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WEIGHT CENTRALIZER EXPECTATIONS WITH FINITE INDEX

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Abstract

Let M be a von Neumann algebra, φ be a faithful, normal semifinite weight on M and M^φ its centralizer. We characterize the conditional expectations $E_\varphi : M \to M^\varphi$ of finite index for a faithful normal strictly semifinite weight φ on a semifinite von Neumann algebra M with finite dimensional center. This result is used to characterize weights φ such that the orbit $U_\varphi = \{\varphi \circ \mathrm{Ad}(u) : u \text{ unitary in } M\}$ can be represented as a submanifold of M_1 (=basic extension of $E_\varphi : M \to M_\varphi$).

1.1 *Introduction*. Let M be a von Neumann algebra and φ a faithful, normal and semifinite weight on M. Denote by σ_t^φ the modular group of φ and by M^φ the centralizer of φ, i.e. the subalgebra of M of fixed points for σ_t^φ. It is well known ([C],[T]) that there exists a faithful and normal conditional expectation $E_\varphi : M \to M^\varphi$ satisfying $\varphi \circ E_\varphi = \varphi$ if and only if the weight φ is semifinite in M^φ. Under these asumptions, the expectation E_φ is unique. Such weights φ are called strictly semifinite [C].

There are two notions of finite index for a conditional expectation between von Neumann algebras [BDH], [Ha]. In many cases they do coincide (for example, if the algebras are factors). In general they do not coincide ([FK]). In this paper we investigate the finiteness (in the two senses) of the expectations E_φ associated with weights φ. Very naturally one is constrained to consider semifinite algebras. In section 4 we state that in the case of factors (or more generally algebras with finite dimensional center) the finite index condition holds if and only if the spectrum of the Radon-Nykodim derivative of φ with respect to a tracial weight (see [PT]) has finite spectrum.

Denote by $\mathscr{U}(M)$ the unitary group of M. Our interest in the inclusions $M^\varphi \subset M$ is motivated by the geometric study of the orbits

$$\mathscr{U}_\varphi = \{\varphi \circ \mathrm{Ad}(u) : u \in \mathscr{U}(M)\}.$$

In 5.1 we show that for the case $M = \mathscr{L}(H)$ the unitary orbit of a faithful

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and normal state is never a topological submanifold of $\mathscr{L}(H)^*$. Moreover, in the case of infinite weights, the orbit can not even be regarded as functionals on a suitable Banach space.

On the other hand, using the appropriate quotient topology, this sets can be regarded as homogeneous spaces (i.e. quotients of Banach–Lie groups with a differentiable structure). Namely, $\mathscr{U}_\varphi \simeq \mathscr{U}(M)/\mathscr{U}(M^\varphi)$.

A reductive structure for a homogeneous space G/K is an invariant supplement for the Lie–Banach algebra $\mathscr{K}$ of K in the Lie–Banach algebra $\mathscr{G}$ of G, where invariant means invariant under the natural action of K in $\mathscr{G}$.

In our case, such a supplement is given by $\mathrm{Ker}(E_\varphi)$. Moreover, by means of the expectation E_φ one may obtain a representation of $\mathscr{U}_\varphi$ as a space of projections in a von Neumann superalgebra M_1 of M. Recall the basic construction associated to a (φ invariant) conditional expectation. This yields a projection e_φ (denoted the Jones projection of E_φ) acting on the Hilbert space H_φ of the GNS triple of φ. As is usual notation , M_1 is the von Neumann algebra generated by e_φ and M. Then e_φ satisfies

i) $e_\varphi m e_\varphi = E_\varphi(m)e_\varphi$, $m \in M$

ii) $\{e_\varphi\}' \cap M = M^\varphi$

iii) $e_\varphi M_1 e_\varphi$ is isomorphic to M^φ via the *-isomorphism $x \mapsto xe_\varphi = e_\varphi x e_\varphi$.

Then, as in [AS2], one has the (one to one) representation

$$\mathscr{U}_\varphi \hookrightarrow \{\text{projections of } M_1\}$$

$$\varphi \circ \mathrm{Ad}(u) \mapsto u^* e_\varphi u$$

which is continuous if $\mathscr{U}_\varphi$ is considered with the quotient topology $\mathscr{U}(M)/\mathscr{U}(M^\varphi)$ and the set of projections with the norm topology of M_1.

If the index of E_φ is finite, then this map is a topological imbedding (i.e. a homeomorphism between $\mathscr{U}_\varphi$ and $\mathscr{U}_M(e_\varphi) = \{u^* e_\varphi u : u \in \mathscr{U}(M)\}$). Moreover, in [AS] it was shown that $\mathscr{U}_M(e_\varphi)$ is a C^∞ submanifold of M_1 if and only if the index of E_φ is finite. Therefore, the finite index condition allows one to regard $\mathscr{U}_\varphi$ as a C^∞ submanifold of the space of projections of M_1 (which is a space with rich geometric structure [CPR] and itself a submanifold of M_1). If the index is infinite, one still has the continuous representation (called basic representation) $\mathscr{U}_\varphi \hookrightarrow \{\text{projections of } M_1\}$.

If the index of E_φ is infinite, the above representation is a homeomorphism if one changes the usual norm of M by the M^φ-Hilbert module norm induced by E_φ. The price payed is that M is not complete with this new norm (completeness being equivalent to the finite index condition). Section 5 is devoted to the description of this application.

1.2 *Preliminary and notations*. Throughout this paper φ will denote a faithful normal semifinite weight and σ_t^φ the modular group of φ. Denote by

$M^\varphi = \{x \in M : \sigma_t^\varphi(x) = x, \forall t \in \mathsf{R}\}$. A strictly semifinite weight is a semifinite weight of M such that $\varphi|_{M^\varphi}$ is also semifinite on M^φ.

As it is standard notation, $\mathcal{N}_\varphi = \{x \in M : \varphi(x^*x) < \infty\}$ and $(H_\varphi, \pi_\varphi, \eta_\varphi)$ denotes the GNS triple for φ , i.e. H_φ is the completion of $\mathcal{N}_\varphi$ to a Hilbert space, $\pi_\varphi : M \longrightarrow \mathcal{L}(H_\varphi)$ is the usual $*$-isomorphism, $\eta_\varphi : \mathcal{N}_\varphi \longrightarrow H_\varphi$ is the canonical imbedding, with

$$\langle \eta_\varphi(y), \eta_\varphi(z) \rangle = \varphi(z^*y) \quad \text{and} \quad \pi_\varphi(z)\eta_\varphi(y) = \eta_\varphi(zy) \text{ for } y, z \in \mathcal{N}_\varphi, z \in M$$

If G is a locally compact group, an invariant mean m on G is a state of $L^\infty(G)$ such that m is invariant under the action of G, i.e. $m(f(t_0.)) = m(f(.))$ for all $t_0 \in G$ and $f \in L^\infty(G)$ (see [Pa]).

Let m be an invariant mean in R, and $\alpha : \mathsf{R} \longrightarrow \mathrm{Aut}(M)$ $(\alpha(t) = \alpha_t,)$ a weakly continuous homomorphism of R on the automorphism group of a von Neumann algebra M represented on $\mathcal{L}(H)$. For $x \in M$ denote by $\int_{\mathsf{R}} \alpha_t(x)dm(t)$ the element of M given by

$$\left\langle \left(\int_{\mathsf{R}} \alpha_t(x)dm(t) \right) \xi, \eta \right\rangle = m(t \mapsto \ < \alpha_t(x)\xi, \eta >) \text{ , for } \xi, \eta \in H$$

This integral does not depend on the Hilbert space H.

Let $E : M \to N$ be a normal conditional expectation and let ψ be a normal state of N. Denote also by ψ the extension of this state to M given by $\psi \circ E$. Let e be the Jones projection of E (and ψ), i.e. the orthogonal projection obtained as the closure of E as an operator on $L^2(M, \psi)$ (with range $L^2(N, \psi)$). As it is also standard, denote by M_1 the algebra generated by M and e in $\mathcal{L}(L^2(M, \psi))$.

After the introduction of Jones paper [Jo] several notions of index for a conditional expectation between von Neumann algebras appeared ([PP], [K],[W]). We will follow the terminology of the paper by Baillet, Denizeau and Havet [BDH], where three notions of index are considered. Let $N \subset M$ be a von Neumann subalgebra and $E : M \to N$ a conditional expectation.

E is said to be of *weak finite index* if there exists a positive real number λ such that $E - \lambda \mathrm{Id}$ is a positive map. In that case put $\mathrm{Ind}_w(E) := \lambda_0^{-1}$ where λ_0 is the supremum of all such λ's (the index is said to be ∞ if no such λ exists).

In such case ([Po], [BDH], [FK]), there exists another positive constant κ such that $E - \kappa \mathrm{Id}$ is completely positive, and there is a family $\{m_i\}_{i \in I} \subset M$ such that

i) $1 = \sum_{i \in I} m_i e m_i^*$, (e the Jones projection associated to E)
ii) $E(m_i^* m_j) = \delta_{ij} p_i$, (p_i projections in N)
iii) and $\sum_{i \in I} m_i m_i^*$ converges ultraweakly in M.

In [BDH] it is shown that the limit of the latter sum belongs to the center of M, and is called the *index of* E, and will be denoted by $\mathrm{Ind}(E)$.

It is said that E has *strongly finite index* if the family $\{m_i\}_{i\in I}$ is finite.

There are conditional expectations that are of finite index and not of strongly finite index [J], [FK]. Nevertheless these two notions of finite index conincide in many cases, for instance if N is a subfactor [BDH].

2. Conditional expectations and invariant means

REMARK 2.1. *Let m be an invariant mean in R, φ a faithful normal semifinite weight on a von Neumann algebra M, and $E_\varphi^m : M \longrightarrow M$ the map*

$$E_\varphi^m(x) = \int_{\mathsf{R}} \sigma_t^\varphi(x)dm(t).$$

Then E_φ^m is a σ_t^φ-invariant conditional expectation with range M^φ which satisfies

$$\varphi(E_\varphi^m(x)) \le \varphi(x) \quad \text{ for all } x \in M_+.$$

Moreover, if φ is strictly semifinite, then E_φ^m is the unique faithful and normal conditional expectation invariant with φ.

PROOF. The fact that E_φ^m is a σ^φ invariant conditional expectation is well known (see for instance 10.12 [S]).

In order to verify that $\varphi\big(E_\varphi^m(x)\big) \le \varphi(x)$ for every $x > 0$, suppose M represented in a standard form. Therefore $\varphi(z) = \sup_i \langle z\xi_i, \xi_i\rangle$ for certain vectors ξ_i, $i \in I$. Then

$$\varphi\big(E_\varphi^m(x)\big) = \sup_i \langle E_\varphi^m(x)\xi_i, \xi_i\rangle = \sup_i \ m(t \mapsto \langle \sigma_t^\varphi(x)\xi_i, \xi_i\rangle) \le \varphi(x)$$

where the last inequality holds because $\langle \sigma_t^\varphi(x)\xi_i, \xi_i\rangle \le \varphi(\sigma_t^\varphi(x)) = \varphi(x)$.

Let us suppose now that φ is strictly semifinite. We will see that under this hypothesis E_φ^m is faithful, normal and $\varphi \circ E_\varphi^m = \varphi$, and therefore coincides with Takesaki's expectation.

PROOF. Let $x \in M_+$. Since φ is strictly semifinite, there exists a normal positive linear functional ψ which is σ_t^φ invariant such that $\psi(x) > 0$. Then we can write $\psi = \varphi(h.)$ whith h a positive operator affiliated to $Z(M^\varphi)$, and for certain $h_\epsilon \in Z(M^\varphi)$ (see [P]),

$$\psi\big(E_\varphi^m(x)\big) = \varphi\big(hE_\varphi^m(x)\big) = \lim_{\epsilon\longrightarrow 0} \varphi\big(h_\epsilon^{1/2}E_\varphi^m(x)h_\epsilon^{1/2}\big).$$

Note that $\varphi(h_\epsilon) < \infty$ (i.e. $h_\epsilon^{1/2} \in \mathcal{N}_\varphi$) and

$$\begin{aligned}\varphi\big(h_\epsilon^{1/2}E_\varphi^m(x)h_\epsilon^{1/2}\big) &= \langle E_\varphi^m(x)\eta_\varphi(h_\epsilon^{1/2}),\eta_\varphi(h_\epsilon^{1/2})\rangle\\ &= m\big(t\mapsto \langle\sigma_t^\varphi(x)\eta_\varphi(h_\epsilon^{1/2}),\eta_\varphi(h_\epsilon^{1/2})\rangle\big)\\ &= m\big(t\mapsto \varphi\big(h_\epsilon^{1/2}\sigma_t^\varphi(x)h_\epsilon^{1/2}\big)\big) = m\big(t\mapsto\varphi\big(\sigma_t^\varphi(h_\epsilon^{1/2}xh_\epsilon^{1/2})\big)\big)\\ &= \varphi\big(h_\epsilon^{1/2}xh_\epsilon^{1/2}\big).\end{aligned}$$

Therefore $\psi(E_\varphi) = \psi(x) > 0$, which proves that $E_\varphi m(x) \neq 0$ and then E_φ^m is faithful.

Let us show that E_φ^m is normal. If $p_i, p \in M$, and $p_i \nearrow p$, then $E_\varphi^m(p_i) \nearrow i \to \sup\ E_\varphi^m(p_i) = q \in M^\varphi$. Clearly $q \leq E_\varphi^m(p)$. Suppose $E_\varphi^m(p) - q \neq 0$. Since φ is strictly semifinite, there exists a normal positive linear functional ψ which is σ_t^φ invariant such that $\psi\big(E_\varphi^m(p) - q\big) > 0$. As before, this ψ verifies $\psi\big(E_\varphi^m(y)\big) = \psi(y)$ for $y \in M$, and then

$$\begin{aligned}\psi\big(E_\varphi^m(p) - q\big) &= \psi\big(E_\varphi^m(p) - E_\varphi^m(q)\big) = \psi(p-q)\\ &= \sup_i\ \psi(p_i) - \sup_i\ \psi\big(E_\varphi^m(p_i)\big) = 0\end{aligned}$$

Therefore $E_\varphi^m\big(\sup_i\ p_i\big) = E_\varphi^m(p)$ which proves the normality of E_φ^m.

It remains to verify that $\varphi\circ E_\varphi^m = \varphi$. Since φ is strictly semifinite there exist normal positive functionals ψ_i with orthogonal supports p_i such that $\varphi = \sum_i \psi_i$. Therefore, $\varphi(p_i.) = \psi_i$ with $p_i \in M^\varphi$, and then $\psi_i \circ \sigma_t^\varphi = \psi_i$. If we represent M in a standard form, there exist ξ_i such that $\psi_i(x) = \langle x\xi_i, \xi_i\rangle$ for all $x \in M$, and then

$$\psi_i\big(E_\varphi^m(x)\big) = \langle E_\varphi^m(x)\xi_i,\xi_i\rangle = m\big(t\mapsto\langle\sigma_t^\varphi(x)\xi_i,\xi_i\rangle\big) = m\big(t\mapsto\psi_i(\sigma_t^\varphi(x))\big) = \psi_i(x).$$

Therefore, $\varphi\big(E_\varphi^m(x)\big) = \sum_i \psi_i\big(E_\varphi^m(x)\big) = \sum_i \psi_i(x) = \varphi(x)$, which completes the proof.

In other words, if φ is a faithful normal strictly semifinite weight then E_φ^m does not depend on the choice of m.

Remark 2.2. If $J \subset \mathsf{R}$ is an interval, put $\Psi_J(x) = |J|^{-1}\int_J x(t)dt$. This positive forms are called Bohr means. Eberlein ([E] Th. 5.2) showed that if x is almost periodic then the net $\{\Psi_J(x)\}_{J\subset\mathsf{R}}$ converges to a unique limit. Using this result, applied to $x(t) = \langle\sigma_t^\varphi(x)\xi,\eta\rangle$, it can be proven directly for almost periodic weights (i.e. weights φ such that the modular operator Δ_φ is diagonalizable) that $\langle E_{\Psi_J}(x)\xi,\eta\rangle$ converges to a unique limit, namely $E_\varphi(x)$. Note that almost periodic weights are strictly semifinite.

Example 2.3. If the weight φ of 2.1 fails to be strictly semifinite, the expectation E_φ^m may be neither normal nor faithful, and depends on the choice of m. Let m be a mean for R obtained as in 2.2. Let $M = \mathscr{L}(L^2(0,1))$, $h = \mu_x$

the operator "multiplication by x" and $\varphi(x) = \mathrm{Tr}(hx)$ where Tr is the trace of M. It is straightforward to verify that $M^{\varphi} = \{h\}' = L^{\infty}(0,1)$ regarded as multiplication operators μ_f in $L^2(0,1)$. Therefore in this case, for all $a \in M$

$$E_{\varphi}^{m}(a) = \int_{\mathsf{R}} \mu_{x^{it}} a \mu_{x^{-it}} dm(t).$$

We will show that these expectations coincide with limit points of the so called von Neumann's operation [U], i.e. expectations obtained as cluster points of the net of expectations

$$\left\{ E_{\Pi}(a) = \sum_{\Delta \in \Pi} \mu_{\chi_{\Delta}} a \mu_{\chi_{\Delta}} : \Pi \text{ finite partitions of } (0,1) \right\}$$

where χ_{Δ} is the characteristic function of $\Delta \in \Pi$, the partitions are directed by inclusion and the topology is that of weak operator convergence at every $a \in M$. In [Su] it was shown that there exists an uncountable set of limit points for this net. Moreover, this limit points are conditional expectations which have the compact operators in their kernels, and therefore can not be normal nor faithful.

Let $\sum_i \alpha_i \chi_{J_i}$ be a step function close to the identity function t in the norm topology of $L^{\infty}(0,1)$. Then for every $a \in \mathscr{L}(L^2(0,1))$ we have that $E_{\varphi}^{m}(a)$ is close to

$$\int_{\mathsf{R}} \sum_{n,m} \alpha_n^{it} \alpha_m^{-it} \mu_{\chi_{J_n}} a \mu_{\chi_{J_m}} dm(t) =$$

$$= \sum_{n \neq m} \mu_{\chi_{J_n}} a \mu_{\chi_{J_m}} \int_{\mathsf{R}} \alpha_n^{it} \alpha_m - it dm(t) + \sum_n \mu_{\chi_{J_n}} a \mu_{\chi_{J_n}}.$$

It is easy to verify that for $n \neq m$ (with invariant means obtained as weak limits of Bohr means) $\int_{\mathsf{R}} \alpha_n^{it} \alpha_m^{-it} dm(t) = 0$. Therefore $E_{\varphi}^{m}(a)$ can be approximated (in the weak operator topology) by the sums

$$\sum_n \mu_{\chi_{J_n}} a \mu_{\chi_{J_n}} = E_{\Pi}(a)$$

if Π is the partition given by $J_1, \ldots, J_n$.

Suarez showed in [Su] that the expectations obtained as limits of the E_{Π} when restricted to the operators which are diagonal in the Fourier basis take scalar values. Denote by $\{e_n\}_{n \in \mathsf{Z}}$ the Fourier basis of $L^2(0,1)$. Then for each invariant mean m of R the linear functional

$$l^{\infty}(\mathsf{Z}) \ni \{\lambda_n\} \mapsto E_{\varphi}^{m}\left(\sum_n \lambda_n e_n \otimes e_n \right) \in \mathsf{C}$$

is an invariant mean for Z. Moreover, any mean in Z may be obtained in this fashion [Su].

3. Conditional expectations with finite index

Kosaki proved (see [W], Prop. 2.5.2) that if $M \supset N$ are factors and $E : M \longrightarrow N$ is a faithful and normal conditional expectation of finite index, then M is finite iff N is finite. Jolissaint [J] generalized this result for non factors (with the finite index condition). Let us state this result.

Proposition 3.1. *Let M be a von Neumann algebra and $E : M \longrightarrow N$ a faithful and normal conditional expectation.*

a) *If N is finite and E is of strongly finite index then M_1 and M are finite.*

b) *If N is semifinite and E is of finite index then M_1 and M are semifinite.*

Proof. a): Let φ be a faithful normal state on N, then $\varphi \circ E$ is a faithful normal state on M. Let us also denote by φ this extension. Let $(H_\varphi, \pi_\varphi, \xi_\varphi)$ be the GNS triple for φ and M and $J_\varphi = J$. Since N is finite, N' is semifinite. Therefore, $M_1 = JN'J$ is also semifinite.

The projection $[N'\xi_\varphi] \in N$ is finite. Recall ([KR] 9.1.2) that $[N'\xi_\varphi] \in N$ is finite iff $[N\xi_\varphi] \in N'$ finite. Therefore $[N\xi_\varphi] = [N.1] = e \in N'$ is finite, and then $JeJ = e$ is a finite projection in M_1. Then there exists a semifinite faithful tracial weight τ on $c(e)M_1 = M_1$ such that $\tau(e) < \infty$ ([P], 5.4.6).

Since $E : M \longrightarrow M^\varphi$ is of strongly finite index, $1 = \sum_{i=1}^k m_i e m_i^*$ for certain $m_i \in M$. Then

$$\tau(m_i e m_i^*) = \tau(m_i e (m_i e^*)) = \tau(e m_i^* m_i e) \leq \tau\big(\|m_i\|^2 e\big) < \infty.$$

This shows that $\tau(1) < \infty$, and therefore τ is a finite trace and M_1 and M are finite.

b): As in the previous point, M_1 is semifinite. Since the index of E is finite, by [BDH] 3.10 there exists a faithful and normal conditional expectation $E_1 : M_1 \to M$. Then it follows that M is semifinite.

Corollary 3.2. *Let φ be a faithful normal strictly semifinite weight in M.*

a) *If φ is finite and $E_\varphi : M \longrightarrow M^\varphi$ has strongly finite index then M is finite.*

b) *If $E_\varphi : M \longrightarrow M^\varphi$ has finite index then M is semifinite.*

4. Characterization of E_φ with finite index

In view of 3.2, if one will look for E_φ with finite index one has to consider semifinite algebras.

Lemma 4.1. *Let M be a semifinite von Neumann algebra, τ a tracial weight* *on M, $\varphi = \tau(h.)$ a faithful normal strictly semifinite weight on M. If $\{p_i\}_{i\in I}$ are orthogonal spectral projections of h with $\sum_{i\in I} p_i = 1$, then*

$$E_\varphi(x) = E_\varphi\left(\sum_{i\in I} p_i x p_i\right).$$

If moreover $\sigma(h) = \{r_i : i \in \mathsf{N}\}$ is discrete, then

$$E_\varphi(x) = \sum_{i\in\mathsf{N}} p_i x p_i,$$

with p_i the spectral projection of h associated to $\{r_i\}$.

Proof. Note that $p_i \in Z(M^\varphi)$ and therefore

$$E_\varphi\left(\sum_i p_i x p_i\right) = \sum_{i\in\mathsf{N}} E_\varphi(p_i x p_i) = \sum_{i\in\mathsf{N}} p_i E_\varphi(x) p_i = \left(\sum_{i\in\mathsf{N}} p_i\right) E_\varphi(x) = E_\varphi(x).$$

Suppose now that $\sigma(h) = \{r_i : i \in \mathsf{N}\}$, then $h = \sum_i r_i p_i$. Put $E(x) = \sum_i p_i x p_i$. E is a faithful and normal conditional expectation onto $\{p_i : i \in \mathsf{N}\}' \cap M = \{h\}' \cap M = M^\varphi$. Note that it is also φ invariant:

$$\begin{aligned}\varphi\big(E(x)\big) &= \varphi\left(\sum_i p_i x p_i\right) = \sum_i \tau(h p_i x p_i) = \sum_i \tau(r_i p_i x) = \tau\left(\sum_i r_i p_i x\right)\\ &= \varphi(x).\end{aligned}$$

Therefore $E = E_\varphi$.

Theorem 4.2. *Let M be a finite von Neumann algebra with finite dimensional center, φ a faithful normal strictly semifinite weight on M and $E_\varphi : M \longrightarrow M^\varphi$ the unique normal and faithful conditional expectation that leaves φ invariant. Let τ be a faithful and normal trace in M and h affiliated to $Z(M^\varphi)$ such that $\varphi = \tau(h.)$. The following statements are equivalent:*

(i) *E_φ has strongly finite index.*
(ii) *E_φ has weakly finite index.*
(iii) *$\sigma(h)$ is finite.*
And in particular, φ is finite.

Proof. As $\dim\big(Z(M)\big) < \infty$, we can suppose $M = \oplus_{1\le i\le n} M_i$, with M_i a finite factor. Then $E_\varphi\big|_{M_i} : M_i \longrightarrow M_i^\varphi$ defines a conditional expectation (because $Z(M) \subset M^\varphi$), $E_\varphi = \sum_{i=1}^n E_\varphi\big|_{M_i}$, and E_φ has strongly finite index iff each $E_\varphi\big|_{M_i}$ does. Moreover, if h is affiliated to $Z(M^\varphi)$, then $h = \sum_{i=1}^n h_i$ with h_i affiliated to $Z(M_i^\varphi)$, and the spectrum of h is finite iff the spectrum of each h_i is finite. Therefore, we can suppose that M is a finite factor.

(i) $\Longrightarrow$ (ii) is trivial.

(iii) $\Longrightarrow$ (i): If $\sigma(h) = \{r_1\}$, then $h = r_1 1$. In this case $\varphi = r_1\tau$, and therefore $E_\varphi = Id$ which has strongly finite index.

Reasoning by induction, we are going to asume that the proposition is valid for $\sigma(h) = \{r_i : 1 \leq i \leq n\}$ and prove it for the case $n+1$. Then, if $\{p_i\}_{1\leq i\leq n}$ are the orthogonal spectral projections of h corresponding to $\{r_i\}$ with $1 \leq i \leq n$, there can be found finite E_φ-orthonormal $m_k \in M$, $1 \leq k \leq z$, such that for all $x \in M$, $x = \sum_{k=1}^{z} m_k E_\varphi(m_k^* x)$. The existence of such family is equivalent to the strongly finite index condition of E_φ (see [BDH]).

Suppose that $\sigma(h) = \{r_i : 1 \leq i \leq n+1\}$, and $\{p_i\}_{1\leq i\leq n+1}$ are the orthogonal spectral projections of h corresponding to $\{r_i\}_{1\leq i\leq n+1} \subset \sigma(h)$. As M is a factor, we can suppose $p_i \preceq p_{i+1}$ for $1 \leq i \leq n$.

We can assume the theorem is valid for the factor $(1-p_{n+1})M(1-p_{n+1})$, the expectation $E_\varphi|_{(1-p_{n+1})M(1-p_{n+1})}$ and the operator $h_n = \sum_{i=1}^{n} r_i p_i$. Since h is affiliated to $Z(M^\varphi)$, then $p_i \in Z(M^\varphi)$, for $1 \leq i \leq n+1$, and then

$$\begin{aligned} E_\varphi\big((1-p_{n+1})x(1-p_{n+1})\big) &= E_\varphi\left(\left(\sum_{i=1}^{n} p_i\right) x \left(\sum_{i=1}^{n} p_i\right)\right) \\ &= \left(\sum_{i=1}^{n} p_i\right) E_\varphi(x) \sum_{i=1}^{n} p_i \\ &= (1-p_{n+1})E_\varphi(x)(1-p_{n+1}) = (1-p_{n+1})E_\varphi(x). \end{aligned}$$

Therefore $E_\varphi\big((1-p_{n+1})M(1-p_{n+1})\big) \subset (1-p_{n+1})M(1-p_{n+1})$. There exist orthonormal $m_i \in (1-p_{n+1})M(1-p_{n+1})$ with $1 \leq i \leq z$, such that

$$\begin{aligned} (1) \quad (1-p_{n+1})x(1-p_{n+1}) &= \sum_{i=1}^{z} m_i E_\varphi(m_i^*(1-p_{n+1})x(1-p_{n+1})) \\ &= \sum_{i=1}^{z} m_i(1-p_{n+1})E_\varphi(m_i^* x) = \sum_{i=1}^{z} m_i E_\varphi(m_i^* x) \end{aligned}$$

for all $x \in M$.

Now as $p_1 \preceq p_2 \preceq \ldots \preceq p_n \preceq p_{n+1}$, for $1 \leq j \leq n+1$ we can write

$$p_j = \left(\sum_{i=1}^{w_j} q_{j,i}\right) + r_j, \text{ with } q_{j,i} \sim p_1 (= q_{1,1}) \text{ for } 1 \leq i \leq w_j, \text{ and } r_j \prec p_1$$

where $\{q_{j,i}\}_{1\leq i\leq w_j}$ and r_j are orthogonal subprojections of p_j.

There exist partial isometries $\{v_{j,i}^{t,h}\}$ in M with $1 \leq j,t \leq n+1$, $1 \leq i \leq w_j$ and $1 \leq h \leq w_t$ such that $q_{j,i} \overset{v_{j,i}^{t,h}}{\sim} q_{t,h}$. Therefore, they verify

$$(v_{j,i}^{t,h})^* = v_{t,h}^{j,i} \ , \ v_{j,i}^{t,h} v_{t,h}^{j,i} = q_{t,h} \ , \ v_{j,i}^{t,h} v_{l,w}^{k,s} = \delta_{(j,i)(k,s)} \ v_{l,w}^{t,h}$$

Then $v_{j,i}^{j,i} = q_{j,i}$, and $v_{j,i}^{t,h}.q_{k,s}.v_{t,h}^{j,i} = v_{j,i}^{t,h} v_{k,s}^{k,s} v_{t,h}^{j,i} = \delta_{(j,i)(k,s)} v_{t,h}^{t,h} = \delta_{(j,i)(k,s)} \ q_{t,h}$.

There also exist partial isometries $\{u_j^{t,i}\}$ with $1 \le j, t \le n+1$ and $1 \le i \le w_t$ such that $r_j \overset{u_j^{t,i}}{\sim} s_{t,i} < q_{t,i}$. They verify

$$r_j = (u_j^{t,i})^* u_j^{t,i}, \text{ and } u_j^{t,i} (u_j^{t,i})^* = s_{t,i} < q_{t,i}$$

We are going to use the following equalities:

$$v_{j,i}^{t,h} p_k v_{t,h}^{j,i} = v_{j,i}^{t,h} \left(\sum_{s=1}^{w_k} q_{k,s} + r_k \right) v_{t,h}^{j,i} = \delta_{jk} q_{t,h}$$

$$(u_j^{t,h})^* p_i u_j^{t,h} = \delta_{it} r_j$$

Using 4.1, for every $x \in M$, $1 \le t \le n$ and $1 \le j \le w_{n+1}$ we have

$$v_{t,1}^{n+1,j} E_\varphi \big(v_{n+1,j}^{t,1} x \big) = v_{t,1}^{n+1,j} \sum_{i=1}^{n+1} p_i v_{n+1,j}^{t,1} x p_i = q_{n+1,j} x p_t. \tag{2}$$

Also, for $1 \le t \le n$ and $1 \le j \le m_t$,

$$\begin{aligned} v_{n+1,1}^{t,j} E_\varphi \big(v_{t,j}^{n+1,1} x \big) &= v_{n+1,1}^{t,j} \sum_{i=1}^{n+1} p_i v_{t,j}^{n+1,1} x p_i \\ &= v_{n+1,1}^{t,j} p_{n+1} v_{t,j}^{n+1,1} x p_{n+1} = q_{t,j} x p_{n+1}. \end{aligned} \tag{3}$$

For $2 \le t \le n+1$,

$$\begin{aligned} (u_t^{n+1,1})^* E_\varphi \big(u_t^{n+1,1} x \big) &= (u_t^{n+1,1})^* \sum_{i=1}^{n+1} p_i u_t^{n+1,1} x p_i 4 \\ &= (u_t^{n+1,1})^* p_{n+1} u_t^{n+1,1} x p_{n+1} \\ &= r_t x p_{n+1}. \end{aligned} \tag{4}$$

And for $1 \le t \le n$,

$$(u_{n+1}^{t,1})^* E_\varphi \big(u_{n+1}^{t,1} x \big) = (u_{n+1}^{t,1})^* \sum_{i=1}^{n+1} p_i u_{n+1}^{t,1} x p_i = (u_{n+1}^{t,1})^* p_t u_{n+1}^{t,1} x p_t = r_{n+1} x p_t \tag{5}$$

Recall that $\sum_{t=1}^{n+1} p_t = 1$ and $p_t = \big(\sum_{i=1} w_t q_{t,i} \big) + r_t$. It follows that

$$x = (1-p_{n+1})x(1-p_{n+1}) + p_{n+1}xp_{n+1} + \left(\sum_{t=1}^{n} p_t\right)xp_{n+1} + p_{n+1}x\left(\sum_{t=1}^{n} p_t\right)$$

$$= (1-p_{n+1})x(1-p_{n+1}) + p_{n+1}xp_{n+1} + \left(\sum_{t=1}^{n}\left(\sum_{j=1}^{w_t} q_{t,j}\right) + r_t\right)xp_{n+1} +$$

$$+ \left(\left(\sum_{j=1}^{w_{n+1}} q_{n+1,j}\right) + r_{n+1}\right)x\left(\sum_{t=1}^{n} p_t\right)$$

$$= (1-p_{n+1})x(1-p_{n+1}) + p_{n+1}xp_{n+1} + \sum_{t=1}^{n}\sum_{j=1}^{w_t} q_{t,j}xp_{n+1} +$$

$$+ \sum_{t=2}^{n} r_t x p_{n+1} + \sum_{j=1}^{w_{n+1}}\sum_{t=1}^{n} q_{n+1,j}xp_t + \sum_{t=1}^{n} r_{n+1}xp_t.$$

Then, using (1),(2),(3),(4) and (5) we obtain

$$x = \sum_{i=1}^{z-1} m_i E_\varphi(m_i^* x) + (m_z + p_{n+1})E_\varphi\big((m_z)^* x + p_{n+1}x\big) +$$
$$+ \sum_{t=1}^{n}\sum_{j=1}^{w_t} v_{n+1,1}^{t,j} E_\varphi\big(v_{t,j}^{n+1,1}x\big) + \sum_{t=2}^{n}(u_t^{n+1,1})^* E_\varphi\big(u_t^{n+1,1}x\big) +$$
$$+ \sum_{j=1}^{w_{n+1}}\sum_{t=1}^{n} v_{t,1}^{n+1,j} E_\varphi\big(v_{n+1,j}^{t,1}x\big) + \sum_{t=1}^{n}(u_{n+1}^{t,1})^* E_\varphi\big(u_{n+1}^{t,1}x\big).$$

It is straightfoward to verify that the union of the sets $\{m_i\}_{1\le i\le z-1}$, $\{m_z + p_{n+1}\}$, $\{v_{n+1,1}^{t,j}\}_{\substack{1\le t\le n\\ 1\le j\le w_t}}$, $\{v_{t,1}^{n+1,j}\}_{\substack{1\le t\le n\\ 1\le j\le w_{n+1}}}$, $\{(u_t^{n+1,1})^*\}_{2\le t\le n}$ and $\{(u_{n+1}^{t,1})^*\}_{1\le t\le n}$ forms an orthonormal family. Therefore E_φ has strongly finite index.

(ii) $\Longrightarrow$ (iii): If $\sigma(h)$ is infinite, for every $n \in \mathsf{N}$ there exist n orthogonal spectral projections $\{p_i\}_{1\le i\le n}$ of h with $\sum_{1\le i\le n} p_i = 1$. Since h is affiliated to $Z(M^\varphi)$, then $p_i \in Z(M^\varphi)$. As M is a factor, we can suppose $p_1 \preceq \ldots \preceq p_n$. Therefore, there exist subprojections $q_i \in M$, $q_i \le p_i$, $1 \le i \le n$ such that $q_i \sim q_1 = p_1$, and n^2 partial isometries $v_{ij} \in M$ that verify:

$v_{lj}v_{lj}^* = q_l$, $v_{lj}^* v_{lj} = q_j$, $v_{ij}^* = v_{ji}$, and $v_{lj}v_{kr} = \delta_{jk}v_{lr}$ with $1 \le l,j,k,r \le n$.
Observe that $p_k v_{ij} p_k = \delta_{ik}\delta_{jk}v_{ij}$.

Let $v = \sum_{l,j=1}^{n} v_{lj}$. Then

$$v^2 = \sum_{l,j,k,r=1}^{n} v_{lj}v_{kr} = \sum_{l,j,r=1}^{n} v_{lr} = n\sum_{l,r=1}^{n} v_{lr} = nv$$

Then $v \ge 0$ and $||v|| = n$.

Using lemma 4.1, $E_\varphi(v) = E_\varphi(\sum_{i\in I} p_i v p_i)$, and then

$$E_\varphi(v) = E_\varphi\left(\sum_{1\le i,j,k\le n} p_k v_{ij} p_k\right) = E_\varphi\left(\sum_{k=1}^n v_{kk}\right) = \sum_{k=1}^n E_\varphi(q_k)$$

But $p_i E_\varphi(q_k) p_i = E_\varphi(p_i q_k p_i) = \delta_{ik} E_\varphi(q_k)$, and then $E_\varphi(q_k) \in p_k M p_k$. Therefore

$$||E_\varphi(v)|| = \left\| \sum_k E_\varphi(q_k) \right\| \ge ||E_\varphi(q_1)|| = ||E_\varphi(p_1)|| = \|p_1\| = 1$$

because $q_1 = p_1 \in M^\varphi$. In other words, for every $n \in \mathbb{N}$ there exists a positive element $v \in M$ such that $||E_\varphi(v)|| \ge \frac{1}{n}||v||$. This proves that E_φ has infinite weak index.

Remark 4.3. With the same hypothesis of 4.2, but for M a factor, it is straightforward to verify that if $\sigma(h)$ consists of n points, then

$$\mathrm{Ind}(E_\varphi) = n.$$

If M has finite dimensional center there exists $\{q_i\}_{1\le i\le w} \subset Z(M)$ such that $E_\varphi = \sum_{i=1}^w E_\varphi\big|_{Mq_i}$, where Mq_i is a finite factor. In that case, $\mathrm{Ind}(E_\varphi\big|_{Mq_i}) = n_i$, where n_i is the number of points of $\sigma(h_i)$ ($h_i = h\big|_{Mq_i} \in L(Mq_i)$). Then the strong index

$$\mathrm{Ind}(E_\varphi) = \sum_{i=1}^w n_i q_i \ \in Z(M).$$

The analogous result for the semifinite case also holds.

Theorem 4.4. *Let M be a semifinite von Neumann algebra with finite dimensional center, φ a faithful normal strictly semifinite weight and $E_\varphi : M \longrightarrow M^\varphi$ the unique normal and faithful conditional expectation that leaves φ invariant. Let τ be a tracial weight in M and h affiliated to $Z(M^\varphi)$ such that $\varphi = \tau(h.)$. The following statements are equivalent:*

(i) *E_φ has weakly finite index.*

(ii) *$\sigma(h)$ is finite.*

Proof. (ii) $\Longrightarrow$ (i): If $\sigma(h)$ is finite, denote by p_i the spectral projections of h. It is a standard fact that the expectation $E_\varphi(x) = \sum_{i=1}^n p_i x p_i$ has weakly finite index.

(i) $\Longrightarrow$ (ii) follows exactly as in the previous theorem.

Remark 4.5. If $\dim\big(Z(M)\big)$ is not finite, the preceeding results are not true. Take for example the finite von Neumann algebra $M = \oplus_{n\in\mathbb{N}} M_2(\mathbb{C})$

(where $M_2(\mathsf{C})$ are the 2×2 matrices). Put $h = \oplus_n h_n$ with $h_n \in M_2(\mathsf{C})$ the diagonal matrix $h_n = \begin{pmatrix} 1/n & 0 \\ 0 & 1+1/n \end{pmatrix}$. Let τ be the trace on M defined by $\tau(\oplus_n x_n) = \sum_n \frac{1}{2^n} \mathrm{tr}(x_n)$, where tr is the canonical trace on $M_2(\mathsf{C})$. Then, $\varphi(.) = \tau\left(\frac{h}{\tau(h)}.\right)$ defines a faithful normal state with $M^\varphi = M \cap \{h\}' = \{\oplus_n x_n : x_n \text{ is a diagonal matrix}\}$. Then if $D\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} z_{11} & 0 \\ 0 & z_{22} \end{pmatrix}$, it is clear that $E_\varphi(\oplus_n z_n) = \oplus_n D(z_n)$.

E_φ has strongly finite index: put $u_1 = 1 \in M$, $u_2 = \oplus_n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $u_3 = \oplus_n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, it is easy to see that for every $x \in M$, $x = \sum_{i=1}^3 u_i E_\varphi(u_i^* x)$ and that $E_\varphi(u_i^* u_j) = 0$ if $i \neq j$.

Observe that if $\varphi(.) = \mu(k.)$ with μ a trace on M and $k = \oplus_n k_n \in M_+^\varphi$. Then the spectrum of k has to be infinite. Indeed, μ is a multiple of tr on each $M_2(\mathsf{C})$ (let us say $\mu(z_n) = \alpha_n \,\mathrm{tr}(z_n)$) and therefore $\frac{1}{2^n}\mathrm{tr}(h_n.) = \alpha_n \,\mathrm{tr}(k_n.)$ for every $n \in \mathsf{N}$. Clearly this equality would not hold if the spectrum of k is finite.

Summarizing, we have found a faithful normal state φ on M, with E_φ of strongly finite index, such that for every trace μ on M, if $\varphi(.) = \mu(k.)$ then the spectrum of k is infinite.

It has been already observed (3.2) that if M is a type III algebra, there are no normal conditional expectations of finite index onto M^φ. Moreover, if M is a factor of type III there cannot exist E_φ with *weakly* finite index.

PROPOSITION 4.6. *Let M be a factor of type* III, *and φ a faithful normal and strictly semifinite weight in M. Then the weak index of E_φ is infinite.*

PROOF. By the result of [AS] cited above, if E_φ had weakly finite index, then M^φ should have finite dimensional center. Let $p_1, ..., p_n$ be the minimal projections of the center of M^φ. Then the expectations

$$E_i : M_{p_i} \to p_i M^\varphi \quad E_i(p_i x p_i) = E_\varphi(p_i x p_i)$$

have weak finite index, for $i = 1, ...n$. But since $p_i M p_i$ and $p_i M^\varphi$ are factors, the expectations E_i have strongly finite index. Therefore $\sum_i E_i : \oplus p_i M p_i \to M^\varphi$ has strongly finite index. Note that $E = (\sum_i E_i) \circ F$, where $F(x) = \sum_i p_i x p_i$. The projections $p_1, ..., p_n$ are equivalent in M. Therefore an argument analogous (but much simpler) to that of ii) $\Rightarrow$ i) in 4.2, shows that F has strongly finite index. Therefore E_φ has strongly finite index, which leads to a contradiction.

5. The geometry of the orbit of a faithful, normal, strictly semifinite weight

Let us introduce a case where the unitary orbits of faithful and normal states are not (topological) submanifolds of the predual (and neither the dual) space of the algebra. In the case of orbits of weights, no natural ambient space is given in order to model the manifold structure of the orbit. This facts justify the use of other topologies other than the usual norm topology.

PROPOSITION 5.1. *Let φ be a faithful, normal state of $\mathscr{L}(H)$ with H infinite dimensional. Then the unitary orbit $\mathscr{U}_\varphi = \{\varphi \circ \mathrm{Ad}(u) : u \in \mathscr{U}(\mathscr{L}(H))\}$ is not a topological submanifold of $\mathscr{L}(H)_*$ (and neither of $\mathscr{L}(H)^*$).*

PROOF. There exists $a \in \mathscr{L}(H)_+$ with $\mathrm{Tr}(a) = 1$ (Tr the usual trace), such that $\varphi(x) = \mathrm{Tr}(ax)$ for all $x \in \mathscr{L}(H)$. The element a is of the form $a = \sum_{i=1}^{\infty} \lambda_i p_i$ with $\dim R(p_i) = m_i < \infty$, and therefore $\sum_{i=1}^{\infty} \lambda_i m_i = 1$. Moreover, the unitary orbit $\mathscr{U}_\varphi \subset \mathscr{L}(H)_*$ with the norm topology identifies with $\mathscr{U}_a = \{u^* a u : u \in \mathscr{U}(\mathscr{L}(H))\} \subset \mathscr{T}(H)$ with the trace norm $\| \ \|_1$, where $\mathscr{T}(H)$ denotes the trace class of $\mathscr{L}(H)$.

Let u_n be unitaries in $\mathscr{L}(H)$ such that $u_n^* a u_n$ converge to b in the usual norm of $\mathscr{L}(H)$.

Then it can be proved that

i) b is compact and positive,

ii) $\sigma(b) = \{\lambda_i : i \in \mathsf{N}\} = \sigma(a)$, $b = \sum_{i=1}^{\infty} \lambda_i q_i$ with $\dim R(q_i) = m_i$, and $\mathrm{Tr}(b) = 1$,

iii) $u_n^* a u_n \to b$ in $\| \ \|_1$.

If $\mathscr{U}_a = \{u^* a u : u \in \mathscr{U}(\mathscr{L}(H))\}$ is a submanifold of $(\mathscr{T}(H), \| \ \|_1) = \mathscr{L}(H)_*$ then $\mathscr{U}_a$ is locally closed in $\mathscr{T}(H)$. That is, each point $c \in \mathscr{U}_a$ has a neighbourhood of the form $\{d \in \mathscr{U}_a : \|c - d\|_1 \leq \varepsilon\}$ which is closed in $\mathscr{T}(H)$. But since the action of the unitaries of $\mathscr{L}(H)$ is isometric on $\mathscr{T}(H)$, the number ε can be chosen the same for all c in $\mathscr{U}_a$. This clearly implies that the orbit $\mathscr{U}_a$ is closed in $\mathscr{T}(H)$. Therefore, by the remarks above, $\mathscr{U}_a$ is closed in the usual norm of $\mathscr{L}(H)$. In his remarkable paper [V], Voiculescu proved, as a by-product of his non-commutative Weyl-von Neumann theorem, that this condition - closedness in norm of the unitary orbit of an operator in $\mathscr{L}(H)$ - implies that the operator generates a finite dimensional C*-algebra. In our case, since a is positive, this implies that the spectrum of a is finite. This leads to a contradiction, since a is also compact and has zero kernel.

Let φ be a faithful, normal, strictly semifinite weight on a von Neumann algebra M. Let $(H_\varphi, \pi_\varphi, \eta_\varphi)$ be the GNS triple as in 1.2, and $E : M \to N$ a φ invariant conditional expectation onto a von Neumann subalgebra $N \subset M$. Let $e \in L(H_\varphi)$ be the Jones projection associated to E, that is e is the orthogonal projection on $\overline{N \cap \mathcal{N}_\varphi} \subset H_\varphi$, and $M_1 = (M, e)'' \subset L(H_\varphi)$ the basic extension. From now on we shall identify $x \in M$ with its image $\pi_\varphi(x) \in \mathcal{L}(H_\varphi)$.

The orbit $\mathcal{U}_\varphi = \{\varphi \circ \mathrm{Ad}(u) : u \in \mathcal{U}(M)\}$ is not "included" in any Banach space. Nevertheless, we shall introduce a representation for $\mathcal{U}_\varphi$ which will allow us to present it as a space of projections of the basic extension of M by E_φ. $\mathcal{U}_\varphi$ will be considered with the quotient topology $\mathcal{U}(M)/\mathcal{U}(M^\varphi)$.

If φ is a faithful, normal, strictly semifinite weight, let e_φ be the Jones projection of $E_\varphi : M \to M^\varphi$. The orbit $\mathcal{U}_M(e_\varphi) = \{u^* e_\varphi u : u \in \mathcal{U}(M)\}$ is a C^∞ homogeneous space [AV]. The continuous map (called the basic representation of $\mathcal{U}_\varphi$)

$$\beta : \mathcal{U}_\varphi \to \mathcal{U}_M(e_\varphi) \subset M_1$$

$$\beta(\varphi \circ \mathrm{Ad}(u)) = u^* e_\varphi u$$

is a bijection that preserves the adjoint and the orbits.

The tangent space $T(\mathcal{U}_M(e_\varphi))_{e_\varphi}$ identifies with the space $\{x e_\varphi - e_\varphi x (= [x, e_\varphi]) : x \in M, x^* = -x\}$.

$\mathcal{U}_M(e_\varphi) \subset M_1$ is a Banach homogeneous space, consisting of projections of M_1. It has a natural connection (see [ALRS]), for example, its geodesics can be easily computed. Namely, the unique geodesic φ_t on $\mathcal{U}_M(e_\varphi)$ with $\varphi_0 = e_\varphi$ and $\frac{d}{dt}\varphi_t\big|_{t=0} = [x, e_\varphi]$ is given by

$$\varphi_t = e^{tx_0} e_\varphi e^{-tx_0}$$

where $x_0 = x - E_\varphi(x)$.

Remark 5.2. a) One has the following commutative diagram

$$\begin{array}{ccc} \mathcal{U}(M) & \xrightarrow{\Pi_\varphi} & \mathcal{U}_\varphi \\ & \Pi_{e_\varphi} \searrow & \downarrow \beta \\ & & \mathcal{U}_M(e_\varphi) \end{array}$$

where $\Pi_\varphi(u) = \varphi \circ \mathrm{Ad}(u)$ and $\Pi_{e_\varphi}(u) = u^* e_\varphi u$, for $u \in \mathcal{U}(M)$. The map Π_{e_φ} has local cross sections, namely (near e_φ):

$$s(u^* e_\varphi u) = (E_\varphi(u) E_\varphi(u^*))^{-1/2} E_\varphi(u) u\,,$$

(i.e. the unitary part of $E_\varphi(u)u$ in its polar decomposition) defined in $\{u^* e_\varphi u : \|u^* e_\varphi u - e_\varphi\| < 1\}$, which takes values in $\mathcal{U}(M)$ and satisfies

$$\Pi_{e_\varphi} \circ s(u^* e_\varphi u) = u^* e_\varphi u \text{ and } s(e_\varphi) = 1.$$

Note that if $\|u^* e_\varphi u - e_\varphi\| < 1$ then also $\|e_\varphi u^* e_\varphi u e_\varphi - e_\varphi\|$ and $\|e_\varphi - e_\varphi u e_\varphi u^* e_\varphi\|$ are strictly less than 1, which implies (using the properties of the basic extension) that $\|E_\varphi(u^*)E_\varphi(u) - 1\| < 1$ and $\|1 - E_\varphi(u)E_\varphi(u^*)\| < 1$, and therefore s is well defined.

b) If M is endowed with the left M^φ-Hilbert module norm induced by E_φ, i.e. $\|x\|_{E_\varphi} = \|E_\varphi(x^*x)\|^{1/2}$, then s is continuous. This is a straightforward verification, using the fact noted in a), that if $u^* e_\varphi u$ is close to e_φ then $E_\varphi(u^*)E_\varphi(u)$ and $E_\varphi(u)E_\varphi(u^*)$ are close to 1. Now, if Π_{e_φ} has continuous local cross section, by means of the diagram above it is easy to see that the basic representation β is a homeomorphism. Therefore $\mathcal{U}_\varphi$ with the quotient topology $(\mathcal{U}(M), \| \ \|_{E_\varphi})/\mathcal{U}(M^\varphi)$ can be regarded as a manifold of projections of M_1.

c) It is known that the equivalence in M of the usual norm with the Hilbert module norm is equivalent to the finite index condition. Therefore if the index of E_φ is finite, then $\mathcal{U}_\varphi$ is homeomorphic to $\mathcal{U}_M(e_\varphi)$, with the usual norms.

We do not know if β is (norm) continuous in general. In other words, if the Hilbert module norm and the usual norm of M can induce the same quotient topology in $\mathcal{U}(M)/\mathcal{U}(M^\varphi)$ in cases other than the finite index situation.

If $\mathrm{Ind}(E_\varphi) < \infty$ one can do more. Let us recall the following result from [AS2].

Theorem 5.3. *Let $N \subset M$ be a von Neumann algebra, $E : M \to N$ a normal and faithful conditional expectation and e and M_1 as before. Then, the following statements are equivalent*

1) *The weak index of E is finite.*

2) $\mathcal{S}_M(e) = \{geg^{-1} : g \text{ invertible in } M\}$ *is an analytic homogeneous Banach space under the action of the invertible elements of M and an analytic submanifold of M_1.*

3) $\mathcal{U}_M(e) = \{ueu^* : u \in \mathcal{U}(M)\}$ *is a C^∞ homogeneous Banach space under the action of the unitary elements of M and a C^∞ submanifold of M_1.*

Corollary 5.4. *Let M be a von Neumann algebra with finite dimensional center and φ a faithful, normal and strictly semifinite weight. The following statements are equivalent:*

1) *E_φ has weakly finite index.*

2) *M is semifinite and $\sigma(h)$ is finite, if h is the Radon–Nikodym derivative of the weight φ with respect to a faithful, normal and semifinite trace on M.*

3) *$\mathcal{U}_M(e_\varphi) = \{ue_\varphi u^* : u \in \mathcal{U}(M)\}$ is a C^∞ Banach homogeneous space under the action of the unitaries of M and a C^∞ submanifold of $M_1 = (M, e)''$.*

If the weak index of E_φ is infinite, $\mathcal{U}_M(e_\varphi)$ and $\mathcal{S}_M(e_\varphi)$ are manifolds of projections of M_1, but not submanifolds of M_1.

Proof. 1) $\Rightarrow$ 2) M is semifinite by 3.2 and by theorem 4.4 $\sigma(h)$ is finite.
2) $\Rightarrow$ 1) by 4.4.
1) $\Leftrightarrow$ 3) by 5.3.

Remark 5.5. As we noted before, these conditions allow us to give a differential structure to $\mathcal{U}_\varphi$ the orbit of a faithful, normal and strictly semifinite weight φ, through the model

$$\varphi \circ \mathrm{Ad}(u) \mapsto u^* e_\varphi u$$

$$\mathcal{U}_\varphi \leftrightarrow \mathcal{U}_M(e_\varphi).$$

Note that if E_φ has finite index, then with the notations above, $M^\varphi = \{h\}' \cap M$. Therefore, $(M^\varphi)' \cap M \subset M^\varphi$. Indeed, since if $p_1, ..., p_n$ are the minimal spectral projections of h, then $(M^\varphi)'$ consists of the elements of M wich have "diagonal" matrices with respect to $p_1, ..., p_n$.

Recall again from [AS2], that if $E : M \to N$ has finite index and $N' \cap M \subset N$, then the mapping

$$\mathcal{U}_M(e) = \{u^* e u : u \in \mathcal{U}(M)\} \longrightarrow \mathcal{O}_E = \{\mathrm{Ad}(u) \circ E \circ \mathrm{Ad}(u^*) : u \in \mathcal{U}(M)\}$$

is a covering map with fibre homeomorphic to $n(E)/\mathcal{U}(N)$, where $n(E) = \{u \in \mathcal{U}(M) : \mathrm{Ad}(u)(N) \subset N\}$.

In our case, we obtain that if φ is a weight on M satisfying the equivalent conditions of 5.4, then the map

$$\mathcal{U}_\varphi \longrightarrow \mathcal{O}_{E_\varphi} = \{\mathrm{Ad}(u) \circ E_\varphi \circ \mathrm{Ad}(u^*) : u \in \mathcal{U}(M)\}$$

is a covering map with fibre (homeomorphic to) the group $n(E_\varphi)/\mathcal{U}(M^\varphi)$. Moreover, since $Z(M)$ is finite dimensional, it can be shown that this group is finite (see [ArS]).

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