

ON DOMINANCY OF MINIMAL ACTIONS OF COMPACT KAC ALGEBRAS AND CERTAIN AUTOMORPHISMS IN $\text{Aut}(\mathcal{A}/\mathcal{A}^\alpha)$

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Abstract

We show that any minimal action of a compact Kac algebra with properly infinite fixed-point algebra is dominant. It is also shown that, for such a minimal action, a certain automorphism in $\text{Aut}(\mathcal{A}/\mathcal{A}^\alpha)$ gives rise to a unitary that lies in the intrinsic group of the dual Kac algebra. A concrete description, in terms of the unitary, of the left co-ideal von Neumann subalgebra determined by the fixed-point algebra of the automorphism is given.

Introduction

In the classification of group actions on (AFD) von Neumann algebras, the case of compact groups has long occupied a central position. One immediately realizes that the situation is already complicated enough and hard to be analyzed even when groups in question are finite ones. However, the classification of them, up to (cocycle) conjugacy, was successfully completed by Jones [J1] after the Connes' breakthrough in [C]. In the infinite case, abelian groups have been first intensively studied by several mathematicians [J2], [OPT], [JT], [T], [KT], etc., and it is completed in [KT]. To the contrary, nonabelian case is far from being completed. To the best of author's knowledge, only partial results have been obtained so far, such as for ergodic actions [W1, 2, 3], and for minimal actions [PW].

In connection with this classification, Izumi-Longo-Popa have recently succeeded in establishing the complete Galois correspondence of minimal actions of compact groups as well as those of compact Kac algebras [ILP], which has been a long-standing problem since 70's (c.f. [NT]). Although we are at this point short of concrete intriguing examples of such actions, we believe that their result is important enough to ensure that minimal actions of compact Kac algebras deserve a further investigation. Thus we pursue this line of research here.

Our aim of this article is (i) to show that every minimal action of a compact Kac algebra is dominant (see Definition 2.1 below); (ii) to prove that, for a minimal action α , a certain automorphism in $\text{Aut}(\mathcal{A}/\mathcal{A}^\alpha)$ gives rise to a group-like element of the dual Kac algebra.

The organization of the paper is as follows. In Section 1, we introduce terminologies and notation used in the following sections. In Section 2, we show that any minimal action of a compact Kac algebra on a factor with properly infinite fixed-point algebra is dominant, hence dual in particular. Part of this statement, i.e., occurrence of every irreducible representation of the Kac algebra in the given minimal action, is referred in the proof of Theorem 4.4 of [ILP]. We give a complete proof to it in Proposition 2.14. In Section 3, we prove that, given such a minimal action α of a compact Kac algebra \mathbb{K} on a factor \mathcal{A} , an automorphism θ in $\text{Aut}(\mathcal{A}/\mathcal{A}^\alpha)$ leaving each of the α -invariant Hilbert spaces in \mathcal{A} (see Section 2) globally invariant gives rise to a unitary $u(\theta)$ in the intrinsic group of the dual Kac algebra. Moreover, we show that the left-coideal von Neumann subalgebra $\mathcal{N}(\theta)$ determined by the intermediate subfactor \mathcal{A}^θ , according to the Galois correspondence of [ILP], is concretely described in terms of the unitary $u(\theta)$. Finally, in Appendix, we collect some general results on actions of Kac algebras on von Neumann algebras that are needed in Section 2-3. We believe that almost all results stated there are well-known to specialists. But the author was unable to locate a literature that actually contains them. So we provide their proofs for completeness of our argument.

1. Terminology and Notation

In this section, we briefly review fundamental results on (compact) Kac algebras, and introduce notation which will be necessary for our discussion that follows. For the general theory of Kac algebras, we refer to [ES1], the notation of which we mainly adopt as well.

From now on, all von Neumann algebras are assumed to have separable preduals.

A Kac algebra is a quadruple $\mathbb{K} = (\mathcal{M}, \Gamma, \kappa, \varphi)$ [ES1, Definition 2.2.5] in which:

(Ki) $(\mathcal{M}, \Gamma, \kappa)$ is a co-involutive Hopf von Neumann algebra [ES1, Definition 1.2.5];

(Kii) φ is a faithful normal semifinite weight on \mathcal{M} , called a Haar measure (weight) of \mathbb{K} ;

(Kiii) $(id_{\mathcal{M}} \otimes \varphi)\Gamma(x) = \varphi(x) \cdot 1 \quad (x \in \mathcal{M}_+)$;

(Kiv) $(id_{\mathcal{M}} \otimes \varphi)((1 \otimes y^*)\Gamma(x)) = \kappa((id_{\mathcal{M}} \otimes \varphi)(\Gamma(y^*)(1 \otimes x)))$
 $(x, y \in N_\varphi)$;

$$(Kv) \quad \kappa \circ \sigma_t^\circ = \sigma_{-t}^\circ \circ \kappa \quad (t \in \mathbb{R}).$$

We say that K is compact if $\varphi(1) < \infty$. In this case, it turns out that φ is a trace with $\varphi \circ \kappa = \varphi$. Let us fix a Kac algebra $K = (\mathcal{M}, \Gamma, \kappa, \varphi)$. We always think of \mathcal{M} as represented on the Hilbert space $L^2(\varphi)$ obtained from φ . Let Λ_φ denote the canonical injection of \mathcal{M} into $L^2(\varphi)$. Then the equation

$$W(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)) = \Lambda_{\varphi \otimes \varphi}(\Gamma(y)(x \otimes 1)) \quad (x, y \in N_\varphi)$$

defines a unitary on $L^2(\varphi) \otimes L^2(\varphi)$, called the fundamental unitary of K [ES1, Proposition 2.4.2], and denoted by $W(K)$ if an unnecessary confusion may occur. It implements Γ : $\Gamma(x) = W(1 \otimes x)W^*$ ($x \in \mathcal{M}$). The set (group) $G(K)$ of unitaries u in \mathcal{M} satisfying $\Gamma(u) = u \otimes u$ is called the intrinsic group of K [DeC], [S].

The main feature of the theory is the construction of the dual Kac algebra $\hat{K} = (\hat{\mathcal{M}}, \hat{\Gamma}, \hat{\kappa}, \hat{\varphi})$ [ES1, Chap 3]. The fundamental unitary $W(\hat{K})$ of \hat{K} is $\Sigma W(K)^* \Sigma$ [ES1, Theorem 3.7.3], where Σ in general stands for the flip (twist) operator: $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$. There are another Kac algebras canonically attached to K , such as K' = the commutant of K , K^σ = the opposite of K , etc. (see [ES1]).

The predual \mathcal{M}_* becomes an involutive Banach algebra. We shall be mainly concerned with (nondegenerate) representations of \mathcal{M}_* . By [ES1, Theorem 3.1.4], any representation μ of \mathcal{M}_* on a Hilbert space \mathcal{H}_μ admits a generator, i.e., there is a unitary V on $\mathcal{H}_\mu \otimes L^2(\varphi)$ such that $\mu(\omega) = (\text{id} \otimes \omega)(V)$. The representation λ that has $W(\hat{K})$ as a generator is called the regular (Fourier) representation of K . It generates the dual Kac algebra $\hat{\mathcal{M}}$.

An action of K on a von Neumann algebra \mathcal{A} [E] is a unital injective $*$ -homomorphism α from \mathcal{A} into $\mathcal{A} \bar{\otimes} \mathcal{M}$ satisfying $(\alpha \otimes \text{id}_{\mathcal{M}}) \circ \alpha = (\text{id}_{\mathcal{A}} \otimes \Gamma) \circ \alpha$. Conjugacy and cocycle conjugacy of actions are defined as in the case of group actions (see [E], [N]). The crossed product $\mathcal{A} \rtimes_\alpha K$ of \mathcal{A} by the action α is by definition the von Neumann algebra generated by $\alpha(\mathcal{A})$ and $\mathbb{C} \otimes \hat{\mathcal{M}}$. On the crossed product, there exists an action $\hat{\alpha}$ of \hat{K} , called the dual action of α [E, Définition II.7]. In the meantime, there is an action $\bar{\alpha}$ of K on $\mathcal{A} \bar{\otimes} \mathcal{L}(L^2(\varphi))$ (or, more generally, on $\mathcal{A} \bar{\otimes} \mathcal{B}$ with \mathcal{B} another von Neumann algebra), defined by $\bar{\alpha} = (\text{id}_{\mathcal{A}} \otimes \sigma) \circ (\alpha \otimes \text{id})$, where $\sigma = \text{Ad } \Sigma$. We call $\bar{\alpha}$ the amplified action of α . On $\mathcal{A} \bar{\otimes} \mathcal{L}(L^2(\varphi))$, there is another action $\tilde{\alpha}$ of K defined by $\tilde{\alpha} = \text{Ad}(1 \otimes W(\hat{K})) \circ \bar{\alpha}$. It is known [ES2, Proposition IV.5] that the bidual action $\hat{\alpha}$ is conjugate to $\tilde{\alpha}$. The fixed-point algebra \mathcal{A}^α of α is defined to be the set $\{x \in \mathcal{A} : \alpha(x) = x \otimes 1\}$.

Finally, theory of compact Kac algebras almost goes parallel to that of compact groups [ES1, §6]. For example, the Peter-Weyl theorem, the Schur's

orthogonality relations, complete reducibility, etc. hold true [ES1, Rtheorem 6.2.5 and 6.2.6]. See also [W].

2. Dominancy of minimal actions

This section is devoted to proving the first main theorem of this note on dominancy of minimal actions of compact Kac algebras. In what follows, we shall retain the notation established in the preceding section. Let us fix a Kac algebra $K = (\mathcal{M}, \Gamma, \kappa, \varphi)$ once and for all.

DEFINITION 2.1. Let α be an action of K on a von Neumann algebras \mathcal{A} .

(1) (Minimality: see [ILP]) We say that α is *minimal* if

(i) the fixed-point algebra \mathcal{A}^α of α has the trivial relative commutant in \mathcal{A} , i.e., $(\mathcal{A}^\alpha)' \cap \mathcal{A} = \mathbf{C}$;

(ii) the linear span of the set $\{(\omega \otimes \text{id}_{\mathcal{M}}) \circ \alpha(x) : \omega \in \mathcal{A}_*, x \in \mathcal{A}\}$ is σ -weakly dense in \mathcal{M} .

(2) (Dominancy: c.f. [CT], [NT]) The action α is said to be *dominant* if

(i) \mathcal{A}^α is properly infinite;

(ii) α is dual, that is, the dual action of an action on some von Neumann algebra (which is necessarily isomorphic to \mathcal{A}^α).

With these definitions, we can state our main theorem.

THEOREM 2.2. *Suppose that K is a compact Kac algebra. If α is a minimal action of K on a factor \mathcal{A} with properly infinite fixed-point algebra, then it is dominant.*

To prove this theorem, we require a series of preparatory results. Our approach to the proof is, however, basically quite the same as that of group action case. Thus it will be our central goal to show that every irreducible representation of \mathcal{M}_* is unitarily equivalent to “the subrepresentation obtained by restricting the action α to some α -invariant Hilbert space in \mathcal{A} .” This can be described in Roberts’ terminology [R] by saying that α has full monoidal spectrum.

In the discussion that follows, we assume that K is a compact Kac algebra with the normalized Haar measure φ . We also fix an action α of K on a von Neumann algebra \mathcal{A} . It is well-known (see [ES2, Proposition II.2]) that $E_\alpha = (\text{id} \otimes \varphi) \circ \alpha : \mathcal{A} \rightarrow \mathcal{A}$ is a faithful normal conditional expectation from \mathcal{A} onto the fixed-point algebra \mathcal{A}^α . Take a faithful normal state ω_0 on \mathcal{A}^α . Then set $\psi = \omega_0 \circ E_\alpha$. From now on, we shall always think of \mathcal{A} as represented on $L^2(\psi)$, the Hilbert space obtained from ψ by the GNS construction.

For each irreducible representation (π, \mathcal{H}_π) of \mathcal{M}_* with generator $V(\pi) \in \mathcal{L}(\mathcal{H}_\pi) \otimes \mathcal{M}$, we define the π -eigenspace $\mathcal{A}^\alpha(\pi)$ to be the set given by

$$\mathcal{A}^\alpha(\pi) = \{ X \in \mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi) : \bar{\alpha}(X) = X_{12}V(\pi)_{23} \},$$

where, in this case, $\bar{\alpha}$ is the amplified action $\bar{\alpha} = (\text{id}_{\mathcal{A}} \otimes \sigma) \circ (\alpha \otimes \text{id}_{\mathcal{L}(\mathcal{H}_\pi)})$. Any element in $\mathcal{A}^\alpha(\pi)$ is called a π -eigenoperator. As in the case of compact group actions, these eigenspaces $\{\mathcal{A}^\alpha(\pi)\}_{\pi \in \text{Irr}(\mathcal{M}_*)}$ shall play a crucial part in our study.

A finite-dimensional subspace D of \mathcal{A} is said to be α -invariant if $\alpha(D) \subseteq D \otimes \mathcal{M}$. If D is α -invariant, then it determines an operator $V \in \text{End}(D) \otimes \mathcal{M}$ satisfying $V_{12}V_{13} = (\text{id} \otimes \Gamma)(V)$. In fact, with $\{v_1, v_2, \dots, v_d\}$ a basis for D , the equation

$$(2.3) \quad \alpha(v_j) = \sum_{i=1}^d v_i \otimes V_{i,j} \quad (j = 1, 2, \dots, d)$$

determines a family $\{V_{i,j}\}$ of elements in \mathcal{M} . Then put

$$(2.4) \quad V = \sum_{i,j=1}^d e_{i,j} \otimes V_{i,j} \in \text{End}(D) \otimes \mathcal{M},$$

where $\{e_{i,j}\}$ denotes the system of matrix units in $\text{End}(D)$, corresponding to the basis $\{v_i\}$. It is easily verified that V is independent of the choice of the basis $\{v_i\}$, and that it satisfies $V_{12}V_{13} = (\text{id} \otimes \Gamma)(V)$. From the last identity, it follows that the equation

$$\pi(\omega) = (\text{id} \otimes \omega)(V) \quad (\omega \in \mathcal{M}_*)$$

defines a (not necessarily $*$ -preserving) homomorphism π of \mathcal{M}_* into $\text{End}(D)$. We call the pair (π, V) constructed in this way the representation associated with the α -invariant subspace D , and denote it by $(\pi_D, V(D))$. From (2.3), we immediately see that

$$(2.5) \quad \pi_D(\omega)v = (\text{id} \otimes \omega) \circ \alpha(v) \quad (v \in D).$$

The following lemma is a Kac algebraic version of Proposition 2.2 of [NT, Chap. IV], for which we do not need to assume minimality of α or \mathcal{A}^α being properly infinite.

LEMMA 2.6. *Let (π, \mathcal{H}_π) be an irreducible representation of \mathcal{M}_* . Then the following are equivalent:*

- (i) $\mathcal{A}^\alpha(\pi)$ contains a non-zero element;
- (ii) there exists an α -invariant subspace D so that π_D is equivalent to π , in the sense that there is an invertible linear operator $T : \mathcal{H}_\pi \rightarrow D$ satisfying $(T \otimes 1)V(\pi) = V(D)(T \otimes 1)$, or equivalently, $T\pi(\omega)T^{-1} = \pi_D(\omega)$ ($\omega \in \mathcal{M}_*$).

PROOF. (i) \Rightarrow (ii). Take a non-zero $X \in \mathcal{A}^\alpha(\pi)$. With $\{\varepsilon_i\}_{i=1}^{d=\dim \mathcal{H}_\pi}$ an ONB (orthonormal basis) for \mathcal{H}_π , and $\{e_{i,j}\}$ the corresponding matrix units of $\mathcal{L}(\mathcal{H}_\pi)$, we write X and $V(\pi)$ in the form

$$X = \sum_{j,k=1}^d X_{j,k} \otimes e_{j,k}, \quad V(\pi) = \sum_{j,k=1}^d e_{j,k} \otimes V(\pi)_{j,k}.$$

Note that, with the notation as above, the identity $\bar{\alpha}(X) = X_{12}V(\pi)_{23}$ is equivalent to

$$(2.7) \quad \alpha(X_{j,k}) = \sum_{i=1}^d X_{j,i} \otimes V(\pi)_{i,k} \quad (1 \leq j, k \leq d).$$

This fact will be often used in the later discussion. Since $X \neq 0$, there are j_0, k_0 such that $X_{j_0, k_0} \neq 0$. With $v_k = X_{j_0, k}$ ($k = 1, 2, \dots, d$), we define D to be the linear span of $\{v_k\}$. By (2.7), D is α -invariant. Moreover, from (2.5) and (2.7), it follows that, for $\omega \in \mathcal{M}_*$, we have

$$\pi_D(\omega)v_j = (\text{id} \otimes \omega) \circ \alpha(v_j) = \sum_{i=1}^d \omega(V(\pi)_{i,j})v_i.$$

Thus, if we define a linear operator $T : \mathcal{H}_\pi \rightarrow D$ by $T\varepsilon_i = v_i$, then we obtain $\pi_D(\omega)T = T\pi(\omega)$. Since π is irreducible and D is a non-zero vector space, T is invertible.

(ii) \Rightarrow (i). A proof can be obtained by tracing back the preceding paragraph. With $\{\varepsilon_i\}$ as in (i) \Rightarrow (ii), set $v_i = T\varepsilon_i$ ($i = 1, 2, \dots, d$). If $e'_{i,j}$ denotes the matrix units of $\text{End}(D)$ determined by the basis $\{v_i\}$, then the identity $(T \otimes 1)V(\pi) = V(D)(T \otimes 1)$ yields $V(D) = \sum_{i,j=1}^d e'_{i,j} \otimes V(\pi)_{i,j}$. Thus we have $\alpha(v_j) = \sum_{i=1}^d v_i \otimes V(\pi)_{i,j}$ (see (2.3) and (2.4)). So, with $X_{j,k} = v_k$ ($1 \leq j, k \leq d$), we get equation (2.7). Therefore, $X = [X_{j,k}] \in \mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi)$ is a π -eigenoperator. Clearly, X is non-zero.

LEMMA 2.8. (c.f. [R], [AKHT], [NT]). *Let (π, \mathcal{H}_π) be an irreducible representation of \mathcal{M}_* . Suppose that \mathcal{A}^α is properly infinite. Then the following are equivalent:*

- (i) $\mathcal{A}^\alpha(\pi)$ contains a unitary;
- (ii) there exists an α -invariant Hilbert space D in \mathcal{A} such that π_D is unitarily equivalent to π .

PROOF. With $d = \dim \mathcal{H}_\pi$, let $\{\varepsilon_i\}_{i=1}^d$ be an ONB for \mathcal{H}_π and $\{e_{i,j}\}$ the corresponding matrix units of $\mathcal{L}(\mathcal{H}_\pi)$.

(i) \Rightarrow (ii). Take a unitary $X = \sum_{i=1}^d X_{i,j} \otimes e_{i,j} \in \mathcal{A}^\alpha(\pi)$. Since \mathcal{A}^α is prop-

erly infinite, there is a Hilbert space K in \mathcal{A}^α with $\dim K = d$. Let $\{w_i\}_{i=1}^d$ be an ONB for K . Then define a unitary V from $L^2(\psi)$ onto $L^2(\psi) \otimes \mathcal{H}_\pi$ by

$$V\xi = \sum_{i=1}^d w_i^* \xi \otimes \varepsilon_i \quad (\xi \in L^2(\psi)).$$

The inverse $V^{-1} = V^*$ is given by $V^{-1}(\sum_{i=1}^d \eta_i \otimes \varepsilon_i) = \sum_{i=1}^d w_i \eta_i$. We put $v_i = V^* X V w_i \in \mathcal{L}(L^2(\psi))$ ($i = 1, 2, \dots, d$). Since $v_i^* v_j = \delta_{ij} \cdot 1$ and $\sum_{i=1}^d v_i v_i^* = 1$, the linear span D of $\{v_i\}$ forms a Hilbert space in $\mathcal{L}(L^2(\psi))$. For any $\xi \in L^2(\psi)$, we have

$$v_i \xi = V^* X V w_i \xi = V^* X(\xi \otimes \varepsilon_i) = V^* \left(\sum_{j=1}^d X_{j,i} \xi \otimes \varepsilon_j \right) = \sum_{j=1}^d w_j X_{j,i} \xi.$$

This shows that $v_j = \sum_{i=1}^d w_i X_{i,j} \in \mathcal{A}$, so that D is a Hilbert space in \mathcal{A} . Since $X_{j,k}$ satisfies (2.7) and w_i belongs to \mathcal{A}^α , we have

$$(2.9) \quad \alpha(v_j) = \sum_{i=1}^d v_i \otimes V(\pi)_{ij} \quad (j = 1, 2, \dots, d).$$

From this, it follows easily that D is α -invariant, and that, with a unitary $U : \mathcal{H}_\pi \rightarrow D$ defined by $U\varepsilon_i = v_i$, one has $(U \otimes 1)V(D) = V(\pi)(U \otimes 1)$ and $\pi_D(\omega)U = U\pi(\omega)$ for any $\omega \in \mathcal{M}_*$.

(ii) \Rightarrow (i). Let K , $\{w_i\}$ and V be as in the preceding paragraph. By assumption, we may choose an ONB $\{v_i\}$ of D so that (2.9) holds. Then define an operator $X \in \mathcal{L}(L^2(\psi)) \otimes \mathcal{L}(\mathcal{H}_\pi)$ by $X = \sum_{i=1}^d V v_i w_i^* V^*$. Since $Vy = (y \otimes 1)V$ for any $y \in \mathcal{A}'$. It follows that $X \in \mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi)$. Simple calculations show that X is a unitary. Moreover, we have

$$X(\xi \otimes \varepsilon_j) = \sum_{k=1}^d w_k^* v_j \xi \otimes \varepsilon_k.$$

Thus

$$X = \sum_{i,j=1}^d w_i^* v_j \otimes e_{ij}.$$

From (2.9) and the fact that $w_i \in \mathcal{A}^\alpha$, it results that the matrix components $\{X_{i,j} := w_i^* v_j\}$ of X satisfies (2.7). Therefore, X is an π -eigenoperator.

For each irreducible representation (π, \mathcal{H}_π) , we define a map β_π from $\mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi)$ into $\mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi) \otimes \mathcal{M}$ by

$$\beta_\pi = \text{Ad}(1 \otimes V(\pi)) \circ \bar{\alpha},$$

where, as usual, $\bar{\alpha}$ is the amplified action: $\bar{\alpha} = (\text{id}_{\mathcal{A}} \otimes \sigma) \circ (\alpha \otimes \text{id}_{\mathcal{L}(\mathcal{H}_\pi)})$. The map β_π is an action of K on $\mathcal{Q} = \mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi)$. One can verify this by a direct computation, but another way to do that is to notice that β_π can be written in the form: $\beta_\pi = \alpha *_{V(\pi)} \gamma_\pi$, where γ_π is an action of K on $\mathcal{L}(\mathcal{H}_\pi)$ defined by $\gamma_\pi(z) = V(\pi)(z \otimes 1)V(\pi)^*$ ($z \in \mathcal{L}(\mathcal{H}_\pi)$) (recall that $V(\pi)$ is an $I_K^{\mathcal{L}(\mathcal{H}_\pi)}$ -cocycle), and $\alpha *_{V(\pi)} \gamma_\pi$ stands for the inner tensor product action of α and γ_π in the sense of Nakagami (see [N, §5, (5.2)]). The following lemma is a simple exercise, so it is left to readers.

LEMMA 2.10. *The π -eigenspace $\mathcal{A}^\alpha(\pi)$ is a $\mathcal{Q}^{\bar{\alpha}}$ - \mathcal{Q}^{β_π} bimodule.*

LEMMA 2.11. *Suppose that the relative commutant of \mathcal{A}^α in \mathcal{A} is trivial. Then both the left and the right support projections of $\mathcal{A}^\alpha(\pi)$ are equal to the identity operator 1 for any π with $\mathcal{A}^\alpha(\pi) \neq \{0\}$.*

PROOF. Let e_1 and e_2 be the left and the right support projection of $\mathcal{A}^\alpha(\pi)$, respectively. Suppose that $\mathcal{A}^\alpha(\pi) \neq \{0\}$.

First, we show that $\mathcal{A}^\alpha(\pi)$ is stable under the adjoint action of the unitary group $\mathcal{U}(\mathcal{A}^\alpha)$ of \mathcal{A}^α . So set $\delta_u = \text{Ad } u$ ($u \in \mathcal{U}(\mathcal{A}^\alpha)$). It is apparant that we have $\alpha \circ \delta_u = (\delta_u \otimes \text{id}_{\mathcal{M}}) \circ \alpha$. Let $\bar{\delta}_u = \delta_u \otimes \text{id}_{\mathcal{L}(\mathcal{H}_\pi)}$. Then, with $X \in \mathcal{A}^\alpha(\pi)$, we have

$$\begin{aligned} \bar{\alpha} \circ \bar{\delta}_u(X) &= (\text{id}_{\mathcal{A}} \otimes \sigma) \circ (\alpha \circ \delta_u \otimes \text{id}_{\mathcal{L}(\mathcal{H}_\pi)})(X) \\ &= (\text{id}_{\mathcal{A}} \otimes \sigma) \circ ((\delta_u \otimes \text{id}_{\mathcal{M}}) \circ \alpha \otimes \text{id}_{\mathcal{L}(\mathcal{H}_\pi)})(X) \\ &= (\delta_u \otimes \text{id}_{\mathcal{L}(\mathcal{H}_\pi)} \otimes \text{id}_{\mathcal{M}}) \circ \bar{\alpha}(X) = \bar{\delta}_u(X)_{12} V(\pi)_{23}. \end{aligned}$$

This proves that $\bar{\delta}_u(\mathcal{A}^\alpha(\pi)) = \mathcal{A}^\alpha(\pi)$. From this, it follows that $\bar{\delta}_u(e_i) = e_i$ for any $u \in \mathcal{U}(\mathcal{A}^\alpha)$ ($i = 1, 2$). By our hypothesis, we have $\mathcal{Q}^{\bar{\delta}} = \mathcal{A}^\delta \otimes \mathcal{L}(\mathcal{H}_\pi) = \otimes \mathcal{L}(\mathcal{H}_\pi)$. Hence $e_i = 1 \otimes q_i$ for some projections $q_i \in \mathcal{L}(\mathcal{H}_\pi)$.

Clearly, we have $e_1 = 1 \otimes q_1 \in \mathcal{Q}^{\bar{\alpha}}$. In view of Lemma 2.10, e_1 is stable under the adjoint action of the unitary group of $\mathcal{Q}^{\bar{\alpha}}$. Hence e_1 belongs to the center of $\mathcal{Q}^{\bar{\alpha}}$, which is the scalar multiples of the identity by assumption. Thus $e_1 = 1$.

It remains to show that $q_2 = 1$. For any $X \in \mathcal{A}^\alpha(\pi)$, since $\bar{\alpha}(X) = X_{12} V(\pi)_{23}$, we have $\bar{\alpha}(X^*X) = V(\pi)_{23}^*(X^*X)_{12} V(\pi)_{23}$, so that $\beta_\pi(X^*X) = (X^*X)_{12}$. Namely, X^*X belongs to \mathcal{Q}^{β_π} , and so does the right support of X . Hence $e_2 \in \mathcal{Q}^{\beta_\pi}$. From this, it follows that

$$1 \otimes q_2 \otimes 1 = e_2 \otimes 1 = \beta_\pi(e_2) = (1 \otimes V(\pi))(1 \otimes q_2 \otimes 1)(1 \otimes V(\pi)^*).$$

Thus $q_2 \otimes 1 = V(\pi)(q_2 \otimes 1)V(\pi)^*$. Since π is irreducible, $q_2 = 1$.

Let us now prove a key proposition to the proof of Theorem 2.3.

PROPOSITION 2.12. *Suppose that the fixed-point algebra \mathcal{A}^α is properly infinite and has the trivial relative commutant in \mathcal{A} . Then, for any irreducible representation (π, \mathcal{H}_π) , $\mathcal{A}^\alpha(\pi) \neq \{0\}$ if and only if $\mathcal{A}^\alpha(\pi)$ contains a unitary.*

PROOF. First observe that $X^*Y \in \mathcal{Q}^{\beta_\pi}$ and $XY^* \in \mathcal{Q}^{\bar{\alpha}}$ whenever $X, Y \in \mathcal{A}^\alpha(\pi)$. From this observation and Lemma 2.10, it follows that the set

$$\mathcal{R} = \left\{ \begin{bmatrix} a & X \\ Y^* & b \end{bmatrix} : a \in \mathcal{Q}^{\bar{\alpha}}, X, Y \in \mathcal{A}^\alpha(\pi), b \in \mathcal{Q}^{\beta_\pi} \right\}$$

is a von Neumann subalgebra of $\mathcal{Q} \otimes M_2(\mathbb{C})$. With

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

we have

$$e_1 \mathcal{R} e_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathcal{Q}^{\bar{\alpha}} \right\} \cong \mathcal{Q}^{\bar{\alpha}} = \mathcal{A}^\alpha \otimes \mathcal{L}(\mathcal{H}_\pi),$$

$$e_2 \mathcal{R} e_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} : b \in \mathcal{Q}^{\beta_\pi} \right\} \cong \mathcal{Q}^{\beta_\pi} \supseteq \mathcal{A}^\alpha \otimes \mathbb{C}.$$

Since $\mathcal{Q}^{\bar{\alpha}}$ and \mathcal{Q}^{β_π} are properly infinite, both e_1 and e_2 are properly infinite projections in \mathcal{R} . In the meantime,

$$e_1 \mathcal{R} e_2 = \left\{ \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} : X \in \mathcal{A}^\alpha(\pi) \right\}, \quad e_2 \mathcal{R} e_1 = \left\{ \begin{bmatrix} 0 & 0 \\ X^* & 0 \end{bmatrix} : X \in \mathcal{A}^\alpha(\pi) \right\}.$$

From this, together with Lemma 2.11, we find that the left support projection of

$$e_1 \mathcal{R} e_2 = \begin{bmatrix} \text{the left support projection of } \mathcal{A}^\alpha(\pi) & 0 \\ 0 & 0 \end{bmatrix} = e_1.$$

Similarly, the left support projection of $e_2 \mathcal{R} e_1$ coincides with e_2 . These facts imply that the central supports of e_1 and e_2 are the same. Hence they are equivalent in \mathcal{R} . So there exists a matrix v

$$v = \begin{bmatrix} a & X \\ Y^* & b \end{bmatrix} \in \mathcal{R}$$

such that $v^*v = e_1$ and $vv^* = e_2$. This means that Y is a unitary in $\mathcal{A}^\alpha(\pi)$.

LEMMA 2.13. *If α is an action satisfying the property (ii) in the definition of minimality (Definition 2.1 (1), (ii)), then $\mathcal{A}^\alpha(\pi) \neq \{0\}$ for any irreducible representation π .*

PROOF. Suppose that there exists an irreducible representation π such that the π -spectral subspace \mathcal{A}_π (see Appendix, Part A) reduces to zero. Then, for any $a \in \mathcal{M}$ of the form $a = (\omega \otimes \text{id}_{\mathcal{M}}) \circ \alpha(x)$ ($\omega \in \mathcal{A}_*$, $x \in \mathcal{A}$), we have

$$\varphi(a\chi_\pi^*) = (\omega \otimes \chi_\pi^* \varphi) \circ \alpha(x) = \omega(E_\pi(x)) = 0,$$

where E_π is the map defined in Part A of Appendix. This implies by assumption that $\chi_\pi = 0$, a contradiction.

PROPOSITION 2.14. *Suppose that α is a minimal action on \mathcal{A} with properly infinite fixed-point algebra. Then $\mathcal{A}^\alpha(\pi)$ contains a unitary for every irreducible representation π . Therefore, there exists an α -invariant Hilbert space D in \mathcal{A} such that π_D is unitarily equivalent to π .*

PROOF. Let π be an irreducible representation of \mathcal{M}_* . From Lemma 2.13, we have that $\mathcal{A}^\alpha(\pi) \neq \{0\}$. Then $\mathcal{A}^\alpha(\pi)$ contains a unitary by Propostion 2.12. The last assertion follows from Lemma 2.8.

With the aid of Proposition 2.14 just obtained above, we are now in a position to demonstrate a proof of Theorem 2.2.

PROOF OF THEOREM 2.2. Let us denote by $\text{Irr} = \text{Irr}(\mathcal{M}_*)$ a complete set of representatives of the unitary equivalence classes of irreducible representations of \mathcal{M}_* .

Thanks to Proposition 2.14, for each (π, \mathcal{H}_π) in $\text{Irr}(\mathcal{M}_*)$ with generator $V(\pi)$, we may choose a unitary $W(\pi)$ in $\mathcal{A}^\alpha(\pi)$. We set $W_\alpha = \sum_{\pi \in \text{Irr}(\mathcal{M}_*)}^\oplus W(\pi)$, which is acting on the Hilbert space $\sum_{\pi \in \text{Irr}(\mathcal{M}_*)}^\oplus L^2(\psi) \otimes \mathcal{H}_\pi$. Note that the Hilbert space $\sum_{\pi \in \text{Irr}(\mathcal{M}_*)}^\oplus L^2(\psi) \otimes \mathcal{H}_\pi$ is canonically isomorphic to $L^2(\psi) \otimes \sum_{\pi \in \text{Irr}(\mathcal{M}_*)}^\oplus \mathcal{H}_\pi$. The inverse U of this isomorphism is characterized by

$$U\left(\eta \otimes \sum_{\pi \in \text{Irr}}^\oplus \xi_\pi\right) = \sum_{\pi \in \text{Irr}}^\oplus (\eta \otimes \xi_\pi).$$

By identifying W_α with $U^*W_\alpha U$, we shall often regard the unitary W_α as acting on $L^2(\psi) \otimes \sum_{\pi \in \text{Irr}(\mathcal{M}_*)}^\oplus \mathcal{H}_\pi$. So, for example, we have

$$(2.15) \quad \left(W_\alpha\left(\xi \otimes \sum_{\pi \in \text{Irr}}^\oplus \xi_\pi\right) \middle| \eta \otimes \sum_{\pi \in \text{Irr}}^\oplus \eta_\pi\right) = \sum_{\pi \in \text{Irr}} (W(\pi)(\xi \otimes \xi_\pi) \middle| \eta \otimes \eta_\pi).$$

It is easily verified that, under the isomorphism $\text{Ad } U^*$, the von Neumann algebra $\sum_{\pi \in \text{Irr}(\mathcal{M}_*)}^\oplus \mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi)$ is $*$ -isomorphic to $\mathcal{A} \bar{\otimes} \sum_{\pi \in \text{Irr}(\mathcal{M}_*)}^\oplus \mathcal{L}(\mathcal{H}_\pi)$. Thus W_α belongs to $\mathcal{A} \bar{\otimes} \sum_{\pi \in \text{Irr}(\mathcal{M}_*)}^\oplus \mathcal{L}(\mathcal{H}_\pi)$. Let $\bar{\alpha}$ be the amplified action $\bar{\alpha} = (\text{id}_{\mathcal{A}} \otimes \sigma) \circ (\alpha \otimes \text{id})$ of \mathbb{K} on $\mathcal{Q} := \mathcal{A} \bar{\otimes} \sum_{\pi \in \text{Irr}(\mathcal{M}_*)}^\oplus \mathcal{L}(\mathcal{H}_\pi)$, or appropriate von Neumann algebras. We also let U_ψ be the canonical implementation of

α on the standard Hilbert space $L^2(\psi)$ (see Part A of Appendix for details). With $\xi, \eta \in L^2(\psi)$ and $f, g \in L^2(\varphi)$, from (2.15), we have

$$\begin{aligned} & \left((\text{id}_{\mathcal{L}} \otimes \omega_{f,g})(\bar{\alpha}(W_\alpha)) \left(\xi \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \xi_\pi \right) \middle| \eta \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \eta_\pi \right) \\ &= \left((\alpha \otimes \text{id})(W_\alpha) \left(\xi \otimes f \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \xi_\pi \right) \middle| \eta \otimes g \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \eta_\pi \right) \\ &= \left((U_\psi)_{12}(W_\alpha)_{13}(U_\psi)_{12}^* \left(\xi \otimes f \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \xi_\pi \right) \middle| \eta \otimes g \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \eta_\pi \right) \\ &= \sum_{\pi \in \text{Irr}} \left((U_\psi)_{12} W(\pi) (U_\psi)_{12}^* \left(\xi \otimes f \otimes \xi_\pi \right) \middle| \eta \otimes g \otimes \eta_\pi \right) \\ &= \sum_{\pi \in \text{Irr}} \left(\bar{\alpha}(W(\pi)) \left(\xi \otimes \xi_\pi \otimes f \right) \middle| \eta \otimes \eta_\pi \otimes g \right) \\ &= \sum_{\pi \in \text{Irr}} \left(W(\pi)_{12} V(\pi)_{23} \left(\xi \otimes \xi_\pi \otimes f \right) \middle| \eta \otimes \eta_\pi \otimes g \right) \\ &= \sum_{\pi \in \text{Irr}} \langle V(\pi)_{23}, \omega_{\xi \otimes \xi_\pi, W(\pi)^*(\eta \otimes \eta_\pi)} \otimes \omega_{f,g} \rangle \\ &= \sum_{\pi \in \text{Irr}} \left(W(\pi) (1 \otimes \pi(\omega_{f,g})) \left(\xi \otimes \xi_\pi \right) \middle| \eta \otimes \eta_\pi \right) \\ &= \left(W_\alpha \left(1 \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \pi(\omega_{f,g}) \right) \left(\xi \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \xi_\pi \right) \middle| \eta \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \eta_\pi \right). \end{aligned}$$

Thus we have shown that

$$(2.16) \quad (\text{id}_{\mathcal{L}} \otimes \omega_{f,g})(\bar{\alpha}(W_\alpha)) = W_\alpha \left(1 \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \pi(\omega_{f,g}) \right).$$

Since the direct sum representation $\sum_{\pi \in \text{Irr}}^{\oplus} \pi$ is quasi-equivalent to the left regular (Fourier) representation λ of K by [ES1], there is a $*$ -isomorphism Φ from $\sum_{\pi \in \text{Irr}}^{\oplus} \mathcal{L}(\mathcal{H}_\pi)$ onto the dual Kac algebra $\hat{\mathcal{M}}$ such that $\Phi \circ \left(\sum_{\pi \in \text{Irr}}^{\oplus} \pi \right) = \lambda$. Set $V_\alpha = (\text{id}_{\mathcal{A}} \otimes \Phi)(W_\alpha)$, which lies in $\mathcal{A} \otimes \hat{\mathcal{M}}$. Since

$$(\text{id}_{\mathcal{A}} \otimes \Phi \otimes \text{id}_{\mathcal{M}}) \circ \bar{\alpha}(W_\alpha) = \bar{\alpha}(V_\alpha),$$

it follows that we have

$$\begin{aligned} (\text{id}_{\mathcal{A}} \otimes \Phi)(\text{id}_{\mathcal{L}} \otimes \omega_{f,g})(\bar{\alpha}(W_\alpha)) &= (\text{id}_{\mathcal{A}} \otimes \text{id}_{\hat{\mathcal{M}}} \otimes \omega_{f,g}) \\ &\quad \circ (\text{id}_{\mathcal{A}} \otimes \Phi \otimes \text{id}_{\mathcal{M}}) \circ \bar{\alpha}(W_\alpha) \\ &= (\text{id}_{\mathcal{A}} \otimes \text{id}_{\hat{\mathcal{M}}} \otimes \omega_{f,g}) \circ \bar{\alpha}(V_\alpha). \end{aligned}$$

In the meantime, since the regular representation λ has $\hat{W} = W(\hat{K})$ as a generator, we have

$$(\text{id}_{\mathcal{A}} \otimes \Phi) \left(1 \otimes \sum_{\pi \in \text{Irr}}^{\oplus} \pi(\omega_{f,g}) \right) = 1 \otimes \lambda(\omega_{f,g}) = (\text{id}_{\mathcal{A}} \otimes \text{id}_{\hat{\mathcal{H}}} \otimes \omega_{f,g})(\hat{W}_{23}).$$

Consequently, by applying $\text{id}_{\mathcal{A}} \otimes \Phi$ to both sides of (2.16), we obtain

$$(\text{id}_{\mathcal{A}} \otimes \text{id}_{\hat{\mathcal{H}}} \otimes \omega_{f,g}) \circ \bar{\alpha}(V_{\alpha}) = (\text{id}_{\mathcal{A}} \otimes \text{id}_{\hat{\mathcal{H}}} \otimes \omega_{f,g})(V_{\alpha})_{12} \hat{W}_{23}.$$

From this, it results that

$$(2.17) \quad \bar{\alpha}(V_{\alpha}) = (V_{\alpha})_{12} \hat{W}_{23}.$$

Now we define an automorphism Θ of $\mathcal{A} \bar{\otimes} \mathcal{L}(L^2(\varphi))$ by $\Theta = \text{Ad } V_{\alpha}$. We also consider the action $\tilde{\alpha}$ of K on $\mathcal{A} \bar{\otimes} \mathcal{L}(L^2(\varphi))$ introduced in Section 1. Then, from (2.17), we see that, for any $X \in \mathcal{A} \bar{\otimes} \mathcal{L}(L^2(\varphi))$, we have

$$\begin{aligned} (\Theta \otimes \text{id}_{\hat{\mathcal{H}}}) \circ \bar{\alpha} \circ \Theta^{-1}(X) &= (V_{\alpha})_{12} \tilde{\alpha}(V_{\alpha}^* X V_{\alpha})(V_{\alpha})_{12}^* \\ &= (V_{\alpha})_{12} \hat{W}_{23} \tilde{\alpha}(V_{\alpha}^* X V_{\alpha}) \hat{W}_{23}^* (V_{\alpha})_{12}^* = \bar{\alpha}(X). \end{aligned}$$

Thus we have proven that $\bar{\alpha}$ is conjugate to $\tilde{\alpha}$. This, together with Corollary B.2, implies that α is conjugate to $\tilde{\alpha}$. Since $\tilde{\alpha}$ is conjugate to the bidual action α of α by Proposition IV.5 of [ES2], we conclude that α is dual.

3. On certain automorphisms in $\text{Aut}(\mathcal{A}/\mathcal{A}^{\alpha})$

This section contains a study of certain special automorphisms in $\text{Aut}(\mathcal{A}/\mathcal{A}^{\alpha})$, with the emphasis on the relation with Galois correspondence of the inclusion $\mathcal{A}^{\alpha} \subseteq \mathcal{A}$. We show that such an automorphism gives rise to a unitary in the intrinsic group of the dual Kac algebra. It is also shown that the left co-ideal von Neumann subalgebra of K determined by the fixed-point subfactor of the automorphism is explicitly described in terms of the unitary.

As in the preceding section, α is a minimal action of a compact Kac algebra K on a factor \mathcal{A} with properly infinite fixed-point algebra. We denote by $\text{Aut}(\mathcal{A}/\mathcal{A}^{\alpha})$ the subgroup of all automorphisms of \mathcal{A} leaving the fixed-point algebra \mathcal{A}^{α} pointwise invariant. Let $\text{Irr} = \text{Irr}(\mathcal{M}_*)$ be as in the proof of Theorem 2.2. Next, for each $(\pi, \mathcal{H}_{\pi}) \in \text{Irr}$ with generator $V(\pi)$, let $\{V(\pi)_{ij}\}_{1 \leq i, j \leq d(\pi)}$ be as in the proof of Lemma 2.6, where $d(\pi) = \dim \mathcal{H}_{\pi}$. In view of Proposition 2.14, we may choose an α -invariant Hilbert space D_{π} in \mathcal{A} and an ONB $\{w(\pi)_i\}_{1 \leq i \leq d(\pi)}$ for D_{π} such that

$$\alpha(w(\pi)_j) = \sum_{i=1}^{d(\pi)} w(\pi)_i \otimes V(\pi)_{i,j}. \quad (1 \leq j \leq d(\pi))$$

Suppose now that $\theta \in \text{Aut}(\mathcal{A}/\mathcal{A}^\alpha)$, which leaves each of the α -invariant Hilbert spaces in \mathcal{A} globally invariant, is given. Then set

$$(3.1) \quad V(\theta(\pi))_{i,j} = \sum_{k=1}^{d(\pi)} (w(\pi)_k | \theta(w(\pi)_i)) V(\pi)_{k,j} \quad (1 \leq i, j \leq d(\pi), \pi \in \text{Irr})$$

For each $\pi \in \text{Irr}$, consider the finite-dimensional Hilbert space $\mathfrak{M}_\pi = \text{span}\{\Lambda_\varphi(V(\pi)_{i,j}) : 1 \leq i, j \leq d(\pi)\}$ of $L^2(\varphi)$. The equation

$$(3.2) \quad u(\theta)\Lambda_\varphi(V(\pi)_{i,j}) = \Lambda_\varphi(V(\theta(\pi))_{i,j})$$

defines a linear transformation $u(\theta)$ on \mathfrak{M}_π . In fact, since $\{\sqrt{d(\pi)}\Lambda_\varphi(V(\pi)_{i,j})\}$ is an ONB for \mathfrak{M}_π , it easily follows that $u(\theta)$ is a unitary. Since $L^2(\varphi) = \sum_{\pi \in \text{Irr}}^\oplus \mathfrak{M}_\pi$, $u(\theta)$ naturally extends to a unitary on $L^2(\varphi)$, which we denote by $u(\theta)$ again.

LEMMA 3.3. *The unitary $u(\theta)$ belongs to the dual Kac algebra \hat{M} .*

PROOF. Note first that the commutant \hat{M}' is generated by the right regular representation λ' of \mathcal{M}_* defined by $\lambda'(\omega) = \hat{J}\lambda(\omega)^*\hat{J}$. Here \hat{J} stands for the modular conjugation of the dual Haar measure $\hat{\varphi}$ which is given by $\hat{J}\Lambda_\varphi(x) = \Lambda_\varphi(\kappa(x^*))$. From Theorem 2.5.3 of [ES1], it is easily verified that

$$\lambda'(\omega)\Lambda_\varphi(x) = \Lambda_\varphi((\text{id}_\mathcal{M} \otimes \omega \circ \kappa) \circ \Gamma(x)) \quad (x \in \mathcal{M}).$$

From this, together with the fact that $\Gamma(V(\pi)_{i,j}) = \sum_{k=1}^{d(\pi)} V(\pi)_{i,k} \otimes V(\pi)_{k,j}$, it follows that, for any $\omega \in \mathcal{M}_*$, we have

$$\begin{aligned} u(\theta)\lambda'(\omega)\Lambda_\varphi(V(\pi)_{i,j}) &= u(\theta)\Lambda_\varphi((\text{id}_\mathcal{M} \otimes \omega \circ \kappa) \circ \Gamma(V(\pi)_{i,j})) \\ &= \sum_{k=1}^{d(\pi)} \omega \circ \kappa(V(\pi)_{k,j})u(\theta)\Lambda_\varphi(V(\pi)_{i,k}) \\ &= \sum_{k,\ell=1}^{d(\pi)} (w(\pi)_\ell | \theta(w(\pi)_i))\omega \circ \kappa(V(\pi)_{k,j})\Lambda_\varphi(V(\pi)_{\ell,k}) \\ &= \sum_{\ell=1}^{d(\pi)} (w(\pi)_\ell | \theta(w(\pi)_i))\Lambda_\varphi((\text{id}_\mathcal{M} \otimes \omega \circ \kappa) \circ \Gamma(V(\pi)_{\ell,j})) \\ &= \sum_{\ell=1}^{d(\pi)} (w(\pi)_\ell | \theta(w(\pi)_i))\lambda'(\omega)\Lambda_\varphi(V(\pi)_{\ell,j}) \\ &= \lambda'(\omega)u(\theta)\Lambda_\varphi(V(\pi)_{i,j}). \end{aligned}$$

This completes the proof.

LEMMA 3.4. *We have*

$$V(\theta(\pi_1))_{i,j} u(\theta) A_\varphi(V(\pi_2)_{k,\ell}) = u(\theta) V(\pi_1)_{i,j} A_\varphi(V(\pi_2)_{k,\ell})$$

for any $\pi_1, \pi_2 \in \text{Irr}(\mathcal{M}_*)$.

PROOF. Let π_1, π_2 be in $\text{Irr}(\mathcal{M}_*)$. We write D_1, D_2 for D_{π_1}, D_{π_2} , respectively for simplicity. It is easy to see that the representation $\pi_{D_1 D_2}$ associated with the α -invariant Hilbert space $D_1 D_2$ is unitarily equivalent to the Kronecker product $\pi_1 \times \pi_2$ (see Theorem 1.4.3 of [ES1]). Moreover, we have

$$(3.5) \quad \begin{aligned} & (\pi_{D_1 D_2}(\omega) w(\pi_1)_j w(\pi_2)_k | w(\pi_1)_m w(\pi_2)_n) \\ & = \omega(V(\pi_1)_{m,j} V(\pi_2)_{n,k}) \quad (\omega \in \mathcal{M}_*). \end{aligned}$$

By using 3.5 and the Fourier expansion with respect to the ONB $\{\sqrt{d(\pi)} A_\varphi(V(\pi)_{i,j}) : 1 \leq i, j \leq d(\pi), \pi \in \text{Irr}\}$, we have

$$\begin{aligned} & V(\theta(\pi_1))_{i,j} u(\theta) A_\varphi(V(\pi_2)_{k,\ell}) \\ & = \sum_{m=1}^{d(\pi_1)} \sum_{n=1}^{d(\pi_2)} (w(\pi_1)_m w(\pi_2)_n | \theta(w(\pi_1)_i w(\pi_2)_k)) V(\pi_1)_{m,j} A_\varphi(V(\pi_2)_{n,\ell}) \\ & = \sum_{m=1}^{d(\pi_1)} \sum_{n=1}^{d(\pi_2)} \sum_{\pi \in \text{Irr}} d(\pi) \sum_{p,q=1}^{d(\pi)} (w(\pi_1)_m w(\pi_2)_n | \theta(w(\pi_1)_i w(\pi_2)_k)) \\ & \quad \times (V(\pi_1)_{m,j} A_\varphi(V(\pi_2)_{n,\ell}) | A_\varphi(V(\pi)_{p,q})) A_\varphi(V(\pi)_{p,q}) \\ & = \sum_{m=1}^{d(\pi_1)} \sum_{n=1}^{d(\pi_2)} \sum_{\pi \in \text{Irr}} d(\pi) \sum_{p,q=1}^{d(\pi)} (w(\pi_1)_m w(\pi_2)_n | \theta(w(\pi_1)_i w(\pi_2)_k)) \\ & \quad \times (\pi_{D_1 D_2}(\varphi V(\pi)_{p,q}^*) w(\pi_1)_j w(\pi_2)_\ell | w(\pi_1)_m w(\pi_2)_n) A_\varphi(V(\pi)_{p,q}) \\ & = \sum_{\substack{\pi \in \text{Irr}; \\ \pi \prec \pi_1 \times \pi_2}} d(\pi) \sum_{p,q=1}^{d(\pi)} (\pi_{D_1 D_2}(\varphi V(\pi)_{p,q}^*) w(\pi_1)_j w(\pi_2)_\ell | \\ & \quad \times \theta(w(\pi_1)_i w(\pi_2)_k)) A_\varphi(V(\pi)_{p,q}). \end{aligned}$$

In the meantime, similar calculations show that $u(\theta) V(\pi_1)_{i,j} A_\varphi(V(\pi_2)_{k,\ell})$ also equals the vector occurring last in the above computation. The details are left to readers.

PROPOSITION 3.6. *The unitary $u(\theta)$ satisfies*

$$W(u(\theta) \otimes u(\theta)) = (u(\theta) \otimes 1)W.$$

In particular, $u(\theta)$ belongs to the intrinsic group $G(\hat{K})$ of \hat{K} .

PROOF. Let π_1, π_2 be in $\text{Irr}(\mathcal{M}_*)$. Then, by Lemma 3.4, we have

$$\begin{aligned} &W(u(\theta) \otimes u(\theta))(\Lambda_\varphi(V(\pi_1)_{i,j}) \otimes \Lambda_\varphi(V(\pi_2)_{k,\ell})) \\ &= \sum_{m=1}^{d(\pi_1)} \sum_{n=1}^{d(\pi_2)} (w(\pi_1)_m | \theta(w(\pi_1)_i)) (w(\pi_2)_n | \theta(w(\pi_2)_k)) \\ &\quad \times W(\Lambda_\varphi(V(\pi_1)_{m,j}) \otimes \Lambda_\varphi(V(\pi_2)_{n,\ell})) \\ &= \sum_{m=1}^{d(\pi_1)} \sum_{n=1}^{d(\pi_2)} \Lambda_{\varphi \otimes \varphi}(\Gamma(V(\pi_2)_{n,\ell})(V(\pi_1)_{m,j} \otimes 1)) \\ &= \sum_{m=1}^{d(\pi_1)} \sum_{n=1}^{d(\pi_2)} \sum_{p=1}^{d(\pi_2)} \Lambda_\varphi(V(\pi_2)_{n,p} V(\pi_1)_{m,j}) \otimes \Lambda_\varphi(V(\pi_2)_{p,\ell}) \\ &= \sum_{p=1}^{d(\pi_2)} u(\theta) V(\pi_2)_{k,p} \Lambda_\varphi(V(\pi_1)_{i,j}) \otimes \Lambda_\varphi(V(\pi_2)_{p,\ell}) \\ &= (u(\theta) \otimes 1) \Lambda_{\varphi \otimes \varphi}(\Gamma(V(\pi_2)_{k,\ell})(V(\pi_1)_{i,j} \otimes 1)) \\ &= (u(\theta) \otimes 1) W(\Lambda_\varphi(V(\pi_1)_{i,j}) \otimes \Lambda_\varphi(V(\pi_2)_{k,\ell})). \end{aligned}$$

Since the set $\{\sqrt{d(\pi)} \Lambda_\varphi(V(\pi)_{i,j}) : 1 \leq i, j \leq d(\pi), \pi \in \text{Irr}\}$ is total in $L^2(\varphi)$, the first assertion follows.

For the second assertion, it suffices to note that $W(\hat{K}) = \Sigma W^* \Sigma$.

In order to relate our results to Galois correspondence established in [ILP], we modify the unitary $u(\theta)$ a little so that it may fit into their theory. Set $v(\theta) = \hat{J}u(\theta)\hat{J}$, which belongs to the Kac algebra \hat{K}' . We emphasize that, in [ILP], it is \hat{K}' that is considered to be the Kac algebra dual to K . By (3.1) and (3.2), we have

$$(3.7) \quad v(\theta) \Lambda_\varphi(V(\pi)_{i,j}) = \Lambda_\varphi(V(\theta(\pi))'_{i,j}),$$

where $V(\theta(\pi))'_{i,j} = \sum_{k=1}^{d(\pi)} (\theta(w(\pi)_j) | w(\pi)_k) V(\pi)_{i,k}$. By Proposition 3.6, $v(\theta)$ belongs to the intrinsic group $G(\hat{K}')$. It follows from Theorem 2.3 of [DeC] that $\beta_\theta = \text{Ad } v(\theta)$ defines an automorphism of $K^\sigma = (\hat{K}')^\wedge$ satisfying $(\beta_\theta \otimes \text{id}_{\mathcal{M}}) \circ \Gamma^\sigma = \Gamma^\sigma \circ \beta_\theta$, whose canonical implementation on $L^2(\varphi)$ is precisely $v(\theta)$. From Lemma 3.4, we find that $u(\theta) V(\pi)_{i,j} u(\theta)^* = V(\theta(\pi))_{i,j}$. By applying $\text{Ad } \hat{J}$ to both sides of this identity, we obtain $v(\theta) V(\pi)_{i,j} v(\theta)^* = V(\theta(\pi))'_{i,j}$. Thus the automorphism β_θ enjoys the following properties:

$$(3.8) \quad (\text{id}_{\mathcal{M}} \otimes \beta_\theta) \circ \Gamma = \Gamma \circ \beta_\theta,$$

$$\beta_\theta(V(\pi)_{i,j}) = V(\theta(\pi))'_{i,j} \quad (1 \leq i, j \leq d(\pi), \pi \in \text{Irr}).$$

LEMMA 3.9. *Let π be in $\text{Irr}(\mathcal{M}_*)$. Then the π -spectral subspace \mathcal{A}_π introduced in Part A of Appendix has the form $\mathcal{A}_\pi = \mathcal{A}^\alpha D_\pi$.*

PROOF. Let D be an α -invariant subspace of \mathcal{A} such that π_D is equivalent to π in the sense of Lemma 2.6. Then, as in the proof of Lemma 2.6, we may choose a basis $\{v_i\}$ of D such that

$$\alpha(v_j) = \sum_{i=1}^{d(\pi)} v_i \otimes V(\pi)_{i,j}.$$

Then we have

$$\alpha(w(\pi)_j v_j^*) = \sum_{i,k=1}^{d(\pi)} w(\pi)_k v_i^* \otimes V(\pi)_{k,j} V(\pi)_{i,j}^*.$$

Since $E_\alpha = (\text{id}_{\mathcal{A}} \otimes \varphi) \circ \alpha$,

$$E_\alpha(w(\pi)_j v_j^*) = \sum_{i,k=1}^{d(\pi)} \varphi(V(\pi)_{k,j} V(\pi)_{i,j}^*) w(\pi)_k v_i^* = \frac{1}{d(\pi)} \sum_{i=1}^{d(\pi)} w(\pi)_i v_i^*.$$

Thus $z = \sum_{i=1}^{d(\pi)} w(\pi)_i v_i^*$ belongs to \mathcal{A}^α , whence $v_i = z^* w(\pi)_i \in \mathcal{A}^\alpha D_\pi$. In view of Proposition A.1, we obtain $\mathcal{A}_\pi \subseteq \mathcal{A}^\alpha D_\pi$. The reverse inclusion is clear.

PROPOSITION 3.10. *We have*

$$(\text{id}_{\mathcal{A}} \otimes \beta_\theta) \circ \alpha = \alpha \circ \theta.$$

PROOF. First, a straightforward calculation yields

$$(3.11) \quad \sum_{i=1}^{d(\pi)} w(\pi)_i \otimes V(\theta(\pi))'_{i,j} = \alpha(\theta(w(\pi)_j)).$$

With $z \in \mathcal{A}^\alpha$, by (3.8) and (3.11), we obtain

$$\begin{aligned} (\text{id}_{\mathcal{A}} \otimes \beta_\theta) \circ \alpha(zw(\pi)_j) &= \sum_{i=1}^{d(\pi)} (\text{id}_{\mathcal{A}} \otimes \beta_\theta)(zw(\pi)_i \otimes V(\pi)_{i,j}) \\ &= \sum_{i=1}^{d(\pi)} zw(\pi)_i \otimes V(\theta(\pi))'_{i,j} \\ &= (z \otimes 1) \alpha(\theta(w(\pi)_j)) = \alpha \circ \theta(zw(\pi)_j). \end{aligned}$$

From this and Lemma 3.9, it follows that the assertion is true for any ele-

ment of the π -spectral subspace \mathcal{A}_π . Since the subspace generated by all the π -spectral subspaces is σ -weakly dense in \mathcal{A} , we obtain the asserted identity.

Now let us consider the fixed-point algebra \mathcal{A}^θ of the automorphism θ . Since $\theta \in \text{Aut}(\mathcal{A}/\mathcal{A}^\alpha)$, the von Neumann algebra \mathcal{A}^θ is an intermediate subfactor of the inclusion $\mathcal{A}^\alpha \subseteq \mathcal{A}$. According to [ILP], there exists a unique left co-ideal von Neumann subalgebra $\mathcal{N}(\theta)$ of \mathcal{M} such that

$$\mathcal{A}^\theta = \{x \in \mathcal{A} : \alpha(x) \in \mathcal{A} \bar{\otimes} \mathcal{N}(\theta)\}.$$

The following theorem illustrates how $\mathcal{N}(\theta)$ can be described in terms of the unitary $v(\theta)$.

THEOREM 3.12. *The left co-ideal von Neumann subalgebra $\mathcal{N}(\theta)$ of \mathcal{M} determined by the intermediate subfactor \mathcal{A}^θ is the fixed-point algebra $\mathcal{M}^{\beta_\theta}$ of the associated automorphism β_θ of \mathcal{M} . Thus we have*

$$\mathcal{N}(\theta) = \{a \in \mathcal{M} : av(\theta) = v(\theta)a\}.$$

PROOF. By (3.8), the fixed-point algebra $\mathcal{M}^{\beta_\theta}$ is a left co-ideal von Neumann subalgebra of \mathcal{M} . Hence, by Theorem 4.4 of [ILP], $\mathcal{M}^{\beta_\theta}$ determines a unique intermediate subfactor \mathcal{B} of the inclusion $\mathcal{A}^\alpha \subseteq \mathcal{A}$. So we have

$$\mathcal{B} = \{x \in \mathcal{A} : \alpha(x) \in \mathcal{A} \bar{\otimes} \mathcal{M}^{\beta_\theta}\}.$$

Now let x be in \mathcal{A}^θ . It then follows from Proposition 3.10 that $\alpha(x)$ belongs to the fixed-point algebra $(\mathcal{A} \bar{\otimes} \mathcal{M})^{\text{id}_{\mathcal{A}} \otimes \beta_\theta} = \mathcal{A} \bar{\otimes} \mathcal{M}^{\beta_\theta}$. Thus $x \in \mathcal{B}$.

Conversely, suppose that $x \in \mathcal{B}$. Then, by Proposition 3.10, we have

$$\alpha \circ \theta(x) = (\text{id}_{\mathcal{A}} \otimes \beta_\theta) \circ \alpha(x) = \alpha(x).$$

Hence $\theta(x) = x$, i.e., $x \in \mathcal{A}^\theta$.

Consequently, $\mathcal{A}^\theta = \mathcal{B}$. Therefore, the corresponding left co-ideal von Neumann subalgebras $\mathcal{N}(\theta)$ and $\mathcal{M}^{\beta_\theta}$ must coincide with each other.

Appendix

This appendix is concerned with some general results on actions of Kac algebras on von Neumann algebras. We believe that most of our results are well-known to experts. But the author was unable to locate a literature that actually contains them. So we provide their proofs for readers' convenience.

We shall keep the notation in Section 1. Throughout this appendix, we fix an action α of K on a von Neumann algebra \mathcal{P} .

Part A. Spectral subspaces. In this subsection, we briefly study the spectral subspaces associated with a compact Kac algebra action. So let us assume

for the time being that K is compact. We use the normalized Haar measure φ .

For each irreducible representation (π, \mathcal{H}_π) of \mathcal{M}_* with generator $V(\pi)$, let χ_π denote the normalized character of π (c.f. [KP], [W]), i.e., $\chi_\pi = d(\pi)(\text{Tr}_{\mathcal{H}_\pi} \otimes \text{id}_{\mathcal{M}})(V(\pi)) \in \mathcal{M}$, where $d(\pi) = \dim \mathcal{H}_\pi$, and $\text{Tr}_{\mathcal{H}}$ in general stands for the usual trace on the finite-dimensional algebra $\mathcal{L}(\mathcal{H})$. Then we define a normal linear map $E_\pi : \mathcal{P} \rightarrow \mathcal{P}$ by

$$E_\pi = (\text{id}_{\mathcal{P}} \otimes \chi_\pi^* \varphi) \circ \alpha.$$

Since $\chi_\pi^* \varphi$ belongs to the center of the Banach algebra \mathcal{M}_* , we have $(\text{id}_{\mathcal{M}} \otimes \chi_\pi^* \varphi) \circ \Gamma = (\chi_\pi^* \varphi \otimes \text{id}_{\mathcal{M}}) \circ \Gamma$. From this, it results that, for any $x \in \mathcal{P}$,

$$\begin{aligned} \alpha(E_\pi(x)) &= (\alpha \otimes \chi_\pi^* \varphi) \circ \alpha(x) = (\text{id}_{\mathcal{P}} \otimes \text{id}_{\mathcal{M}} \otimes \chi_\pi^* \varphi) \circ (\alpha \otimes \text{id}_{\mathcal{M}}) \circ \alpha(x) \\ &= (\text{id}_{\mathcal{P}} \otimes \text{id}_{\mathcal{M}} \otimes \chi_\pi^* \varphi) \circ (\text{id}_{\mathcal{P}} \otimes \Gamma) \circ \alpha(x) \\ &= (\text{id}_{\mathcal{P}} \otimes \chi_\pi^* \varphi \otimes \text{id}_{\mathcal{M}}) \circ (\text{id}_{\mathcal{P}} \otimes \Gamma) \circ \alpha(x) \\ &= (\text{id}_{\mathcal{P}} \otimes \chi_\pi^* \varphi \otimes \text{id}_{\mathcal{M}}) \circ (\alpha \otimes \text{id}_{\mathcal{M}}) \circ \alpha(x) = (E_\pi \otimes \text{id}_{\mathcal{M}}) \circ \alpha(x). \end{aligned}$$

This shows that

$$\alpha \circ E_\pi = (E_\pi \otimes \text{id}_{\mathcal{M}}) \circ \alpha.$$

Moreover, by using the fact that $\chi_\pi^* \varphi$ is a projection: $\chi_\pi^* \varphi * \chi_\pi^* \varphi = \chi_\pi^* \varphi$, we can prove that $E_\pi \circ E_\pi = E_\pi$. From these properties of E_π , it follows that the range $\mathcal{P}_\pi = E_\pi(\mathcal{P})$ is a σ -weakly closed subspace of \mathcal{P} . We call \mathcal{P}_π the π -spectral subspace of \mathcal{P} with respect to α .

Let D be an α -invariant subspace of \mathcal{P} such that π_D is equivalent to π (see Section 2). Then, as in the proof Lemma 2.6, we may choose a basis $\{v_i\}_{i=1}^{d=d(\pi)}$ of D so that we have

$$\alpha(v_j) = \sum_{i=1}^d v_i \otimes V(\pi)_{i,j},$$

where $V(\pi)_{i,j}$'s of course denote the matrix elements of $V(\pi)$ with respect to the ONB of \mathcal{H}_π determined by $\{v_i\}$. From this, we have

$$E_\pi(v_j) = \sum_{i=1}^d \varphi(\chi_\pi^* V(\pi)_{i,j}) v_i = d(\pi) \sum_{i,k=1}^d \varphi(V(\pi)_{k,k}^* V(\pi)_{i,j}) = \sum_{i,k=1}^d \delta_{i,k} \delta_{j,k} v_i = v_j.$$

Here we used the fact that $\{\sqrt{d(\pi)} V(\pi)_{i,j}\}_{1 \leq i,j \leq d}$ forms an orthonormal system in $L^2(\varphi)$ (see [ES1]). Hence D is contained in \mathcal{P}_π . Therefore, the π -spectral subspace \mathcal{P}_π contains the subspace \mathcal{D}_π of \mathcal{P} generated by the family of α -invariant subspaces D with the property that π_D is equivalent to π .

Conversely, let $x \in \mathcal{P}_{\pi}$, and consider the subspace D_x defined by

$$D_x = \{ (\text{id}_{\mathcal{P}} \otimes \omega) \circ \alpha(x) : \omega \in \mathcal{M}_* \}.$$

Since $E_{\pi}(x) = x$, x lies in D_x . We assert that D_x is finite-dimensional. For this, we need to introduce the canonical implementation of α . Let ω_0 be a faithful normal state on \mathcal{P}^{α} , and E_{α} the faithful normal conditional expectation from \mathcal{P} onto \mathcal{P}^{α} given by $E_{\alpha} = (\text{id}_{\mathcal{P}} \otimes \varphi) \circ \alpha$ (E_{α} is nothing but E_{ι} in the above notation, where ι is the trivial representation of \mathcal{M}_*). Set $\psi = \omega_0 \circ E_{\alpha}$, which is a faithful normal state on \mathcal{P} . As in Section 2, we regard \mathcal{P} as represented on the Hilbert space $L^2(\psi)$ obtained from ψ by the GNS construction. Let Λ_{ψ} be the canonical injection $\Lambda_{\psi} : \mathcal{P} \rightarrow L^2(\psi)$. Then, by Section III of [ES2], the operator U_{ψ} on $L^2(\psi) \otimes L^2(\varphi)$ defined by

$$U_{\psi}(\Lambda_{\psi}(y) \otimes \Lambda_{\varphi}(a)) = \Lambda_{\psi \otimes \varphi}(\alpha(y)(1 \otimes a)) \quad (y \in \mathcal{P}, a \in \mathcal{M})$$

is a unitary in $\mathcal{L}(L^2(\psi)) \bar{\otimes} \mathcal{M}$. Among others, it satisfies (i) $(U_{\psi})_{12}(U_{\psi})_{13} = (\text{id}_{\mathcal{L}(L^2(\psi))} \otimes \Gamma)(U_{\psi})$; (ii) $(\text{id}_{\mathcal{L}(L^2(\psi))} \otimes \omega \circ \kappa)(U_{\psi}) = (\text{id}_{\mathcal{L}(L^2(\psi))} \otimes \omega)(U_{\psi}^*)$ ($\omega \in \mathcal{M}_*$); (iii) $\alpha(y) = U_{\psi}(y \otimes 1)U_{\psi}^*$ ($y \in \mathcal{P}$). The unitary U_{ψ} is called the canonical implementation on $L^2(\psi)$. From (i), (ii) and 1.5.2 of [ES1],

$$\mu_{\alpha}(\omega) = (\text{id} \otimes \omega)(U_{\psi}) \quad (\omega \in \mathcal{M}_*)$$

defines a nondegenerate representation μ_{α} of \mathcal{M}_* on $L^2(\psi)$. With the notation just introduced, for any $a \in \mathcal{M}$ and $y \in \mathcal{P}$, we have

$$\begin{aligned} (\Lambda_{\psi}((\text{id}_{\mathcal{P}} \otimes \varphi_a)(\alpha(x))) | \Lambda_{\psi}(y)) &= (\psi \otimes \varphi)((y^* \otimes a)\alpha(x)) \\ &= (\Lambda_{\psi \otimes \varphi}(\alpha(x)) | \Lambda_{\psi}(y) \otimes \Lambda_{\varphi}(a^*)) \\ &= (U_{\psi}(\Lambda_{\psi}(x) \otimes \Lambda_{\varphi}(1)) | \Lambda_{\psi}(y) \otimes \Lambda_{\varphi}(a^*)) \\ &= (\mu_{\alpha}(\varphi_a)\Lambda_{\psi}(x) | \Lambda_{\psi}(y)), \end{aligned}$$

where φ_a is defined by $\varphi_a(b) = \varphi(ab)$. Thus we have shown that $\Lambda_{\psi}((\text{id}_{\mathcal{P}} \otimes \varphi_a)(\alpha(x))) = \mu_{\alpha}(\varphi_a)\Lambda_{\psi}(x)$. Since φ_a 's form a dense subspace of \mathcal{M}_* , it follows that

$$\begin{aligned} \text{(A.1)} \quad \Lambda_{\psi}((\text{id}_{\mathcal{P}} \otimes \omega)(\alpha(x))) &= \mu_{\alpha}(\omega)\Lambda_{\psi}(x), \\ \Lambda_{\psi}(D_x) &= \{ \mu_{\alpha}(\omega)\Lambda_{\psi}(x) : \omega \in \mathcal{M}_* \}. \end{aligned}$$

Since $\Lambda_{\psi}(x) = \Lambda_{\psi}(E_{\pi}(x)) = \Lambda_{\psi}((\text{id} \otimes \chi_{\pi}^* \varphi)(\alpha(x))) = \mu_{\alpha}(\chi_{\pi}^* \varphi)\Lambda_{\psi}(x)$, $\Lambda_{\psi}(D_x)$ is essentially the subspace $\{ \mu_{\alpha}(\omega * \chi_{\pi}^* \varphi)\Lambda_{\psi}(x) : \omega \in \mathcal{M}_* \}$. By using the Fourier expansion, we find that $\omega * \chi_{\pi}^* \varphi = d(\pi) \sum_{i,j=1}^{d(\pi)} \omega(V(\pi)_{i,j})(V(\bar{\pi})_{i,j} \varphi)$ for any $\omega \in \mathcal{M}_*$. From this, it follows that $\Lambda_{\psi}(D_x)$ is generated by $\{ \mu_{\alpha}(V(\bar{\pi})_{i,j} \varphi)\Lambda_{\psi}(x) \}_{1 \leq i,j \leq d(\pi)}$. In particular, D_x itself is finite-dimensional.

Finally, we claim that $x \in \mathcal{D}_{\pi}$. By A.1, $\Lambda_{\psi}(D_x)$ is a μ_{α} -invariant subspace,

so that $\mu_\alpha|_{A_\psi(D_x)}$ can be decomposed into irreducible representations as follows: $A_\psi(D_x) = \sum_{l \in I}^\oplus A_\psi(D_l)$, $\mu_\alpha|_{A_\psi(D_l)} \cong \pi$. Consequently, D_x is an algebraic direct sum: $D_x = \sum_{l \in I}^\oplus D_l$. In particular, x can be written in the form: $x = \sum_{l \in I} x_l$ ($x_l \in D_l$). Hence, in order to show that $x \in \mathcal{D}_\pi$, we may assume that $\mu_\alpha|_{A_\psi(D_x)} \cong \pi$. In this case, we may choose a basis $\{v_i\}$ of D_x such that

$$\mu_\alpha(\omega)A_\psi(v_j) = \sum_{i=1}^d \omega(V(\pi)_{i,j})A_\psi(v_i).$$

This, together with (A.1), implies that $(\text{id}_{\mathcal{P}} \otimes \omega) \circ \alpha(v_j) = \sum_{i=1}^d \omega(V(\pi)_{i,j})v_i$, which is in turn equivalent to

$$\alpha(v_j) = \sum_{i=1}^d v_i \otimes V(\pi)_{i,j}.$$

Hence D_x is an α -invariant subspace of \mathcal{P} such that π_D is equivalent to π . Therefore, $x \in \mathcal{D}_\pi$ as claimed.

We summarize, as a proposition, the results of the preceding discussion.

PROPOSITION A.1. *For each irreducible representation π , the π -spectral subspace \mathcal{P}_π with respect to the action α coincides with the subspace generated by the α -invariant subspaces D of \mathcal{P} with the property that π_D is equivalent to π .*

Part B. Remarks on the actions $\bar{\alpha}$ and $\tilde{\alpha}$ This subsection is concerned with some relations among the actions α , $\bar{\alpha}$ and $\tilde{\alpha}$. In what follows, K is a general (not necessarily compact) Kac algebra.

PROPOSITION B.1. *Suppose that \mathcal{P} is properly infinite. Then there exists a $*$ -isomorphism Ψ from \mathcal{P} onto $\mathcal{P} \bar{\otimes} \mathcal{F}_\infty$ so that, through this Ψ , α is cocycle conjugate to the action $\bar{\alpha}$ defined in Section 1, where \mathcal{F}_∞ denotes the countably decomposable infinite factor of type I.*

PROOF. By assumption, we may take a sequence $\{e_n\}_{n=1}^\infty$ of projections in \mathcal{P} such that $e_n \sim 1$ and $\sum_{n=1}^\infty e_n = 1$. Then, for each $n \in \mathbb{N}$, choose a partial isometry $v_n \in \mathcal{P}$ with $v_n^*v_n = e_n$ and $v_nv_n^* = 1$. Let $\{u_{i,j}\}_{i,j=1}^\infty$ be a system of matrix units in \mathcal{F}_∞ . It is well-known that, with $x_{i,j} = v_i^*xv_j$, the equation

$$\Psi(x) = \sum_{i,j=1}^\infty x_{i,j} \otimes u_{i,j} \quad (x \in \mathcal{P})$$

defines a $*$ -isomorphism Ψ from \mathcal{P} onto $\mathcal{P} \bar{\otimes} \mathcal{F}_\infty$. We put $\beta = (\Psi \otimes \text{id}_{\mathcal{M}}) \circ \alpha \circ \Psi^{-1}$, which is, by definition, an action of K on $\mathcal{P} \bar{\otimes} \mathcal{F}_\infty$. Next we define an element U of $\mathcal{P} \bar{\otimes} \mathcal{F}_\infty \bar{\otimes} \mathcal{M}$ by

$$U = \sum_{n=1}^{\infty} (w_n \otimes 1) \beta(w_n^*),$$

where $w_n = \pi(v_n)$. Then U is a unitary. Indeed, we have

$$\begin{aligned} U^*U &= \sum_{n,m=1}^{\infty} \beta(w_n)(w_n^* \otimes 1)(w_m \otimes 1)\beta(w_m^*) = \sum_{n,m=1}^{\infty} \delta_{n,m} \beta(w_n w_m^*) \\ &= \sum_{n=1}^{\infty} \beta(w_n w_n^*) = 1. \end{aligned}$$

Similarly, we obtain $UU^* = 1$. Moreover, on $\mathcal{P} \bar{\otimes} \mathcal{F}_{\infty} \bar{\otimes} \mathcal{M} \bar{\otimes} \mathcal{M}$, one has

$$\begin{aligned} U_{123}(\beta \otimes \text{id}_{\mathcal{M}})(U) &= \sum_{n,m=1}^{\infty} (w_n \otimes 1 \otimes 1)(\beta(w_n^*) \otimes 1)(\beta \otimes \text{id}_{\mathcal{M}})((w_m \otimes 1)\beta(w_m^*)) \\ &= \sum_{n,m=1}^{\infty} (w_n \otimes 1 \otimes 1)(\beta(w_n^* w_m) \otimes 1)(\beta \otimes \text{id}_{\mathcal{M}})(\beta(w_m^*)) \\ &= \sum_{n=1}^{\infty} (w_n \otimes 1 \otimes 1)(\beta \otimes \text{id}_{\mathcal{M}})(\beta(w_n^*)) \\ &= \sum_{n=1}^{\infty} (w_n \otimes 1 \otimes 1)(\text{id}_{\mathcal{P} \bar{\otimes} \mathcal{F}_{\infty}} \otimes \Gamma)(\beta(w_n^*)) \\ &= (\text{id}_{\mathcal{P} \bar{\otimes} \mathcal{F}_{\infty}} \otimes \Gamma)(U). \end{aligned}$$

This shows that U is a β -cocycle [E]. Since $w_n^*(1 \otimes u_{k,\ell})w_m = \delta_{n,k} \delta_{\ell,m} \cdot 1$, we can easily verify

$$U\beta(1 \otimes u_{k,\ell})U^* = 1 \otimes u_{k,\ell} \otimes 1 = \bar{\alpha}(1 \otimes u_{k,\ell})$$

for any $k, \ell \in \mathbb{N}$. Meanwhile, from $w_n^*(x \otimes 1)w_m = \delta_{n,m} \Psi(x)$ for any $x \in \mathcal{P}$, it follows that

$$\begin{aligned} U\beta(x \otimes 1)U^* &= \sum_{n,m=1}^{\infty} (w_n \otimes 1)\beta(w_n^*(x \otimes 1)w_m)(w_m^* \otimes 1) \\ &= \sum_{n=1}^{\infty} (w_n \otimes 1)(\Psi \otimes \text{id}_{\mathcal{M}})(\alpha(x))(w_n^* \otimes 1) \\ &= \sum_{n=1}^{\infty} (w_n \otimes 1)(\Psi \otimes \text{id}_{\mathcal{M}})(\alpha(x))(w_n^* \otimes 1). \end{aligned}$$

Note that $\sum_{n=1}^{\infty} (w_n \otimes 1)(\Psi \otimes \text{id}_{\mathcal{M}})(y \otimes a)(w_n^* \otimes 1) = y \otimes 1 \otimes a$ for any $y \in \mathcal{P}$ and $a \in \mathcal{M}$. This, together with the above computation, yields

$$U\beta(x \otimes 1)U^* = \alpha(x)_{13} = \bar{\alpha}(x \otimes 1).$$

Thus we have proven that $\text{Ad } U \circ \beta = \bar{\alpha}$. Namely, β is cocycle conjugate to $\bar{\alpha}$. Therefore, α is cocycle conjugate to $\bar{\alpha}$ by construction.

COROLLARY B.2. *With the notation in Proposition B.1, we now assume that the fixed-point algebra \mathcal{P}^α of α is properly infinite. Then α is conjugate to $\bar{\alpha}$ through the isomorphism Ψ .*

PROOF. In the proof of Proposition B.1, we can take e_n and v_n from the inside of \mathcal{P}^α by assumption. Then w_n 's belong to the fixed-point algebra of β , so that the β -cocycle U in this case is the identity operator. This proves the assertion.

COROLLARY B.3. *Suppose that \mathcal{P} is properly infinite. Consider the induced action $\tilde{\alpha}$ of K on $\mathcal{P} \otimes \mathcal{L}(L^2(\varphi))$ defined in Section 1, i.e., $\tilde{\alpha} = \text{Ad}(1 \otimes W(\hat{K})) \circ \bar{\alpha}$. Then α is cocycle conjugate to $\tilde{\alpha}$.*

PROOF. Since the unitary $1 \otimes W(\hat{K})$ is an $\bar{\alpha}$ -cocycle by Théorème IV.3. (ii) of [ES2], the assertion follows from Proposition B.1.

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