paper we show that stable rank of the C^*-algebras of simply connected, amenable Lie groups of type I is estimated by complex dimension of the spaces of all characters of these groups. It is extended to the connected case, by which we show that product formula of stable rank holds for the group C^*-algebras of connected, amenable Lie groups of type I. For further estimation of stable rank of those group C^*-algebras, we need the conditions of the radicals of those groups.

1. Introduction

Stable rank of C^*-algebras, that is, non commutative complex dimension, was initiated by M.A. Rieffel [9] to study the stability problems such as determination of the cancellation property of finitely generated projective modules over irrational rotation C^*-algebras. He also raised an interesting problem such as describing stable rank of the C^*-algebras of Lie groups in terms of geometrical structure of groups. In this direction, A.J-L. Sheu [10] computed stable rank of the C^*-algebras of certain simply connected, nilpotent Lie groups. Next, H. Takai and the author [12] succeeded in the computation of stable rank of the C^*-algebras of simply connected, nilpotent Lie groups. Moreover, we [13] extended our results to the case of simply connected, solvable Lie groups of type I.

On the other hand, the author [11] estimated stable rank of the reduced C^*-algebras of semi-simple Lie groups by real rank of their groups, and extended it to the case of reductive Lie groups and partially to the case of non amenable Lie groups of type I.

In this article we show that stable rank of the C^*-algebras of simply connected, amenable Lie groups of type I is estimated by complex dimension of the spaces of all their 1-dimensional representations which is homeomorphic to the fixed point subspaces of the chracters of their radicals under the adjoint actions of their semi-simple parts. We extend it to the connected case,

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by which we prove product formula of stable rank for the group C^*-algebras of connected, amenable Lie groups of type I. Moreover, in the special case where the radicals of those groups are commutative, we estimate completely stable rank of their group C^*-algebras by complex dimension of the spaces of all characters of these groups and that of the orbit spaces of the characters of their radicals under the adjoint actions of these groups.

2. Preliminaries

We review some basic properties of covariant representations of C^*-crossed products.

Let $\mathfrak{A}$ be a C^*-algebra, G a locally compact group and $(\mathfrak{A}, G, \alpha)$ a C^*-dynamical system. A covariant representation of $(\mathfrak{A}, G, \alpha)$ is the couple (π, U) of a unitary representation π of G on a Hilbert space H and a $*$-representation of $\mathfrak{A}$ on the same space with the property that $U_g \pi(a) U_g^* = \pi(\alpha_g(a))$, for $a \in \mathfrak{A}, g \in G$. Recall that there is a bijection between covariant representations of $(\mathfrak{A}, G, \alpha)$ and non-degenerate representations of the crossed product $\mathfrak{A} \rtimes_\alpha G$. In particular, the set of all irreducible representatons of $\mathfrak{A} \rtimes_\alpha G$ corresponds to a subclass of covariant representations of $(\mathfrak{A}, G, \alpha)$ (cf.[7]).

Let G_0 be a closed subgroup of G and Γ the right coset space $G_0 \backslash G$. Let (π_0, L) be a covariant representation of $(\mathfrak{A}, G_0, \alpha|_{G_0})$ on a separable Hilbert space H_0. Take the induced representation $U = \mathrm{ind}_{G_0 \uparrow G} L$ of L to G. Let $L^2(G, H_0)$ be the Hilbert space of all H_0-valued, square integrable, measurable functions on G with respect to a left Haar measure. The representation space H of U is considered as a closed subspace of $L^2(G, H_0)$. A representation π of $\mathfrak{A}$ on H is defined by $(\pi(a)\xi)(g) = \pi_0(\alpha_g^{-1}(a))\xi(g)$, for $a \in \mathfrak{A}$, $\xi \in H, g \in G$. The couple (π, U) is a covariant representation of $(\mathfrak{A}, G, \alpha)$ induced by the covariant representation (π_0, L) of $(\mathfrak{A}, G_0, \alpha|_{G_0})$ (cf.[14]).

A projective representation L of G on a separable Hilbert space H is the map from G to the group of unitaries on H with the property that

(1) $L_e = 1_H$, where e is the identity of G and 1_H is the identity operator on H.

(2) $L_{gh} = \sigma(g, h) L_g L_h$ for any $g, h \in G$, where $\sigma(g, h)$ is in the one torus $\mathbb{T}$.

(3) The function $g \to \langle L_g \xi | \eta \rangle$ is a Borel function on G for each $\xi, \eta \in H$.

Then L is said to be a σ-representation of G (cf.[4]).

If $\mathfrak{A}$ is of type I and G acts smoothly on $\hat{\mathfrak{A}}$, then every covariant representation (π, U) of $\mathfrak{A} \rtimes_\alpha G$ is induced by some covariant representation (π_ρ, L_ρ) of $(\mathfrak{A}, G_\rho)$ for $\rho \in \hat{\mathfrak{A}}$ such that $\pi_\rho = \rho \otimes 1_{H_\rho}$ and $L_\rho = L_\rho^1 \otimes L_\rho^2$, where 1_{H_ρ} is the identity representation of $\mathfrak{A}$ on the representaton space H_ρ of ρ

and L_ρ^1, L_ρ^2 are $\sigma_\rho, \sigma_\rho^{-1}$-representations of the stabilizer G_ρ of ρ with $L_\rho^1(g)\rho = \rho \circ \alpha_g$ for any $g \in G_\rho$ respectively (cf.[14]).

Stable rank of a unital C^*-algebra $\mathfrak{A}$ denoted by $\mathrm{sr}(\mathfrak{A})$, is the least positive integer n such that, for every $(a_i)_{i=1}^n$ in $\mathfrak{A}^n$ and $\varepsilon > 0$, there exists an element $(b_i)_{i=1}^n$ in $\mathfrak{A}^n$ with $\sum_{i=1}^n b_i^* b_i$ invertible in $\mathfrak{A}^n$ and $\|a_i - b_i\| < \varepsilon$ $(1 \le i \le n)$. If no such integers exist, then we let $\mathrm{sr}(\mathfrak{A}) = \infty$. For a non unital C^*-algebra $\mathfrak{A}$, let $\mathfrak{A}^+$ be its unitization and define its stable rank by $\mathrm{sr}(\mathfrak{A}^+)$ (cf.[9]).

We recall the following estimate of the stable rank obtained by combining Theorem 4.3, 4.4, and 4.11 in Rieffel's paper [9]: For any C^*-algebra and its any closed ideal $\mathfrak{I}$,

$$\mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \le \mathrm{sr}(\mathfrak{A}) \le \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}),$$

where $\vee$ means maximum and $\mathrm{csr}(\cdot)$ is connected stable rank (cf.[9]). This formula is used over and over again in this paper.

3. Main theorems

Let G be a Lie group and $\hat{G}$ its spectrum consisting of all irreducible representations of G identified up to unitary equivalence. Denote by $\hat{G}_1$ the subspace of $\hat{G}$ consisitng of all 1-dimensional representations, i.e. characters of G. Let $C^*(G)$ be the group C^*-algebra of G whose spectrum $C^*(G)^\wedge$ is identified with $\hat{G}$.

Let G be a simply connected Lie group and $\mathfrak{G}$ its Lie algebra. Let $\mathfrak{G} = \mathfrak{R} \oplus \mathfrak{S}$ be the Levi decomposition where $\mathfrak{R}$ is the radical, i.e. the largest solvable ideal of $\mathfrak{G}$, and $\mathfrak{S}$ is the semi-simple Lie subalgebra of $\mathfrak{G}$. Let R, S be the simply connected Lie subgroups of G corresponding to $\mathfrak{R}, \mathfrak{S}$ respectively. Let $\tilde{G} = R \rtimes_\alpha S$ be the semi-direct product of R by the adjoint action α of S. Then its Lie algebra $\tilde{\mathfrak{G}}$ is isomorpic to $\mathfrak{R} \oplus \mathfrak{S}$. Then we have $G \cong \tilde{G}$, since two simply connected Lie groups with their Lie algebras isomorphic are isomorphic (See [2]). Hence, $C^*(G)$ is isomorphic to $C^*(R) \rtimes_\alpha S$.

Let $\mathfrak{G}^*$ be the real dual space of $\mathfrak{G}$. Then G acts on $\mathfrak{G}^*$ by the coadjoint action Ad^*. Denote by $(\mathfrak{G}^*)^G$ the subspace of all fixed points of $\mathfrak{G}^*$ under Ad^*.

We consider the case that $\hat{G}$ is equal to the reduced dual of G, that is, G is amenable. Then S is compact. Since it is semi-simple, $\hat{S}_1 = \{1_S\}$, where 1_S is the trivial representation of S. Denote by $\hat{\alpha}$ the action of S on $\hat{R}$ defined by $\hat{\alpha}_s(\pi) = \pi \circ \alpha_s, s \in S, \pi \in \hat{R}$. Since $\hat{R}_1$ is S-invariant and closed in $\hat{R}$ (cf.[13]), we have the following exact sequence:

$$0 \to \mathfrak{I}_R \rtimes S \to C^*(R) \rtimes_\alpha S \to C_0(\hat{R}_1) \rtimes_{\hat{\alpha}} S \to 0, \tag{1}$$

where $\mathfrak{I}_R$ is the closed ideal of $C^*(R)$ corresponding to $\hat{R} \setminus \hat{R}_1$. Denote by $\hat{R}_1^S$

the set of all fixed points of $\hat{R}_1$ under $\hat{\alpha}$. Since $\hat{R}_1^S$ is S-invariant and closed in $\hat{R}_1$, the following exact sequence is obtained:

$$0 \to C_0(\hat{R}_1 \setminus \hat{R}_1^S) \rtimes S \to C_0(\hat{R}_1) \rtimes_{\hat{\alpha}} S \to C_0(\hat{R}_1^S) \rtimes S \to 0, \tag{2}$$

with $C_0(\hat{R}_1^S) \rtimes S \cong C_0(\hat{R}_1^S) \otimes C^*(S)$ and $C^*(S) \cong \oplus_{\hat{S}} M_n(\mathsf{C})$.

Lemma 3.1. *Let G be a simply connected Lie group, R its radical with $S = G/R$, and $\mathfrak{R}$ the Lie algebra of R. Then $\hat{R}_1^S$ is isomorphic to $(\mathfrak{R}^*)^G$ as a topological group.*

Proof. Note that $\hat{R}_1$ is isomorphic to $(\mathfrak{R}^*)^R$ via $\chi \to d\chi/2\pi i$, where $\chi \in \hat{R}_1$ and $d\chi$ is the derivative of χ ([13; Lemma 2.1]). Since $\hat{\alpha}_s(\chi)(r) = \chi(srs^{-1}), r \in R, s \in S$ and $\mathrm{Ad}^*(s)\varphi(X) = \varphi(\mathrm{Ad}(s^{-1})X), X \in \mathfrak{R}, \varphi \in \mathfrak{R}^*$, then

$$d(\hat{\alpha}_s(\chi))(X) = \frac{d}{dt}\chi(s \exp tXs^{-1})|_{t=0} = \frac{d}{dt}\chi(\exp t(\mathrm{Ad}(s)X))|_{t=0} = d\chi(\mathrm{Ad}(s)X),$$

which implies that the adjoint orbit space $\hat{R}_1/S = \hat{R}_1/G$ is homeomorphic to the coadjoint orbit space $(\mathfrak{R}^*)^R/S = (\mathfrak{R}^*)^R/G$ as orbit spaces with respective quotient topologies. In particular, it implies the conclusion.

Using Pukanszky's results [8], we have the following:

Lemma 3.2. *If G is a simply connected, solvable Lie group, then every element of $\hat{G}$ is one or infinite dimensional.*

Proof. Let $[G, G]$ be the commutator subgroup of G, which is a connected, nilpotent Lie group. Let $[G, G]^\sim$ be the universal covering group of $[G, G]$. Then we have the canonical map from $([G, G])^\wedge$ to $([G, G]^\sim)^\wedge$ induced by the quotient map from $[G, G]^\sim$ to $[G, G]$. We know that any element of $([G, G]^\sim)^\wedge$ is one or infinite dimensional ([13; Lemma 2.5]). So is any element of $([G, G])^\wedge$.

For each $\pi \in ([G, G])^\wedge$, there is a closed Lie subgroup K_π of G with $K_\pi \supset [G, G]$ and there exists an element of ρ of $\hat{K}_\pi$ with $\rho|_{[G,G]} = \pi$ such that the induced representation $U_\pi = \mathrm{ind}_{K_\pi \uparrow G}\rho$ of ρ to G is a factor representation of G. Every factor representation of G is quasi equivalent to a factor representation constructed in this way ([cf.[8]).

Note that $\hat{G}_1 \cong (G/[G, G])^\wedge$ ([13; Lemma 2.3]). Thus, if $\pi \in ([G, G])^\wedge$ is trivial and $K_\pi = G$, then ρ is in $\hat{G}_1$. If π is a non trivial one dimensional representation, then $K_\pi \neq G$. Hence U_π is infinite dimensional. If π is infinite dimensional, then so is U_π. Therefore any factor representation of G is one or infinite dimensional. In particular, every element of $\hat{G}$ is one or infinite dimensional.

Lemma 3.3. *In the situation* (2) *above,*

$$\dim_{\mathbf{C}} \hat{R}_1^S \leq \mathrm{sr}(C_0(\hat{R}_1) \rtimes_{\hat{\alpha}} S) \leq 2 \vee \dim_{\mathbf{C}} \hat{R}_1^S,$$

where $\dim_{\mathbf{C}}(\cdot) = [\dim(\cdot)/2] + 1$ *with* $[\cdot]$ *Gauss symbol and* $\vee$ *means maximum.*

Proof. Since $\hat{R}_1$ is a T_2-space (cf.[13]) and S is compact, the orbit space $\hat{R}_1/S$ is a T_2-space. By [1; Proposition 3,9] and its remark, S acts smoothly on $\hat{R}_1$, that is, any quasi-orbit on $\hat{R}_1$ by S is transitive. Let S_χ be the stabilizer of S at $\chi \in \hat{R}_1$. Since S_χ is a Lie subgroup of S, then $\dim S_\chi < \dim S$ for any $\chi \in \hat{R}_1 \setminus \hat{R}_1^S$. Thus, from the Takesaki's result in section 2, every irreducible representation of $C_0(\hat{R}_1 \setminus \hat{R}_1^S) \rtimes S$ is unitarily equivalent to an irreducible representation acting on the infinite dimensional sapce of $L^2(S_\chi \backslash S)$ for some χ in $\hat{R}_1 \setminus \hat{R}_1^S$ (cf.[14]). Hence every element of $(C_0(\hat{R}_1 \setminus \hat{R}_1^S) \rtimes S)^\wedge$ is infinite dimensional. Since S has the trivial representation,

$$\mathrm{sr}(C_0(\hat{R}_1^S) \otimes (\oplus_{\hat{S}} M_n(\mathbf{C}))) = \sup_{\hat{S}}(\{(\dim_{\mathbf{C}} \hat{R}_1^S - 1)/n\} + 1) = \dim_{\mathbf{C}} \hat{R}_1^S,$$

where $\{x\}$ is the least integer $\geq x$. Applying the same methods in ([13; Lemma 3.2]), we have the conclusion.

The solvable case of the next result was proved by H. Takai and the author [13].

Lemma 3.4. *Let G be a simply connected, amenable Lie group of type I. Then*

$$\hat{G}_1 \cong \hat{R}_1^S \times \{1_S\}, \quad \dim_{\mathbf{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq 2 \vee \dim_{\mathbf{C}} \hat{G}_1.$$

Proof. First of all, we show from (1) that every element in $(\mathfrak{I}_R \rtimes S)^\wedge$ is infinite dimensional. By Lemma 3.2, any element of $\hat{\mathfrak{I}}_R$ is infinite dimensional. Let π be an elemant of $(\mathfrak{I}_R \rtimes S)^\wedge$. Suppose that π is finite dimensional. Take a covariant representation (ρ, U) of $(\mathfrak{I}_R, S)$ corresponding to π. Then ρ is finite dimensional, so are its irreducible components, which is the contradiction. In particular, $\hat{G}_1 \cong \hat{R}_1^S \times \{1_S\}$.

Let $\mathfrak{I}_G$ be a closed $*$-ideal of $C^*(G)$ such that $\hat{\mathfrak{I}}_G = (\mathfrak{I}_R \rtimes S)^\wedge \cup (C_0(\hat{R}_1 \setminus \hat{R}_1^S) \rtimes S)^\wedge$. Then the following exact sequence is obtained:

$$0 \to \mathfrak{I}_G \to C^*(G) \to \mathfrak{D} \to 0, \quad \mathfrak{D} = C_0(\hat{R}_1^S) \otimes C^*(S).$$

We apply the methods in ([13; Lemma 3.2]) to the exact sequence above. Take a composition series $\{\mathfrak{I}_i\}_{i=1}^\infty$ with $\mathfrak{I}_0 = \{0\}$ of $C^*(G)$ such that $\mathfrak{I}_i/\mathfrak{I}_{i-1}$ $(i \geq 1)$ are of continuous trace. Consider the following exact sequence:

$$0 \to \mathfrak{E}_i \to \mathfrak{I}_i \to \mathfrak{D}_i \to 0$$

for every $i \geq 1$, where $\mathfrak{E}_i, \mathfrak{D}_i$ are the closed $*$-ideals and quotients of $\mathfrak{I}_i$ corresponding to $\hat{\mathfrak{I}}_G \cap \hat{\mathfrak{I}}_i$, $\hat{\mathfrak{D}} \cap \hat{\mathfrak{I}}_i$ respectively. Next consider the finite composition series $\{\mathfrak{I}_j \cap \mathfrak{E}_i\}_{j=1}^i$ of $\mathfrak{E}_i$. Put $\mathfrak{L}_j = \mathfrak{I}_j \cap \mathfrak{E}_i$ $(1 \le j \le i)$ and $\mathfrak{L}_0 = \{0\}$. Then the following exact sequences are obtained:

$$0 \to \mathfrak{L}_j/\mathfrak{L}_{j-1} \to \mathfrak{I}_i/\mathfrak{L}_{j-1} \to \mathfrak{I}_i/\mathfrak{L}_j \to 0, \quad (1 \le j \le i). \tag{3}$$

Since $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ $(1 \le j \le i)$ are of continuous trace, so are

$$(\mathfrak{L}_j + \mathfrak{I}_{j-1})/\mathfrak{I}_{j-1} \cong \mathfrak{L}_j/(\mathfrak{L}_j \cap \mathfrak{I}_{j-1}) = \mathfrak{L}_j/\mathfrak{L}_{j-1}, \quad (1 \le j \le i).$$

By definition, $\hat{\mathfrak{L}}_j$ is contained in $\hat{\mathfrak{I}}_G$ for $(1 \le j \le i)$. Hence, every element of $(\mathfrak{L}_j/\mathfrak{L}_{j-1})^\wedge$ $(1 \le j \le i)$ is infinite dimensional. Thus, applying Nistor's result [5; Lemma 2] to (3), we have that $\mathrm{sr}(\mathfrak{I}_i/\mathfrak{L}_{j-1}) \le 2 \vee \mathrm{sr}(\mathfrak{I}_i/\mathfrak{L}_j)$, $(1 \le j \le i)$. Since $\mathfrak{I}_i = \mathfrak{I}_i/\mathfrak{L}_0$ and $\mathfrak{D}_i = \mathfrak{I}_i/\mathfrak{L}_i$, we get $\mathrm{sr}(\mathfrak{I}_i) \le 2 \vee \mathrm{sr}(\mathfrak{D}_i)$. Hence, $\mathrm{sr}(\mathfrak{I}_i) \le 2 \vee \mathrm{sr}(\mathfrak{D})$. By the same argument, the above inequality holds for any $i \ge 1$.

Put $m = 2 \vee \mathrm{sr}(\mathfrak{D})$. Let $(a_i)_{i=1}^m$ be an element of $(C^*(G)^+)^m$. Then there exists an element $(b_i)_{i=1}^m$ of $(\mathfrak{I}_j^+)^m$ such that $\|a_i - b_i\| < \varepsilon/2$ $(1 \le i \le m)$ for some j. Since $\mathrm{sr}(\mathfrak{I}_j) \le m$, there is an element $(c_i)_{i=1}^m$ of $(\mathfrak{I}_j^+)^m$ such that $\sum_{i=1}^m c_i^* c_i$ is invertible in $\mathfrak{I}_j^+$ and $\|b_i - c_i\| < \varepsilon/2$ $(1 \le i \le m)$. Hence $\|a_i - c_i\| < \varepsilon$ $(1 \le i \le m)$ and $\sum_{i=1}^m c_i^* c_i$ is invertible in $C^*(G)^+$.

Proposition 3.5. *Let G be a connected, amenable Lie group of type* I. *Then*

$$\dim_{\mathbf{C}} \hat{G}_1 \le \mathrm{sr}(C^*(G)) \le 2 \vee \dim_{\mathbf{C}} \hat{G}_1.$$

Proof. Let $\tilde{G}$ be the universal covering group of G. Consider the map $\Phi : \hat{G} \to \tilde{G}^\wedge$ defined by $\Phi(\pi) = \pi \circ q$, where q is the quotient map from $\tilde{G}$ to G. Then q induces a surjective $*$-homomorphism $\tilde{q}$ from $C^*(\tilde{G})$ to $C^*(G)$. We denote by $\tilde{\pi}$ the element in $C^*(G)^\wedge$ corresponding to $\pi \in \hat{G}$. Let K be a closed set in $\tilde{G}^\wedge$. Then $K \cap \Phi(\hat{G}) = \{\pi \circ q \in \tilde{G}^\wedge \mid \cap_{\rho \in K} \ker \tilde{\rho} \subset \ker \tilde{\pi} \circ \tilde{q}\}$. Put $L = q^{-1}(K \cap \Phi(\hat{G}))$, then $L = \{\pi \in \hat{G} \mid \cap_{\pi' \in L} \ker \tilde{\pi}' \subset \ker \tilde{\pi}\}$. In fact, if $\cap_{\pi' \in L} \ker \tilde{\pi}' \subset \ker \tilde{\pi}$ for $\pi \in \hat{G}$, we get

$$\cap_{\rho \in K} \ker\ \tilde{\rho} \subset \cap_{\rho \in K \cap \Phi(\hat{G})} \ker \tilde{\rho} = \cap_{\pi' \in L} \ker \tilde{\pi}' \circ \tilde{q} \subset \ker \tilde{\pi} \circ \tilde{q}.$$

Hence π is in L. So Φ is continuous. If R is the radical of $\tilde{G}$ with $S = \tilde{G}/R$, then $\hat{R}_1^S \times \hat{S}$ is closed in $\tilde{G}^\wedge$, so is $\Phi^{-1}((\hat{R}_1^S \times \hat{S}) \cap \Phi(\hat{G}))$ in $\hat{G}$. Thus, in the same way as in Lemma 3.4, we have the conclusion.

Applying Proposition 3.5, we have the product formula of stable rank as follows:

Corollary 3.6. *If G, H are two connected, amenable Lie groups of type* I, *then*

$$\mathrm{sr}(C^*(G) \otimes C^*(H)) \leq \mathrm{sr}(C^*(G)) + \mathrm{sr}(C^*(H)).$$

PROOF. Since G, H are of type I, $(G \times H)^{\wedge}$ is homeomorphic to $\hat{G} \times \hat{H}$. $C^*(G \times H)$ is isomorphic to $C^*(G) \otimes C^*(H)$. By Lemma 3.5,

$$\mathrm{sr}(C^*(G) \otimes C^*(H)) \leq 2 \vee \dim_{\mathsf{C}}(\hat{G}_1 \times \hat{H}_1)$$
$$\leq 2 \vee (\dim_{\mathsf{C}} \hat{G}_1 + \dim_{\mathsf{C}} \hat{H}_1) \leq \mathrm{sr}(C^*(G)) + \mathrm{sr}(C^*(H)).$$

LEMMA 3.7. *If G is the semi-direct product $\mathsf{R} \rtimes S$ with S a connected, compact Lie group, then $G \cong \mathsf{R} \times S$.*

PROOF. Suppose that the action of S on R is non trivial. Let X be a non-zero element of the Lie algebra of R. Take an element Y of the Lie algebra of S such that $\mathrm{ad}(Y)X = \alpha X$ with $\alpha > 0$. Then $\mathrm{Ad}(\exp tY)X = e^{t\alpha}X$ for t in R. Hence $\mathrm{Ad}(S)X$ is non compact, which is impossible since S is compact.

LEMMA 3.8. *Let G be a simply connected, amenable Lie group of type* I. *Then*

$$\mathrm{sr}(C^*(G)) \leq (2 \vee \dim_{\mathsf{C}} \hat{G}_1) \wedge (\dim R \vee 1),$$

where $\wedge$ is minimum.

PROOF. Note that $\dim_{\mathsf{C}} \hat{G}_1 = \dim_{\mathsf{C}} \hat{R}_1^S \leq \dim R$. If $G = S$, that is, $\dim R = 0$, then $\mathrm{sr}(C^*(G)) = 1$. If $\dim R = 1$, then, by Lemma 3.7, $G \cong \mathsf{R} \times S$. Then $C^*(G) \cong C_0(\mathsf{R}) \otimes C^*(S)$. Hence, $\mathrm{sr}(C^*(G)) = 1$.

REMARK. If G is a simply connected, solvable Lie group of type I, that is $G = R$, then we have that ([13; Theorem 3.9]) $\mathrm{sr}(C^*(G)) = (2 \vee \dim_{\mathsf{C}} \hat{G}_1) \wedge \dim G$.

We have the complete description of stable rank of the group C^*-algebras of the following special case in terms of groups:

PROPOSITION 3.9. *Let G be a simply connected, amenable Lie group. If its radical R is commutative, then G is of type* I *and*

$$\mathrm{sr}(C^*(G)) = (2 \wedge \dim_{\mathsf{C}}(\hat{R}_1/G)) \vee \dim_{\mathsf{C}} \hat{G}_1.$$

PROOF. We use the fact that $\hat{R} = \hat{R}_1 \cong (\mathfrak{R}^*)^R = \mathfrak{R}^*$ (cf.[13]) where $\mathfrak{R}^*$ is the real dual space of the Lie algebra $\mathfrak{R}$ of R. Since the adjoint representation Ad of S on $\mathfrak{R}$ is completely reducible, so is the coadjoint representation Ad^* of S on $\mathfrak{R}^*$. Thus we obtain that

$$\mathfrak{R}^* = V_0 \times V_1 \times \cdots \times V_n,$$

where $V_0 \times \{0\} \times \cdots \times \{0\} = (\mathfrak{R}^*)^S$ is the fixed point subspace under Ad^*, and V_i $(1 \leq i \leq n)$ are S-invariant subspaces of $\mathfrak{R}^*$ such that the restrictions

$\mathrm{Ad}^*|_{V_i}$ are irreducible and the subspaces of all fixed points in V_i are zero $(1 \le i \le n)$. From Lemma 3.7, $V_i \approx \mathbb{R}^{n_i}$ for some $n_i \ge 2$ $(1 \le i \le n)$.

By averaging over the compact connected group S, one can find a metric on V_i invariant under $\mathrm{Ad}^*|_{V_i}(S)$, and hence $\mathrm{Ad}^*|_{V_i}(S)$ is contained in the special orthogonal group of V_i (cf.[6; Theorem 2, p.131]). Then we have that any orbit in $V_i \setminus \{0\}$ is homeomorphic to a subspace of the $(n_i - 1)$-dimensional sphere S^{n_i-1} of V_i.

Then by Fourier transform and the above observation,

$$C^*(G) \cong C^*(R) \rtimes S \cong C_0(\hat{R}_1) \rtimes S \cong C_0(\mathfrak{R}^*) \rtimes S = C_0(V_0 \times V_1 \times \cdots \times V_n) \rtimes S.$$

Since $(\mathfrak{R}^*)^S$ is S-invariant and closed in $\mathfrak{R}^*$,

$$0 \to C_0(\mathfrak{R}^* \setminus (\mathfrak{R}^*)^S) \rtimes S \to C_0(\mathfrak{R}^*) \rtimes S \to C_0((\mathfrak{R}^*)^S) \otimes C^*(S) \to 0.$$

Since $V_0 \times \{0\} \times (V_k \setminus \{0\}) \times \{0\} \times \cdots \times \{0\}$ $(1 \le k \le n)$ are S-invariant and closed in $(\mathfrak{R}^* \setminus (\mathfrak{R}^*)^S) \setminus \sqcup_{i=1}^{k-1}(V_0 \times \{0\} \times V_i \times \{0\} \times \cdots \times \{0\})$, then we have the following exact sequences:

$$\begin{aligned} 0 &\to C_0((\mathfrak{R}^* \setminus (\mathfrak{R}^*)^S) \setminus \sqcup_{i=1}^{k}(V_0 \times \{0\} \times V_i \times \{0\} \times \cdots \times \{0\})) \rtimes S \\ &\to C_0((\mathfrak{R}^* \setminus (\mathfrak{R}^*)^S) \setminus \sqcup_{i=1}^{k-1}(V_0 \times \{0\} \times V_i \times \{0\} \times \cdots \times \{0\})) \rtimes S \\ &\to C_0(V_0 \times \{0\} \times (V_k \setminus \{0\}) \times \{0\} \times \cdots \times \{0\}) \rtimes S \to 0, (1 \le k \le n) \end{aligned}$$

Moreover, from the property of $\mathrm{Ad}^*|_{V_k}$ above, it implies that

$$\begin{aligned} &C_0(V_0 \times \{0\} \times (V_k \setminus \{0\}) \times \{0\} \times \cdots \times \{0\}) \rtimes S \\ &\quad \cong C_0(V_0) \otimes C_0(\mathbb{R}) \otimes (C(S^{n_k-1}) \rtimes S). \end{aligned}$$

Then put $W_1 = (\mathfrak{R}^* \setminus (\mathfrak{R}^*)^S) \setminus \sqcup_{i=1}^{n}(V_0 \times \{0\} \times V_k \times \{0\} \times \cdots \times \{0\})$. Define

$$I_l = \{(i_1, \cdots, i_l) \mid 1 \le i_1 < \cdots < i_l \le n\}, \quad (2 \le l \le n-1).$$

Then define inductively $(2 \le l \le n-1)$

$$W_l = W_{l-1} \setminus \sqcup_{(i_1,\cdots,i_l) \in I_l}(V_0 \times \{0\} \times V_{i_1} \times \{0\} \times \cdots \times V_{i_l} \times \{0\} \times \cdots \times \{0\}),$$

$$W_{n-1} = V_0 \times (V_1 \setminus \{0\}) \times \cdots \times (V_n \setminus \{0\}).$$

Let J is a proper subset of I_l and $J^+ = J \cup \{(k_1, \cdots, k_l)\}$ with $(k_1, \cdots, k_l) \in I_l \setminus J$. Since $V_0 \times \{0\} \times (V_{k_1} \setminus \{0\}) \times \{0\} \times \cdots \times (V_{k_l} \setminus \{0\}) \times \{0\} \times \cdots \times \{0\}$ is S-invariant and closed in $W_{l-1} \setminus \sqcup_{(i_1,\cdots,i_l) \in J}(V_0 \times \{0\} \times V_{i_1} \times \{0\} \times \cdots \times V_{i_l} \times \{0\} \times \cdots \times \{0\})$, it follows that

$$\begin{aligned}
0 \to\; & C_0(W_{l-1} \setminus \sqcup_{(i_1,\cdots,i_l)\in J^+}(V_0 \times \{0\} \times V_{i_1} \times \{0\} \\
& \times \cdots \times V_{i_l} \times \{0\} \times \cdots \times \{0\}))\rtimes S \\
\to\; & C_0(W_{l-1} \setminus \sqcup_{(i_1,\cdots,i_l)\in J}(V_0 \times \{0\} \times V_{i_1} \times \{0\} \\
& \times \cdots \times V_{i_l} \times \{0\} \times \cdots \times \{0\}))\rtimes S \\
\to\; & C_0(V_0 \times \{0\} \times (V_{k_1} \setminus \{0\}) \times \{0\} \\
& \times \cdots \times (V_{k_l} \setminus \{0\}) \times \cdots \times \{0\})\rtimes S \to 0,
\end{aligned}$$

Then from the property of $\mathrm{Ad}^*|_{V_{k_i}}$ $(1 \le i \le l)$ above,

$$\begin{aligned}
& C_0(V_0 \times \{0\} \times (V_{k_1} \setminus \{0\}) \times \{0\} \times \cdots \times (V_{k_l} \setminus \{0\}) \times \cdots \times \{0\})\rtimes S \\
& \cong C_0(V_0 \times \mathsf{R}^l) \otimes (C(S^{(n_{k_1}-1)} \times \cdots \times S^{(n_{k_l}-1)})\rtimes S).
\end{aligned}$$

Put $S_{(k_1,\cdots,k_l)} = S^{(n_{k_1}-1)} \times \cdots \times S^{(n_{k_l}-1)}$, $(1 \le l \le n)$. Since S is compact, then $S_{(k_1,\cdots,k_l)}/S$ is a T_2-space. By the definition of induced covariant representations in section 2 ([14; Theorem 6.1]), every element of $(C(S_{(k_1,\cdots,k_l)})\rtimes S)^\wedge$ is infinite dimensional, and $C(S_{(k_1,\cdots,k_l)})\rtimes S$ is of type I. If $\psi \in S_{(k_1,\cdots,k_l)}$, then its orbit O_ψ is S-invariant and closed in $S_{(k_1,\cdots,k_l)}$. So $C_0(O_\psi)\rtimes S$ is a quotient of $C(S_{(k_1,\cdots,k_l)})\rtimes S$. Note that $C_0(O_\psi)\rtimes S \cong C_0(S/S_\psi)\rtimes S$. Since S is second countable, there is a measurable cross section from S/S_φ to S. Using the Green's imprimitivity theorem [3; Corollary 2.10], $C_0(S/S_\psi)\rtimes S \cong C^*(S_\psi)\otimes \mathsf{K}(L^2(S/S_\psi))$.

Then we have that $C^*(G)$ is of type I and every element of $C^*(G)^\wedge \setminus ((\mathfrak{R}^*)^S \times \hat{S})$ is infinite dimensional. By the same methods in Lemma 3.4,

$$\dim_{\mathsf{C}}(\mathfrak{R}^*)^S \le \mathrm{sr}(C^*(G)) \le 2 \vee \dim_{\mathsf{C}}(\mathfrak{R}^*)^S.$$

If $\dim_{\mathsf{C}}(\mathfrak{R}^*)^S \ge 2$, then $\mathrm{sr}(C^*(G)) = \dim_{\mathsf{C}}(\mathfrak{R}^*)^S$.

If $\dim(\mathfrak{R}^*)^S = 1$ and $\dim_{\mathsf{C}}(\mathfrak{R}^*/S) = 1$, then $C^*(G) \cong C_0(\mathsf{R}) \otimes C^*(S)$. So $\mathrm{sr}(C^*(G)) = 1$. If $\dim(\mathfrak{R}^*)^S = 1$ and $\dim_{\mathsf{C}}(\mathfrak{R}^*/S) \ge 2$, then by the above observation, $C^*(G)$ contains a subquotient of the form $C_0(D) \otimes \mathsf{K}$ with $\dim D \ge 2$. Hence, $\mathrm{sr}(C^*(G)) = 2$.

If $\dim(\mathfrak{R}^*)^S = 0$ and $\dim_{\mathsf{C}}(\mathfrak{R}^*/S) = 1$, then $\mathfrak{R}^* = V_1$. Since $C^*(S)$ has connected stable rank one ([cf.[9]), $\mathrm{sr}(C^*(G)) = 1$. If $\dim(\mathfrak{R}^*) = 0$ and $\dim_{\mathsf{C}}(\mathfrak{R}^*/S) \ge 2$, then $C^*(G)$ contains a subquotient of the form $C_0(D) \otimes \mathsf{K}$ with $\dim D \ge 2$. Hence, $\mathrm{sr}(C^*(G)) = 2$.

Since $\hat{G}_1 \cong (\mathfrak{R}^*)^S$ by Lemma 3.4 and $(\hat{R}_1/G) \cong (\mathfrak{R}^*/S)$ by the proof of Lemma 3.1, we have the conclusion.

Applying Lemma 3.8 and Proposition 3.9 in the situation (1), we have the following main theorem:

THEOREM 3.10. *Let G be a simply connected, amenable Lie group of type* I. *Then*

$$(2 \wedge \dim_{\mathsf{C}}(\hat{R}_1/G)) \vee \dim_{\mathsf{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq (2 \vee \dim_{\mathsf{C}} \hat{G}_1) \wedge (\dim R \vee 1),$$

where R is the radiacl of G.

PROOF. It suffices to show the first inequality. By the same methods in Proposition 3.9, we obtain $\mathrm{sr}(C_0(\hat{R}_1) \rtimes_{\hat{\alpha}} S) = (2 \wedge \dim_{\mathsf{C}}(\hat{R}_1/S)) \vee \dim_{\mathsf{C}} \hat{R}_1^S$. Moreover, $\hat{R}_1/S \cong \hat{R}_1/G$, $\hat{R}_1^S \cong \hat{G}_1$.

REMARK. If G is the direct product of the real $ax + b$ group and a semi-simple compact group, then $\mathrm{sr}(C^*(G)) = 2$. On the other hand, the above inequalities give $1 \leq \mathrm{sr}(C^*(G)) \leq 2$. We conjecture that, in the above formula, if the radical is non commutative, then $\mathrm{sr}(C^*(G)) = 2 \vee \dim_{\mathsf{C}} \hat{G}_1$.

4. Examples

EXAMPLE 4.1. If G is the direct product $R \times S$ with R a simply connected, solvable Lie group and S compact, then

$$C^*(G) \cong C^*(R) \otimes C^*(S) \cong \oplus_{\hat{S}} C^*(R) \otimes M_n(\mathsf{C}).$$

Since S has the trivial representation, $\mathrm{sr}(C^*(G)) = \mathrm{sr}(C^*(R))$.

EXAMPLE 4.2. Let $G = \mathsf{R}^n \rtimes \mathrm{Spin}(n), (n \geq 2)$ where $\mathrm{Spin}(n)$ is the universal covering group of $\mathrm{SO}(n)$. Let α be the action of $\mathrm{SO}(n)$ on R^n defined by $\alpha_g(t) = g \cdot t$ for $g \in \mathrm{SO}(n), t \in \mathsf{R}^n$, where $\cdot$ means the matrix multiplication. The action $\tilde{\alpha}$ of $\mathrm{Spin}(n)$ on R^n is defined by $\tilde{\alpha}_g(t) = \alpha_{q(g)}(t)$ for $g \in \mathrm{Spin}(n), t \in \mathsf{R}^n$, where q is the quotient map from $\mathrm{Spin}(n)$ to $\mathrm{SO}(n)$. Put $R = \mathsf{R}^n$. Note that the Lie algebra $\mathfrak{R}$ of R consists of vector fields of the form $X = \sum_{i=1}^n t_i \frac{d}{dx_i}$ for $t_i \in \mathsf{R}$, with $\exp X = (t_1, \cdots, t_n)$. Let $g \cdot t = (s_1, \cdots, s_n)$. Then $\mathrm{Ad}(g)X = \sum_{i=1}^n s_i \frac{d}{dx_i}$. Let $\mathfrak{R}^*$ be the real dual space of $\mathfrak{R}$. Then $\mathfrak{R}^* \cong \hat{R} \cong \mathsf{R}^n$ as a topological (vector) group. Then via the Fourier transform,

$$C^*(G) \cong C^*(R) \rtimes \mathrm{Spin}(n) \cong C_0(\hat{R}) \rtimes_{\tilde{\alpha}'} \mathrm{Spin}(n),$$

where $\tilde{\alpha}'$ is the action of $\mathrm{Spin}(n)$ on $C_0(\hat{R})$ induced by $\tilde{\alpha}$. Let $\mathscr{F}$ be the Fourier transform from $C^*(R)$ to $C_0(\hat{R})$. We check the action $\tilde{\alpha}'$ explicitly as follows: for a rapidly decreasing C^∞-function f on $\hat{R}$, $\chi_t \in \hat{R}$ which corresponds to $t \in \mathsf{R}^n$, and $g \in \mathrm{Spin}(n)$,

$$\tilde{\alpha}'_g(f)(\chi_t) = (\mathscr{F} \circ \tilde{\alpha}_g \circ \mathscr{F}^*)(f)(\chi_t) = f(\chi_{q(g)^{-1} \cdot t}).$$

Then we check the action $\tilde{\alpha}''$ of $\mathrm{Spin}(n)$ on $\hat{R}$ as follows:

$$\tilde{\alpha}''_g(\chi_t)(f) = \chi_t(\tilde{\alpha}'_{g^{-1}}(f)) = \tilde{\alpha}'_{g^{-1}}(f)(\chi_t) = f(\chi_{q(g)\cdot t}) = \chi_{q(g)\cdot t}(f)$$

for $f \in C_0(\hat{R})$. Hence, $\hat{R}/\mathrm{Spin}(n) \approx \mathsf{R}_+ \cup \{\chi_0\}$ with $\{\chi_0\} = (\hat{R})^{\mathrm{Spin}(n)} = \hat{G}_1$. Since $\{\chi_0\}$ is Spin(n)-invariant and closed in $\hat{R}$,

$$0 \to C_0(\hat{R} \setminus \{\chi_0\}) \rtimes \mathrm{Spin}(n) \to C_0(\hat{R}) \rtimes \mathrm{Spin}(n) \to C^*(\mathrm{Spin}(n)) \to 0.$$

Let $\chi \in \hat{R} \setminus \{\chi_0\}$. Since $\mathrm{SO}(n)_\chi \cong \mathrm{SO}(n-1)$, then $\mathrm{SO}(n)/\mathrm{SO}(n)_\chi$ is homeomorphic to S^{n-1}. Then Spin(n) acts on $\mathrm{SO}(n)/\mathrm{SO}(n)_\chi$ by the left multiplication. As in the proof of Proposition 3.9,

$$\begin{aligned} C_0(\hat{R} \setminus \{\chi_0\}) \rtimes \mathrm{Spin}(n) &\cong C_0(\mathsf{R}_+ \times S^{n-1}) \rtimes \mathrm{Spin}(n) \\ &\cong C_0(\mathsf{R}_+) \otimes (C(\mathrm{SO}(n)/\mathrm{SO}(n-1)) \rtimes \mathrm{Spin}(n)). \end{aligned}$$

Note that $\mathrm{Spin}(n)/\mathrm{Spin}(n)_\chi \cong \mathrm{SO}(n)/\mathrm{SO}(n)_\chi$ and Spin(n) acts on $\mathrm{Spin}(n)/\mathrm{Spin}(n)_\chi$ by the left multiplication. It follows that

$$C(\mathrm{SO}(n)/\mathrm{SO}(n)_\chi) \rtimes \mathrm{Spin}(n) \cong C(\mathrm{Spin}(n)/\mathrm{Spin}(n)_\chi) \rtimes \mathrm{Spin}(n).$$

Since Spin(n) is second countable, then by [3; Corollary 2.10],

$$C(\mathrm{Spin}(n)/\mathrm{Spin}(n)_\chi) \rtimes \mathrm{Spin}(n) \cong C^*(\mathrm{Spin}(n)_\chi) \otimes \mathsf{K}(L^2(\mathrm{Spin}(n)/\mathrm{Spin}(n)_\chi)).$$

Since $C^*(\mathrm{Spin}(n)) \cong \oplus_{(\mathrm{Spin}(n))^\wedge} M_n(\mathsf{C}), C^*(\mathrm{Spin}(n)_\chi) \cong \oplus_{(\mathrm{Spin}(n)_\chi)^\wedge} M_n(\mathsf{C})$ and $M_n(\mathsf{C})$ has connected stable rank one, we get

$$\begin{aligned} \mathrm{sr}(C^*(G)) &= \mathrm{sr}(C_0(\mathsf{R}_+) \otimes C^*(\mathrm{Spin}(n)_\chi) \otimes \mathsf{K}) \\ &= \sup_{(\mathrm{Spin}(n)_\chi)^\wedge} (\{(\mathrm{sr}(C_0(\mathsf{R}_+) \otimes \mathsf{K}) - 1)/n\} + 1) = 1. \end{aligned}$$

Example 4.3. Let $G = \mathsf{R}^{n+m} \rtimes (\mathrm{Spin}(n) \times \mathrm{Spin}(m))$, $(n, m \geq 2)$ with the action α defined by $\alpha_{(s,t)}(x, y) = (\tilde{\alpha}_s(x), \tilde{\alpha}_t(y))$ for $s \in \mathrm{Spin}(n), t \in \mathrm{Spin}(m)$, $x \in \mathsf{R}^n, y \in \mathsf{R}^m$, where $\tilde{\alpha}$ is as in Example 4.2. Then $C^*(G) \cong (C_0(\mathsf{R}^n) \rtimes \mathrm{Spin}(n)) \otimes (C_0(\mathsf{R}^m) \rtimes \mathrm{Spin}(m))$. By the same analysis in Example 4.2, $\hat{G}_1 \cong (\mathsf{R}^{n+m})^{(\mathrm{Spin}(n) \times \mathrm{Spin}(m))} = \{(0,0)\}$ and

$$\mathsf{R}^{n+m}/(\mathrm{Spin}(n) \times \mathrm{Spin}(m)) \approx (\mathsf{R}_+ \cup \{0\}) \times (\mathsf{R}_+ \cup \{0\}),$$

where R^{n+m} is identified with its spectrum. By Proposition 3.9, we have $\mathrm{sr}(C^*(G)) = 2$.

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