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LOCALLY INJECTIVE AUTOMORPHIC MAPPINGS IN R^n

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Dedicated to the memory of Lars Ahlfors

1. Introduction

In this paper we continue our studies in the theory of quasimeromorphic automorphic mappings, which we started in [7] and [8]. The notation will be as in [8]. In particular, a continuous mapping $f : D \to \overline{\mathsf{R}}^n = \mathsf{R}^n \cup \{\infty\}$, where D is domain in R^n, is said to be K-*quasimeromorphic*, or shortly *quasimeromorphic*, if either f is a constant map, or else if the set $f^{-1}(\infty)$ is discrete in D, f belongs to the Sobolev class $W^{1,n}_{\mathrm{loc}}(D \setminus f^{-1}(\infty))$ and its weak derivatives satisfy

$$|f'(x)|^n \le KJ(x,f) \tag{1.1}$$

a.e. in D for some constant $K \in [1, \infty)$. Here $f'(x)$ denotes the Jacobian matrix, $|f(x)|$ its operator norm, and $J(x,f)$ the Jacobian of f at x. The smallest K for which (1.1) holds a.e. in D is called the *dilatation* of f and will be denoted by $K(f)$.

If $\infty \notin f(D)$, then f is said to be *quasiregular*, and if f is injective, we say that f is *quasiconformal*.

Let G be a group of Möbius transformations which acts discontinuously on D. A quasimeromorphic map $f : D \to \overline{\mathsf{R}}^n$ is said to be *automorphic* with respect to G, if f is G-invariant, i.e. if $f \circ g = f$ for all $g \in G$. We say that f is K-*automorphic*, if f is K-quasimeromorphic and automorphic for some Möbius group G.

In [9], we showed that if G is a Möbius group which acts discontinuously on the unit ball $B^n = \{x \in \mathsf{R}^n : |x| < 1\}$ and if $\mathrm{vol}(B^n/G) < \infty$, then G carries non-constant automorphic mappings. Later Tukia [15] showed that the existence of non-constant automorphic maps holds if G is only torsion-free. Tukia's result follows also from Peltonen [12]. On the other hand, it has been shown in [14] that, contrary to the two dimensional case, some Möbius groups which act discontinuously on B^n, $n \ge 3$, do not carry non-constant automorphic mappings. These groups have elements of arbitrarily large or-

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der. We do not know whether automorphic maps exist if there is a finite upper bound for the order of those elements in G which are of finite order.

In this paper we will be mostly concerned with the existence of locally injective automorphic mappings. In this respect, there is a remarkable difference between $n = 2$ and $n \geq 3$. This has been demonstrated already by Zorich [18], who showed that there is no counterpart in R^n, $n \geq 3$, to the simplest and most important locally injective automorphic map in C, namely, the exponential function e^z.

We will provide necessary conditions for the existence of locally injective automorphic mappings in X, where $X = B^n$ or $X = H^n = \{x \in \mathsf{R}^n : x_n > 0\}$, and $n \geq 3$, and prove the existence in some special cases.

Locally injective automorphic mappings can be used in the study of Fatou's problem on the existence of radial limits of a bounded quasiregular mapping $f : B^n \to \mathsf{R}^n$, for $n \geq 3$. It is still unknown if a bounded quasiregular map $f : B^n \to \mathsf{R}^n$, $n \geq 3$, has radial limits a.e. on ∂B^n or even at one point.

Martio and Rickman [5] showed, however, that if $f : B^n \to \mathsf{R}^n$, $n \geq 2$, is a bounded quasiregular mapping and if its multiplicity function

$$N(r) = N(r,f) = \sup\{\# f^{-1}(y) \cap B^n(r) : y \in \mathsf{R}^n\},$$

where $B^n(r) = \{x \in \mathsf{R}^n : |x| < r\}$, satisfies the growth restriction

$$N(r,f) \leq \frac{C}{(1-r)^s}, \quad 0 < r < 1, \tag{1.2}$$

for some constants $C > 0$ and $0 < s < n - 1$, then the *singular set* $E(f)$, i.e. the set of points on ∂B^n where f has no radial limits, is of zero $(n-1)$-measure. We will show here that if, in addition, f is locally injective and if $n \geq 3$, then the Hausdorff dimension $\dim E(f)$ of $E(f)$ satisfies $\dim E(f) \leq s$, and this estimate is best possible. For the sharpness of this estimate, we construct for every $n \geq 3$ a sequence of numbers $s_m \in (0, n-1)$ and of locally injective maps $f_m : H^n \to B^n$, $m = 1, 2, \ldots$, such that f_m satisfies (1.2) with $s = s_m$, $\dim E(f_m) = s_m$ and $s_m \to n - 1$.

Recently Koskela, Manfredi and Villamor [4, Theorem 4.1] have obtained a bound for $\dim E(f)$ using the theory of $\mathscr{A}$-harmonic functions. Their bound is weaker than ours but, on the other hand, no local injectivity is assumed and the boundedness of f is replaced by a weaker assumption $|f(x)| \leq c(1 - |x|)^b$, see Remark 5.2 below.

2. Existence of locally injective automorphic mappings

Given a Möbius group G acting on X, $X = B^n$ or $X = H^n$, $n \geq 3$, one may ask: Under what conditions on G, there exists a locally injective automorphic mapping $f : X \to \overline{\mathsf{R}}^n$? In this section we provide necessary conditions, and prove the existence in two special cases. More examples will be brought in the following sections. We always assume that G is non-trivial, i.e. $G \neq \{\mathrm{id}\}$.

2.1. *Radius of injectivity of a Möbius group.* By a Möbius transformation we mean a sense-preserving conformal automorphism of $\overline{\mathsf{R}}^n$, $n \geq 2$. A Möbius transformation g, $g \neq id$, is classified as follows: g is *parabolic* if it has a unique fixed point in $\overline{\mathsf{R}}^n$, it is *elliptic* if it is conjugate to some sense-preserving orthogonal transformation $T : \mathsf{R}^n \to \mathsf{R}^n$, and it is *loxodromic* otherwise. If g acts on X, $X = B^n$ or $X = H^n$, then g is an hyperbolic isometry. If, in addition, g is parabolic, then its unique fix point lies on ∂X, if it is elliptic it fixes at least one point in X, and if it is loxodromic it fixes exactly two points in ∂X and no other point elsewhere.

Let G be a Möbius group which acts on X, $X = B^n$ or $X = H^n$. The *translation length* of g, $g \in G \setminus \{\mathrm{id}\}$, is defined by

$$l(g) = \inf \{k(x, g(x)) : x \in X\}.$$

Here k denotes the hyperbolic distance function in X. Then, clearly, $l(g) > 0$ if g is loxodromic and $l(g) = 0$ otherwise. The *injectivity radius* of G is defined by

$$r(G) = \tfrac{1}{2} \inf \{l(g) : g \in G \setminus \{\mathrm{id}\}\}.$$

Then, obviously, $r(G) = 0$ if G has elliptic or parabolic elements, and $r(G) \geq 0$ if G is *purely loxodromic*.

Suppose that G is torsion-free and acts discontinuously on X. Then $M = X/G$ is a hyperbolic manifold. Its *injectivity radius* $r(M)$ is defined as the supremum over all $r > 0$, for which all sets $B(x, r) = \{y \in M : k(x, y) < r\}$, $x \in M$, are topological balls. Then $2r$ equals the infimum of the length of closed geodesics in M, and $r(M) = r(G)$.

2.2. *A universal radius of injectivity for locally injective quasimeromorphic mappings.* Let $f : B^n \to \overline{\mathsf{R}}^n$ be a locally injective map. The *injectivity radius* of f is defined by

$$r(f) = \sup \{r > 0 : f|B^n(r) \ \text{ is injective}\}.$$

The following theorem, which is quoted from [10], see also [6], asserts the existence of a universal injectivity radius for locally injective quasimeromorphic mappings $f : B^n \to \overline{\mathsf{R}}^n$, $n \geq 3$, which depends only on n and $K(f)$.

2.3. THEOREM. *Given $n \geq 3$ and $K \geq 1$, there exists a constant $r = r(n, K) \in (0, 1]$, such that $r(f) \geq r(r, K)$, whenever $f : B^n \to \overline{\mathsf{R}}^n$ is a locally injective K-quasimeromorphic map.*

2.4. Lemma. *Suppose that* $f : X \to \overline{\mathsf{R}}^n$, $n \geq 3$, *is automorphic with respect to* G. *If* G *contains parabolic or elliptic elements, then* f *is not locally injective in* X.

Proof. Suppose that G contains an elliptic element g. Then g fixes some point a in X. Hence, by continuity of g, given any neighborhood U of a, there is a point x in $U \setminus \{a\}$, such that $g(x) \in U$. Hence $f(g(x)) = f(x)$, proving that f is not injective in any neighborhood U of a. This argument applies to $n = 2$, as well.

Suppose now that G has a parabolic element g and that $g(a) = a$ for some point $a \in \partial X$. By conjugating G and f by a Möbius transformation which maps X onto H^n and a to ∞, we may assume that $X = H^n$, $a = \infty$ and that $f : H^n \to \overline{\mathsf{R}}^n$ is K-automorphic. Denote $g(0) = b$, then $b \in \partial H^n \setminus \{0\}$ and $h(x) = g(x) - b$, being an euclidean isometry, cf. [11, IV.C.6], which fixes 0, is an orthogonal transformation. Now $h(\partial H^n) = \partial H^n$, hence $h(te_n) = te_n$, and $g(te_n) = te_n + b$ for all $t > 0$. Here e_n is the unit vector which is normal to ∂H^n.

Choose $R > |b|/r(n, K)$, where $r(n, K)$ is the constant in Theorem 2.3, and consider the restriction of f to the ball $B^n(Re_n, R)$. If f was locally injective, then, by Theorem 2.3, it would be injective in $B^n(Re_n, r)$ where $r = Rr(n, K) > |b|$. But $|g(Re_n) - Re_n| = |b| < r$ and $f(g(Re_n)) = f(Re_n)$, and therefore f is not locally injective in this case, too.

2.5. Lemma. *Given* $n \geq 3$ *and* $K > 1$, *there exists a positive constant* $l(n, K)$, *such that* $l(g) > l(n, K)$, *whenever* $f : X \to \overline{\mathsf{R}}^n$, $X = B^n$ *or* $X = H^n$, *is locally injective and* K*-automorphic with respect to a Möbius group* G, *and* g *is a loxodromic element in* G.

Proof. The claim easily follows from Theorem 2.3 after a suitable conjugation of G and f.

2.6. Theorem. *Let* G *be a Möbius group acting on* X, $X = B^n$ *or* $X = H^n$, $n \geq 3$, *and suppose that there exists a locally injective map* $f : X \to \overline{\mathsf{R}}^n$, *which is* K*-automorphic with respect to* G. *Then* G *is discrete and purely loxodromic with* $r(G) \geq \rho > 0$, *for some constant* $\rho = \rho(n, K)$. *Furthermore,* X/G *is non-compact and has infinite hyperbolic volume.*

Proof. Since f is a local homeomorphism, G must act discontinuously on X, and hence G is discrete. By Lemma 2.4, G has neither elliptic elements nor parabolic elements, and thus it is purely loxodromic. Also, Lemma 2.5 implies that $l(g) \geq l(r, K)$ for all $g \in G$, and hence, $r(G) \geq \rho(r, K)$, where $\rho(n, K) = l(n, K))/2 > 0$.

Now, if $M = X/G$ was compact, then the lift $\tilde{f} : M \to \overline{\mathsf{R}}^n$ of f would be a covering map, and since $\overline{\mathsf{R}}^n$ is simply connected it would imply that $\tilde{f}$ is injective, which is impossible. Thus M is not compact. This argument applies to $n = 2$ as well. Finally $\mathrm{vol}(M) = \infty$, since otherwise G would contain parabolic elements in view of the fact that M is non-compact, see [11]. The proof is complete.

2.7. COROLLARY. *Let M be an hyperbolic n-manifold, $n \geq 3$. If M is compact, or if $r(M) = 0$, then M cannot be immersed quasiregularily in S^n.*

2.8. REMARK. Theorem 2.6 and Corollary 2.7 are false in two dimensions. The modular function is analytic, locally injective and it is automorphic with respect to a group G, which contains parabolic elements and H^2/G is of finite volume. Also, $r(H^2/G) = 0$, and H^n/G can be immersed conformally in S^2.

The following example shows that a purely loxodromic group G which acts on H^2, may have zero injectivity radius and still carry locally injective automorphic maps.

2.9. EXAMPLE. Choose real numbers $0 < x_1 < x_2 < \ldots$, and $0 < r_1 < r_2 < \ldots$, such that the discs $D_k = \{z \in \mathsf{C} : |z - x_k| < r_k\}$, $k = 1, 2, \ldots$, are disjoint, $0 \notin \overline{D}$, and such that the hyperbolic distance d_k between the disc D_k and the disc $D_{-k} = \{z \in \mathsf{C} : |z + x_k| < r_k\}$ is smaller than $1/k$. This can be achieved by choosing the radii r_k sufficiently large. Now, for $k = 1, 2, \ldots$, let g_k denote the inversion in ∂D_k followed by the reflection in the imaginary axes, and let $G = \langle g_k : k = 1, 2, \ldots \rangle$. Then G is purely loxodromic acting on H^2 with $l(g_k) < 1/k$, and thus with $r(G) = 0$, and $D = H^2 \setminus \cup(\overline{D}_k \cup \overline{D}_{-k})$ is a fundamental domain for the action of G on H^2.

Next, let φ be conformal map of D onto H^2 with $\varphi(0) = 0$, $\varphi(\infty) = \infty$ and $\varphi(i) = i$. Then, φ extends conformally to $\overline{D} \cap H^2$, and it preserves symmetry with respect to the imaginary axes. Finally, let $f(z) = \varphi(z)^2$ if $z \in \overline{D} \cap H^2$ and $f(z) = f(g(z))$ if $g(z) \in D$, $g \in G$. Then f is a locally univalent analytic function in H^2, which is automorphic with respect to G, and $r(G) = 0$.

It should be noted that the same construction holds when some of the discs D_k are tangent, where in this case G has parabolic elements, too.

We do not know if the conditions of Theorem 2.6 are sufficient for the existence of locally injective automorphic maps. However, there are two special cases, where groups satisfying the conditions of Theorem 2.6, do carry locally injective automorphic maps. These two cases are present below.

2.10. THEOREM. *Let G be a finitely generated purely loxodromic group which acts on H^3. If G keeps*

$$H^2 = \{x \in \mathsf{R}^3 : x_3 = 0,\ x_2 > 0\}$$

invariant, then G carries bounded locally injective automorphic mappings $f : H^3 \to \mathsf{R}^3$.

Proof. Since G is finitely generated and acts on H^2, there is a finite sided (hyperbolic) polyhedron P in H^2, which is a fundamental domain for the action of G on H^2. The polyhedron may have free arcs on ∂H^2. This happens if H^2/G is not compact. Since P is finitely sided and since G does not contain parabolic elements nor elliptic element, there exists a C^∞ compact surface S in R^3, possibly with finitely many non-degenerate boundary components, and a C^∞ Lipschitz map (in the Euclidean metric) f of the closure of P in $\overline{H}^2$ onto S, such that $f|P$ is an embedding and such that $f(b) = f(b')$ if $b, b' \in \partial P$ are equivalent under G. Note that such a map would not exist if G had parabolic element, in particular, it could not be Lipschitz.

We now extend f to H^3. For $0 \le \theta \le \pi$, let H^2_θ denote the euclidean half plane in H^3 whose boundary lies on $R = \partial H^2$, and which forms an angle θ with H^2. Let $R_\theta : H^2 \to H^2_\theta$ be the rotation about R. Then

$$D = \cup\{R_\theta(P) : 0 < \theta < \pi\}$$

is a fundamental domain for the action of G on H^3, see Figure 1.

For $z \in \overline{P}$ let $n(z)$ denote the unit normal vector to S at the point $f(z)$. We orientate all the normal sectors $n(z)$ in the same way. We now extend f to $\overline{D}$ by letting,

$$f(R_\theta(z)) = f(z) + \lambda \operatorname{dist}(z, R)\theta n(z),\ z \in \overline{P},\ 0 \le \theta \le \pi, \tag{2.11}$$

where $\lambda > 0$ is a constant. Since the curvature of S is bounded and $f|\overline{P}$ is smooth and Lipschitz, f will be injective and quasiconformal if λ is chosen to be sufficiently small. Furthermore, since G acts on each half plane $R_\theta,\ 0 \le \theta \le \pi$, in the same way, $f(b) = f(b')$ for any $b, b' \in \partial D$ which are equivalent under G.

Finally, we extend f to H^3 by letting $f(x) = f(g(x))$, if $g(x) \in \overline{D}$. Then f is automorphic, locally injective and bounded. Indeed, if the sides of P are paired by the loxodromic generators $g_1, \dots, g_{2\alpha}$, then

$$\cup(g_i(\overline{P}) \cup g_i^{-1}(P))$$

is a neighborhood of P, and since $f|P$ is injective it follows that $f|H^2$ is locally injective, and thus, so is $f|H^2_\theta$ for any θ, and hence f is locally injective in H^3. The image $f(H^3)$ is a bounded domain in R^3, which is topologically $S \times R$.

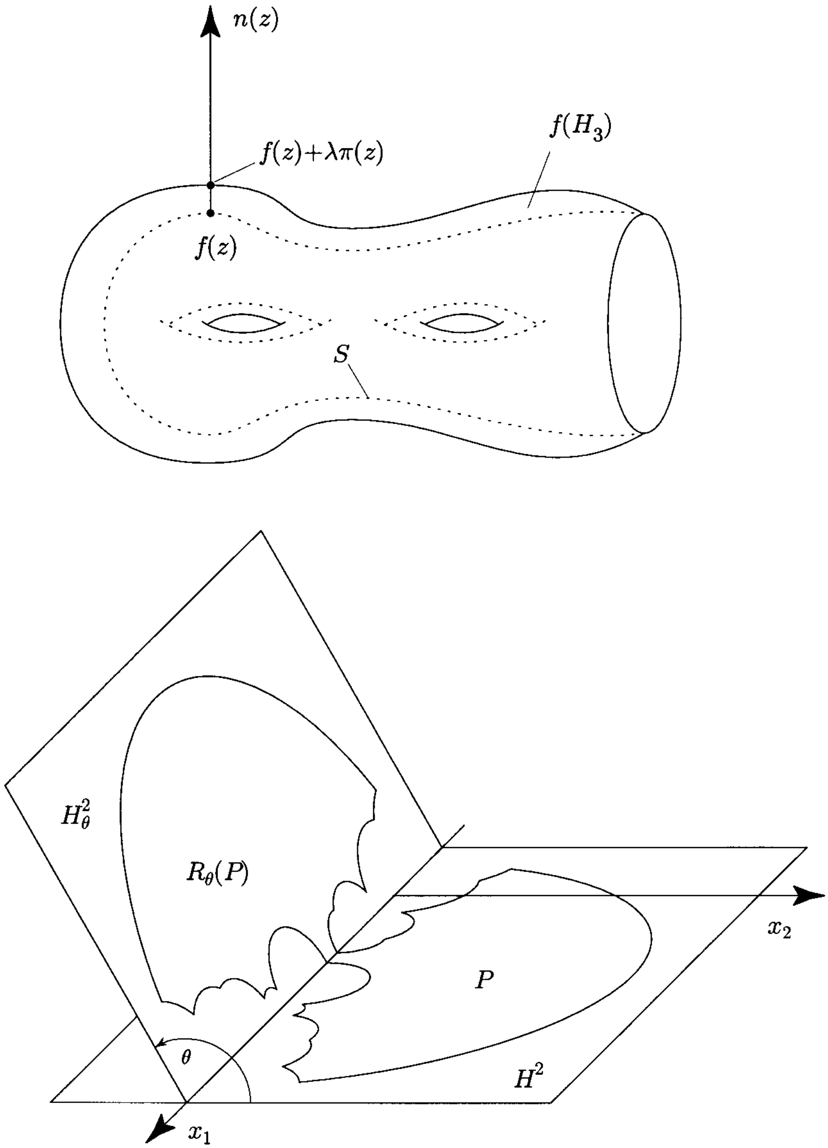

Figure 1. The mapping f in 2.10.

2.12. REMARK. 1. We do not know if Theorem 2.10 is true if G is infinitely generated and otherwise satisfies the same assumptions.

2. Note that the mapping f in the proof of Theorem 2.10 is locally injective in H^3, and if H^2/G is compact, then f has a continuous extension to $\partial H^3 \setminus \mathsf{R}$ but not to R. It is an open problem, if the line R is removable for every locally injective quasimeromorphic map $f : U \setminus \mathsf{R} \to \overline{\mathsf{R}^3}$, where U is a neighborhood of R.

We now prove the existence of locally injective automorphic maps for certain finitely generated Schottky groups which act on H^n, $n \geq 2$.

2.13. THEOREM. *For $i = 1, \dots, k$, let B_i and B_i' be balls in R^n, $n \geq 2$, centered at ∂H^n, such that for some $\lambda > 1$, all balls $\lambda B_i, B_i'$ are mutually disjoint, and g_i a loxodromic transformation acting on H^n such that $g_i(B_i) = \overline{\mathsf{R}}^n \setminus \overline{B}_i'$, $i = 1, \dots, k$. Then the group $G = \langle g_1, \dots, g_k \rangle$ carries locally injective automorphic mappings.*

PROOF. The domain $D = H^n \setminus \cup(\overline{B}_i \cup \overline{B}_i')$ is a fundamental domain for the action G on H^n. We will first construct a function $f : \overline{D} \to \mathsf{R}^n$ which is continuous in $\overline{D}$ and quasiconformal in D, and then extend it by the elements of G.

For $i = 1, \dots, k$, let $U_i = (H^n \cap \lambda B_i) \setminus \overline{B}_i$. Let $f : \overline{D} \to \mathsf{R}^n$ be a continuous function such that $f|\overline{D} \setminus \cup U_i = id$, for $i = 1, \dots, k$, the map $f|U_i$ is quasiconformal, and such that for every point $x \in \partial B_i \cap H^n$, $f(x)$ is the unique point on $\partial B_i'$ which is equivalent to x under G. For each i, $f(U_i)$ is a bounded set topologically equivalent to a product of an $(n-1)$-disc and a line segment, see Figure 2.

Now, extend f to H^n by the elements of G, so that $f(x) = f(g(x))$ if $g(x) \in \overline{D} \setminus \partial H^n$. Then f is automorphic and locally injective. By composing f with a suitable Möbius transformation, it can be made bounded.

The following examples show that a Möbius groups G, acting on H^n, $n \geq 3$, may not carry locally injective automorphic mappings, but still may have a G-invariant local homeomorphism onto a bounded set in R^n. The examples are described in H^3, but can be modified to any dimension ≥ 3.

2.14. EXAMPLE. Consider H^3 as $\mathsf{C} \times (0, \infty)$ and let $f : H^3 \to \mathsf{R}^3$ be defined by

$$f(z, x_3) = (e^z, x_3), \; z \in \mathsf{C}, \; 0 < x_3 < \infty.$$

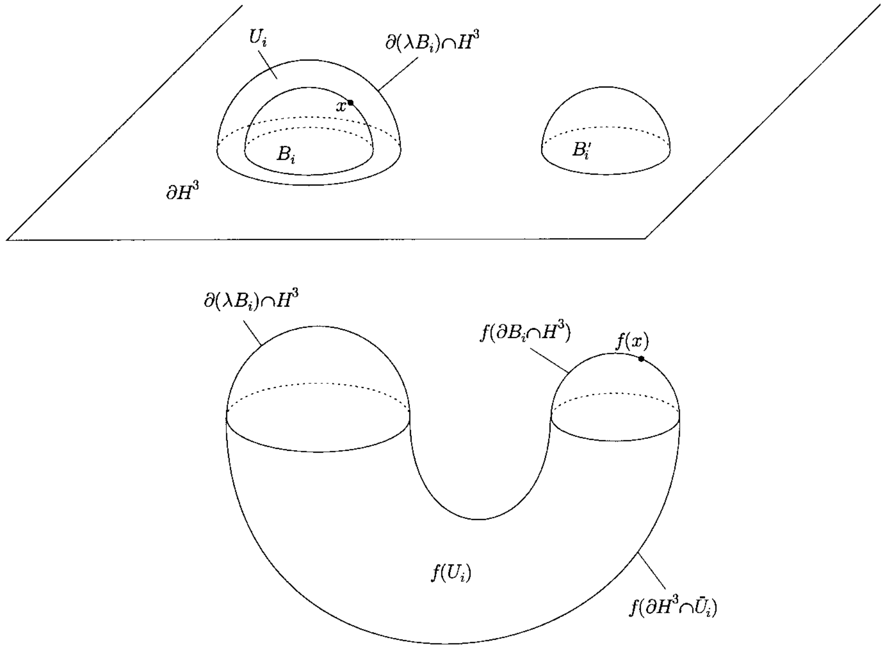

Figure 2. The construction in 2.13.

Then f is a local homeomorphism, mapping H^3 onto $H^3 \setminus \{x_3\text{-axis}\}$, and f is invariant under the parabolic cyclic group G which is generated by $(z, x_3) \to (z + 2\pi i, x_3)$. By Theorem 2.6, G does not carry any locally homeomorphic automorphic map.

2.15. Example. Consider H^3 as $\mathsf{C} \times (0, \infty)$, and let G be the parabolic group which is generated by the two transformations $(z, x_3) \to (z + 1, x_3)$ and $(z, x_3) \to (z + i, x_3)$. Then G acts discontinuously on H^3, and H^3/G is topologically equivalent to $M = T_2 \setminus \overline{T}_1$ where T_1 and T_2, $T_1 \subset T_2$, are open solid tori in R^3, which have the same core. By Theorem 2.6, G does not carry automorphic mappings. However, G does carry a G-invariant C^∞-local homeomorphism f of H^3 onto M. The map f can be constructed as follows. Let $\varphi : (0, \infty) \to (0, \lambda)$ be a C^∞ strictly increasing function, where $\lambda > 0$ is a constant. Then $\Phi(z, t) = (z, \varphi(t))$, $z \in \mathsf{C}$, $0 < t < \infty$, maps H^3 onto $\mathsf{C} \times (0, \lambda)$. Choose a solid torus T_1 in R^3 such that ∂T_1 is a smooth C^∞ torus, and let $F : \mathsf{C} \to \partial T_1$ be a C^∞ doubly periodic map with periods 1 and i such that F is injective on the open unit square which has vertices at $0, 1, i$ and $1 + i$. Let $n(z)$ denote the unit normal vector to ∂T_1 at the point $F(z)$. Then for λ sufficiently small $f(z, x_3) = F(z) + \varphi(x_3)n(z)$ is a C^∞ local homeomorphism of H^3 onto the region D which is bounded by ∂T_1 and ∂T_2.

3. Infinitely generated Schottky groups and radial limits

3.1. *Fatou's problem.* Fatou's theorem asserts that a bounded analytic function in the unit disc $|z| < 1$ in C has radial limits a.e. on $|z| = 1$. This theorem is false for bounded quasiregular mappings in $|z| < 1$, due to the fact that the boundary extension of a quasiconformal self-map of $|z| < 1$ may carry a set of measure 2π to a set of measure zero, as was shown by Beurling and Ahlfors [2].

Not much is known about the existence of radial limits of a bounded quasiregular mapping f in B^n, when $n > 2$. It is not known, for instance, if f has a radial limit anywhere in ∂B^n.

In the following example, the assumption that f is bounded is replaced by the weaker assumption that f omits a non-degenerate continuum. In this example $f : H^n \to \mathsf{R}^n$, $n \geq 3$, is locally injective and automorphic with respect to an infinitely generated Schottky group G, which acts on H^n, $\mathsf{R}^n \setminus f(H^n) \subset \mathsf{R}^n \setminus \{x \in \mathsf{R}^n : x_n = 0,\ x_1 = 0\}$, and f has no radial limits on a dense set in ∂H^n. The description of the construction will be given for $n = 3$. Its extension to $n = 2$ or to higher dimensions is straightforward and will be omitted.

3.2. *The group G.* Fix $\alpha \in (0, \frac{\pi}{2})$. Given $a = (a_1, a_2) \in R^2 = \partial H^3$, $a_2 > 0$ and $r > 0$, let $B(a, r) = \{x \in \mathsf{R}^3 : |x - a| < r\}$, $U(a, r) = B^3(a', r') \cap H^3$, where

$$a' = (a_1, a_2, r \tan \alpha) \text{ and } r' = r / \cos \alpha.$$

Then

$$B(a, r) \cap H_3 \subset U(a, r) \text{ and } \partial B(a, r) \cap \partial U(a, r) \subset \partial H^3.$$

If $U = U(a, r)$ and $B = B(a, r)$, $a \in \partial H^3$, $r > 0$, are contained in the half space $x_2 > 0$, we let

$$B^* = I(B) \text{ and } U^* = I(U)$$

where I denotes the reflection in the plane $x_2 = 0$. Now, select a collection $\mathscr{B}$ of balls $B = B(a, 1)$ of radius 1, with a in the half plane

$$H^2 = \{x \in \partial H^3 : x_2 > 0\},$$

such that the corresponding domains $U(a, 1)$ are in the half space $x_2 > 0$ and are mutually disjoint. Make the selection maximal in the sense that no ball $B(a, 1)$ satisfying the above conditions can be added.

Now add to the collection $\mathscr{B}$ more balls $B(a, r)$ of smaller and smaller radius r so that the corresponding domains $U(a, r)$ are mutually disjoint and such that the set

$$E = \partial H^3 \setminus \bigcup_{B \in \mathscr{B}} B$$

has no interior points.

Finally, for every ball B, $B \in \mathscr{B}$, let g_B denote the loxodromic transformation obtained by a reflection in ∂B followed by the reflection I in the plane $x_2 = 0$, and let

$$G = \langle g_B : B \in \mathscr{B} \rangle.$$

Then G is an infinitely generated Schottky group which acts discontinuously on H^3, and the domain

$$D = H^3 \setminus \bigcup_{B \in \mathscr{B}} (\overline{B} \cup \overline{B}^*)$$

is a fundamental domain for the action of G on H^3.

3.3. *The map $f : H^3 \to \mathsf{R}^3$.* Let φ be a homeomorphism of $\overline{D} \cap H^3$ onto H^3, which is the identity map in $H^3 \setminus \cup(U \cup U^*)$, and which in $U \setminus \overline{B}$ and in $U^* \setminus \overline{B}^*$, $B \in \mathscr{B}$, satisfies the symmetry condition $I \circ \varphi = \varphi \circ I$, and such that $\varphi | U \setminus \overline{B}$ is conjugate by a Möbius transformation to the winding map, which is given in cylinder coordinates by

$$g_\alpha : (r, \vartheta, x_3) \to \left(r, \frac{\pi \vartheta}{2\alpha}, x_3 \right).$$

One can get the conjugation as follows: Fix $B = B(a, r)$, and choose a point b in the circle $C = \partial B \cap \partial H^3$ and let T be a Möbius transformation with $T(b) = \infty$, which maps the circle C onto the x_3 axis, and maps $\partial B \cap H^3$ onto the half plane H^2. Then $T^{-1} \circ g_\alpha \circ T$ maps $U \setminus \overline{B}$ quasiconformally onto U, it is the identity on ∂U and its dilatation depends only on α. For $x \in U^* \setminus \overline{B}^*$, we let $\varphi(x) = I \circ \varphi \circ I$. Then the symmetry condition is satisfied, and since the angle between ∂U and ∂B and between ∂U^* and ∂B^* is the same for all B, the map φ is quasiconformal in D.

Now, let ψ be the winding map which maps $H3$ onto $\mathsf{R}^3 \setminus H^2$, and which is given in cylinder coordinates by $(r, \vartheta, x_1) \to (r, 2\varphi, x_1)$, and let $f = \psi \circ \varphi$. Then f maps D quasiconformally onto $\mathsf{R}^3 \setminus H^2$, and it has a continuous extension on $\overline{D} \cap H^3$. In view of the symmetry of φ, f agrees with the action of G on ∂D.

Finally, extend f to H^3 by the element of G by letting $f(x) = f(g(x))$ if $g(x) \in D$. Then f is quasiregular in H^3, it maps H^3 onto $\mathsf{R}^3 \setminus E$ where

$$E = \partial H^3 \setminus \bigcup_{B \in \mathscr{B}} B$$

and it is locally injective and G-invariant.

Also, the map f has no radial limit along any vertical line l_x which ends at a fixed point x of a generator g of G. Indeed, suppose that g is obtained by a reflection in ∂B followed by the reflection I in $x_2 = 0$. For $k = 1, 2, \ldots$, let $l_{x,k} = l_x \cap g^{-k}(D)$, where g^{-k} denotes the k-th iteration of g^{-1}. Then $g^k(l_{x,k})$ is a circular arc joining ∂B and ∂B^* and

$$d(f(l_{x,k})) = d(f(g^k(l_{x,k}))) > \operatorname{dist}(B, B^*).$$

Therefore f has no limit along l_x, and hence at any point in the orbit of x. The orbit of the fixed points of the generators of G are dense in ∂H^3, and thus f has no radial limit on a dense set in ∂H^3.

4. Quasiconformal groups and radial limits

The purpose in this section is to show the existence of locally injective bounded quasiregular mappings in H^n, $n \geq 3$, which do not have a radial limit on a set E whose Hausdorff dimension is arbitrarily close to $n-1$. Each of the maps will be invariant under a certain finitely generated Schottky type quasiconformal group, which acts on H^n.

4.1. Theorem. *Given $\varepsilon > 0$ and $n \geq 3$, there exists a bounded quasiregular mapping $f : H^n \to \mathsf{R}^n$, which is locally injective and has no radial limits on a set of Hausdorff dimension $> n - 1 - \varepsilon$.*

Proof. We will present the proof for $n = 3$. Almost the same proof applies in all dimensions $n > 3$.

The proof is carried out in two steps. We first construct a quasiconformal group G, which acts discontinuously on H^3, whose limit set Λ has Hausdorff dimension $> 2 - \varepsilon$. In the next step we construct a bounded automorphic mapping $f : H^3 \to \mathsf{R}^3$ which has no radial limits at Λ.

4.2. *The construction of the group G.* Fix $\delta > 0$ and a positive integer m, and consider the $(2m)^2$ closed cubes Q^1_{ij}, $-m \leq i \leq m$, $i \neq 0$, $j = 1, \ldots, 2m$, in R^3, which are of side length $2 - 2\delta$, are parallel to the coordinate axes and are centered as follows. For $i = 1, \ldots, m$ and $j = 1, \ldots, 2m$, Q^1_{ij} is centered at the point $(2i-1, 2j, 0)$, and Q^1_{-ij} is centered at $(1-2i, 2j, 0)$, see Figure 3.

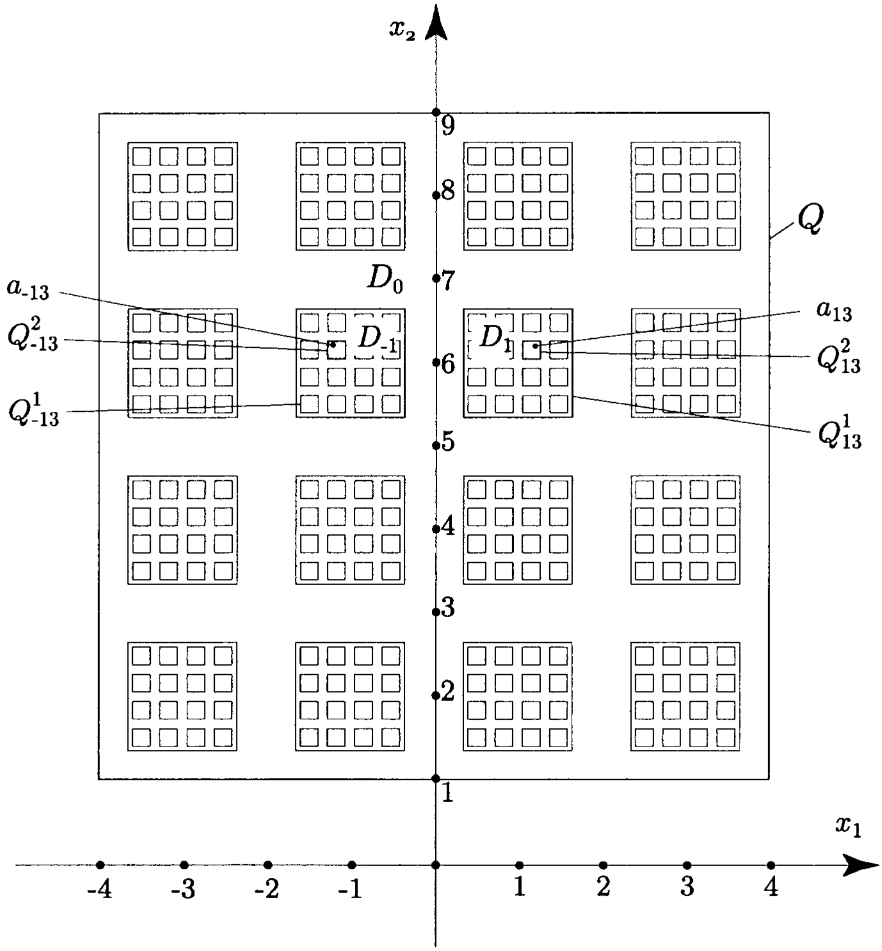

Figure 3. Generations in 4.2.

The cubes Q^1_{ij} are of the first generation. The cubes of the second generation are obtained from those of the first generation by similarities. They will all be centered in $\mathsf{R}^2 = \partial H^3$, and will be of side length $(1-\delta)^2/m$. More specifically, let Q be the closed cube of side length $4m$, which is parallel to the coordinate axes, and which is centered at the point $(0, 2m+1, 0)$. For each pair (i,j) let a_{ij} (see Figure 3) be the center of the similarity $T_{ij} : \mathsf{R}^3 \to \mathsf{R}^3$,

$$T_{ij}(x) = a_{ij} + (x - a_{ij})(1-\delta)/2m$$

which maps Q onto the cube $(1-\delta)^{-1}Q^1_{ij}$. Then the cubes of the second generation are $T_{ij}(Q^1_{\alpha\beta})$, $-m \le \alpha \le m$, $\beta = 1, \dots, 2m$, $-m \le i \le m$,

$j = 1, \ldots, 2m$. A repeated application of each of the $4m^2$ similarities T_{ij} gives all cubes of higher generations.

For each i and j we construct now a quasiconformal map $g_{ij} : \overline{\mathsf{R}}^3 \to \overline{\mathsf{R}}^3$, which acts on H^3, which maps $\overline{\mathsf{R}}^3 \setminus Q^1_{-ij}$ onto int Q^1_{ij}, and which has two fixed points: one at a_{ij} and the other one at a_{-ij}. In Q^1_{ij}, the map g_{ij} is defined by

$$g_{ij}(x) = T_{ij}(x) = a_{ij} + (x - a_{ij})(1 - \delta)/2m,$$

and in $Q^2_{-ij} = T_{-ij}(Q^1_{-ij}),\ g_{ij}$ is defined by

$$g_{ij}(x) = T^{-1}_{-ij}(x) = a_{-ij} + 2m(x - a_{-ij})/(1 - \delta).$$

Clearly, $g_{ij}(Q^2_{-ij}) = Q^1_{-ij}$ and $g_{ij}(Q^1_{ij}) = Q^2_{ij}$.

We now define g_{ij} in $D_{-1} = \text{int } Q^1_{-ij} \setminus Q^2_{-ij}$ and in $D_0 = \mathsf{R}^3 \setminus (Q^1_{-ij} \cup Q^1_{ij})$ so that $g_{ij}(D_{-1}) = D_0$ and $D_1 = g_{ij}(D_0) = \text{int } Q^1_{ij} \setminus Q^2_{ij}$. The regions D_0 and D_1 depend, of course, on i and j.

Let $I : \mathsf{R}^3 \to \mathsf{R}^3$ denote the reflection with respect to the plane $x_1 = 0$. Then $I(Q^1_{-i,j}) = Q^1_{i,j}$. We require that $g_{ij}(x) = I(x)$ for all $x \in \partial Q^1_{-ij}$, and that $g_{ij}(x) = T_{ij}(x)$ for

$$x \in \hat{Q}^1_{ij} = \cup\{Q^1_{\alpha\beta} : (\alpha, \beta) \neq (-i,j)\}.$$

Then $g_{ij}(\partial Q^1_{-ij}) = \partial Q^1_{ij}$ and $g_{ij}(Q^1_{\alpha\beta}) = T_{ij}(Q^1_{\alpha\beta})$ for $(\alpha, \beta) \neq (-i,j)$. This defines g_{ij} on ∂D_0 and on cubes of the first generation which are in D_0. Then g_{ij} is extended quasiconformally over to the rest of D_0.

We now define g_{ij} in $D_{-1} = \text{int } Q^1_{-ij} \setminus Q^2_{-ij}$ by letting

$$g_{ij}(x) = I(g^{-1}_{ij}(I(x)),\ x \in D_{-1}.$$

Then g_{ij} has the following properties

(i) g_{ij} is a sense preserving homeomorphism of $\overline{\mathsf{R}}^3$ onto itself which preserves H^3

(ii) g_{ij} is conformal in $Q^1_{ij} \cup Q^2_{-ij} \cup \hat{Q}^1_{ij}$ and quasiconformal elsewhere

(iii) $g_{ij}(x) = I(x)$ for all points x in ∂Q_{-ij}.

We now let G denote the group generated by the $2m^2$ transformations $g_{ij},\ i = 1, \ldots, m,\ j = 1, \ldots, 2m$. Then G is a Schottky type group which acts discontinuously on H^3. The domain $D = H^3 \setminus \cup Q^1_{ij}$ is a fundamental domain for the action of G on H^3. Two points on $H^3 \cap \partial D$ are equivalent under G iff they are symmetric with respect to the plane $x_1 = 0$. The limit set Λ of G is the intersection of all cubes which are obtained by applying the elements of G to the closed cubes Q^1_{ij} of the first generation. Each of these cubes belongs to a certain generation according to its size, so that a cube in the k-th generation is of side length $(2-2l)(1-\delta)^{k-1}(2m)^{1-k}$. Such a cube contains $(2m)^2-1$ cubes of the $(k+1)$-th generation.

4.3. *The construction of the map* $f : H^3 \to \mathsf{R}^3$. We first construct a special quasiconformal map φ_1 of the fundamental domain D onto H^3. For i and j let U_{ij} denote the set of points in D which are at a distance $< \delta$ from Q^1_{ij}. Let $\varphi_1(x) = x$ for all $x \in \overline{D} \setminus \cup U_{ij}$, and extend φ_1 homeomorphically to each $\overline{U}_{ij}$ in such a way that φ_1 sends $\partial Q^1_{ij} \cap \overline{H}^3$ onto $Q^1_{ij} \cap \{x_3 = 0\}$ and φ_1 is quasiconformal in U_{ij}. Clearly φ_1 can be organized in such a way that it preserves symmetry in the plane $x_2 = 0$. Then $\varphi_1 : \overline{D} \to H^3$ is a homeomorphism which is quasiconformal in D.

Now, let $Q(t),\ t > 0$, denote the cube

$$Q(t) = \{x \in \mathsf{R}^3 : |x_i| \le t,\ i = 1,2,3\}.$$

It is easy to see that there is a quasiconformal map φ_2 of $\overline{\mathsf{R}}^3$ such that $\varphi_2(x) = x$ for $x \in Q(4m)$ and $\varphi_2(H^3) \subset Q(5m)$; a map φ_2 can be constructed distorting a Möbius transformation sending H^3 onto a ball. This extra map φ_2 is added in order to make the final map f bounded.

Next let $\varphi_3 : \overline{H}^3 \to \overline{\mathsf{R}}^3$ be the winding map which is given in cylinder coordinates by

$$\varphi_3(r, \vartheta, x_2) = (r, 2\vartheta, x_2).$$

Here $x_1 = r\cos\vartheta$ and $x_3 = r\sin\vartheta,\ 0 \le \vartheta \le \pi$. Then $\varphi_3 \circ \varphi_2 \circ \varphi_1(D) \subset Q(5m)$.

Now $f = \varphi_3 \circ \varphi_2 \circ \varphi_1$ is bounded and quasiconformal in D, it is continuous in $\overline{D}$ and agrees on point which are symmetric and hence equivalent under G.

We now extend f by the elements of G by letting $f(x) = f(g(x))$ if $g(x) \in D,\ g \in G$. Thus f is bounded; it is invariant under G, and since G is quasiconformal and $f|D$ is quasiconformal, f is quasiregular. Clearly, f is locally injective, and has no radial limit at any point in Λ. By choosing m sufficiently large $\dim \Lambda > 2 - \varepsilon$. The details of this computation will be brought at the end of Section 5, where we compare $\dim \Lambda$ to the multiplicity functions $N(r,f)$. The proof is complete.

5. Locally injective bounded quasiregular maps and radial limits

For $C > 0$ and $s \ge 0$, let $F(C,s)$ denote the class of all quasiregular maps $f : B^n \to B^n$ such that for $0 < r < 1$,

$$N(r) \le C(1-r)^{-s}$$

where $N(r) = \sup\{\# f^{-1}(y) \cap B^n(r) : y \in B^n\}$. It has been shown in [5] that if $f \in F(C,s)$ for some $C > 0$ and $0 \le s < n-1$, then $m_{n-1}(E) = 0$ where

$E = E(f)$ is the set in ∂B, where f does not have radial limits. We now show that if, in addition, f is locally injective in Bn, and $n \geq 3$, then $\dim E \leq s$. Here $\dim E$ denotes the Hausdorff dimension of E. With the aid of the examples in Section 4, we will show that the result is sharp for infinitely many values of s, $s < n - 1$, which accumulate at $n - 1$.

5.1. Theorem. *Let $f : B^n \to B^n$, $n \geq 3$, be a locally injective map in the class $F(C, s)$ for some $C > 0$ and $0 < s < n - 1$. Then $\dim E \leq s$. Furthermore, there are sequences $C_m > 0$, and s_m, $m = 1, 2, \ldots$, with $s_m < n - 1$ and $s_m \to n - 1$ as $m \to \infty$, and a sequence of locally injective quasiregular maps $f_m : B^n \to B^n$ such that $f_m \in F(C_m, s_m)$ and $\dim E(f_m) = s_m$.*

5.2. Remark. In Theorem 5.1 we assume that f is a mapping into B^n, i.e. that f is bounded. This assumption can be replaced by a weaker assumption, that f omits a point in R^n. Indeed, if $f : B^n \to \mathsf{R}^n \setminus \{y\}$, $n \geq 3$, is a locally injective quasiregular map, then f satisfies a growth estimate $|f(x)| \leq C(1 - |x|)^{-b}$ for some $C < \infty$ and $b > 0$, [17, Theorem 11.27]. By composing f with a suitable quasiconformal mapping this estimate can be used as in [4, p. 765] to obtain an upper bound for $M(f\tilde{\Gamma}_k)$ which is weaker than in (5.9) below. However, this suffices for the conclusion of the theorem. The authors thank the referee for this observation.

Proof. For $y \in S^{n-1} = \partial B^n$, $r > 0$ and $p > 0$ let, $S(y, r) = \{x \in S^{n-1}; q(x, y) < r\}$, where q is the spherical distance function on S^{n-1}, $pS(y, r) = S(y, pr)$, and $S(r) = S(e, r)$, where $e = (0, \ldots, 0, 1) \in \mathsf{R}^n$. Then

$$d(S(r)) < 2r \text{ and } m_{n-1}(S(r)) = \lambda r^{n-1}$$

for some constant $\lambda = \lambda(n)$, where d stands for the diameter in R^n. Finally, for $y \in S^{n-1}$ and $k = 1, 2, \ldots$, let γ_y denote the path $\gamma_y(t) = ty$, $0 \leq t \leq 1$, and $\gamma_{y,k} = \gamma_y|[1 - 2^{-k}, 1 - 2^{-k-1}]$, $\gamma = \gamma_e$ and $\gamma_k = \gamma_{e,k}$. The following standard auxiliary lemma is needed; its proof follows from Theorem 2.3 and from the fact that quasiconformal maps are locally quasisymmetric see [16, 2.4].

5.3. Lemma. *There exists a constant $\delta_0 = \delta_0(n, K) > 0$ such that $d(f(\gamma_{y,k})) \geq \delta_0 r_0$ for all $y \in S(2^{-k}\delta_0)$, whenever $f : B^n \to B^n$, $n \geq 3$, is locally injective and K-quasiregular, and $d(f(\gamma_k)) \geq r_0$.*

The proof for the theorem can now be completed as follows. We will show that given $t \in (s, n - 1)$, there is a set E_0 in S^{n-1}, which depends on t, such that $E \subset E_0$ and such that the t-Hausdorff measure $\mathscr{H}^t(E_0) = 0$. This will imply that $\mathscr{H}^t(E) = 0$ for all $t \in (s, n - 1)$, and, hence, that $\dim E \leq s$.

Given $t \in (s, n - 1)$ choose $\alpha \in (0, 1)$ such that

$$\beta = \alpha^{-n} 2^{s-t} < 1. \tag{5.4}$$

For $k = 1, 2, \ldots$ let

$$\begin{aligned} A_k &= \{y \in S^{n-1} : d(f(\gamma_{y,k}) \geq \alpha^k\}, \\ \tilde{A}_k &= \{y \in S^{n-1} : d(f(\gamma_{y,k})) \geq \delta_0 \alpha^k\}, \end{aligned} \tag{5.5}$$

$$E_k = \bigcup_{j=k}^{\infty} A_j \quad \text{and} \quad E_0 = \bigcap_{k=1}^{\infty} E_k, \tag{5.6}$$

where δ_0 is the constant in Lemma 5.3. Then $E_1 \supset E_2 \supset \cdots$, and $E(f) \subset E_0$. Indeed, if $x \in S^{n-1} \setminus E_0$, then $x \in S^{n-1} \setminus E_k$ for some k, and hence, $d(f(\gamma_{x,i})) < \alpha^i$ for all i, $i \geq k$, and since $0 < \alpha < 1$, this implies that $f|\gamma_x$ satisfies the Cauchy condition, and hence that $\lim_{t \to 1} f(tx)$ exists.

For $k = 1, 2, \ldots$, let $\tilde{\Gamma}_k$ denote the family of all paths $\gamma_{y,k}$, $y \in \tilde{A}_k$, and $f\tilde{\Gamma}_k$ the family of all paths $f \circ \gamma_{y,k}$, $\gamma_{y,k} \in \tilde{\Gamma}_k$. Then by the K_0-modulus inequality, see [13],

$$M(\tilde{\Gamma}_k) \leq K_0(f) N(1 - 2^{-k-1}) M(f\tilde{\Gamma}_k), \tag{5.7}$$

where M stands for the n-modulus.

Now, $f \in F(C, s)$, therefore

$$N(1 - 2^{-k-1}) \leq C 2^{s(k+1)}. \tag{5.8}$$

As for $M(f\tilde{\Gamma}_k)$, note that since $d(f(\gamma_{y,k})) > \delta_0 \alpha^k$ for $y \in \tilde{A}_k$ the function

$$\rho(x) = \begin{cases} \delta_0^{-1} \alpha^{-k}, & x \in B^n \\ 0, & x \in \mathsf{R}^n \setminus \overline{B}^n \end{cases}$$

is admissible for $f\tilde{\Gamma}$, and hence

$$M(f\tilde{\Gamma}_k) \leq \Omega_n \delta_0^{-n} \alpha^{-nk}, \tag{5.9}$$

where $\Omega_n = m_n(B^n)$. As for $M(\tilde{\Gamma}_k)$, we have

$$M(\tilde{\Gamma}_k) = m_{n-1}(\tilde{A}_k) \left(\log \frac{1 - 2^{-k-1}}{1 - 2^{-k}} \right)^{1-n} \geq m_{n-1}(\tilde{A}_k) 2^{(n-1)k}, \tag{5.10}$$

where the last inequality follows from $\log x \leq x - 1$.

Then (5.7)–(5.10) imply

$$m_{n-1}(\tilde{A}_k) \leq c 2^{sk} \cdot 2^{-(n-1)k} \alpha^{-nk}, \tag{5.11}$$

where c stands, here as well as in the sequel, for a constant which depends on n and s.

Next, by Lemma 5.3, $S(y, \delta_0 2^{-k}) \subset \tilde{A}_k$, whenever $y \in A_k$. Now by Besicovitch Covering Theorem, for every $k = 1, 2, \ldots$ there are finitely many points $y_{k,1}, y_{k,2}, \ldots, y_{k,p_k}$ in A_k such that the p_k spherical balls $S(y_{k,i}, \delta_0 2^{-k})$, $i = 1, \ldots, p_k$, are disjoint and that

$$A_k \subset \bigcup_{i=1}^{p_k} 10S(y_{k,i}, \delta_0 2^{-k}) \tag{5.12}$$

where $\lambda S(y, r)$ means $S(y, \lambda r)$; note that $S(y, 2r) \supset B^n(y, r) \cap \partial B^n(0, 1)$ for any $y \in \partial B^n(0, 1)$ and that we have used the Covering Theorem for euclidean balls. Then

$$p_k \lambda(\delta_0 2^{-k})^{n-1} \leq \sum_{i=1}^{p_k} m_{n-1} S(y_{k,i}, \delta_0 2^{-k}) \leq m_{n-1}(\tilde{A}_k),$$

and hence, in view of (5.11)

$$p_k \leq c 2^{sk} \alpha^{-nk}. \tag{5.13}$$

We now compute $\mathscr{H}^t(E_0)$. Let $\delta > 0$. Choose k such that $20\delta_0 2^{-k} < \delta$. Then

$$d(10S(y_{j,i}, \delta_0 2^{-j})) < \delta, \; i = 1, \ldots, p_j, \; j \geq k. \tag{5.14}$$

Also

$$E_0 = \bigcap_{i=1}^{\infty} E_i \subset E_k = \bigcup_{j=k}^{\infty} A_j.$$

Therefore, by (5.12)

$$\mathscr{H}^t_\delta(E_0) \leq \mathscr{H}^t_\delta(E_k) \leq \sum_{j=k}^{\infty} \sum_{i=1}^{p_j} d(10S(y_{j,i}, \delta_0 2^{-j}))^t \leq \sum_{j=k}^{\infty} p_j (20\delta 2^{-j})^t.$$

By (5.13) and (5.4)

$$\begin{aligned} \mathscr{H}^t_\delta(E_0) &\leq c \sum_{j=k}^{\infty} (2^s \alpha^{-n})^j 2^{-jt} = c \sum_{j=k}^{\infty} (\alpha^{-n} 2^{-(t-s)})^j \\ &= c \sum_{j=k}^{\infty} \beta^j = c \frac{\beta^k}{1-\beta}. \end{aligned}$$

Hence $\mathscr{H}^t(E_0) = \lim_{\delta \to 0} \mathscr{H}^t_\delta(E_0) = 0$. Consequently $\mathscr{H}^t(E) = 0$, and therefore $\dim E \leq s$.

5.14. *Sharpness.* In order to show that the upper bound for the Hausdorff dimension of the singular set, which is presented in Theorem 5.1 is best possible, we will construct quasiregular maps $F_m : B^n \to B^n$, $n \geq 3$, $m = 1, 2, \ldots$, such that

$$N(r, F_m) \leq \frac{C_m}{(1-r)^{s_m}}$$

and $\dim E(F_m) = s_m$, for some sequence s_m, which tends monotonically to $n-1$, and for some constants C_m.

Fix $\delta \in (0,1)$, integers $m \geq 1$ and $n \geq 3$ and a Möbius transformation T, such that $T(B^n) = H^n$, $T(0) = (0, 2m+1, \ldots, 2m+1, 1)$, $T(e_1) = 0$ and $T(-e_1) = \infty$, and consider the map $F = F_{m,\delta} = f \circ T$, where $f = f_{m,\delta} : H^n \to B^n$ is the map which is mentioned in Theorem 4.1, and which is described in details in the proof of Theorem 4.2 for $n = 3$. More specifically, f is a locally injective quasiregular map, which is invariant under a Schottky type quasiconformal group G, which acts on H^n.

The group G is generated by $(2m)^{n-1}/2$ transformations

$$g_\alpha, \ \alpha = (j_1, \ldots, j_{n-1}), \ j_1 = 1, \ldots, m, \ j_k = 1, \ldots, 2m, \ k = 2, \ldots, n-1,$$

where g_α is a quasiconformal automorphism of $\overline{R}^n$ which acts on H^n and maps the interior of a closed cube Q_α onto the exterior of a closed cube $Q_{\alpha'}$, $\alpha' = -j_1, j_2 \ldots, j_{n-1}$. Q_α and $Q_{\alpha'}$ have side length $2 - 2\delta$, they are parallel to the coordinate axes and centered in ∂H^n at the point $(2j_1 - 1, 2j_2, \ldots, 2j_{n-1}, 0)$ and $(1 - 2j_1, 2j_2, \ldots, 2j_{n-1}, 0)$, respectively, and, thus, are symmetric with respect to the plane $x_1 = 0$. Furthermore, $g_\alpha | Q_\alpha$ and $g_\alpha^{-1} | Q_{\alpha'}$ are similarity maps. The cubes $Q_\alpha^1 = Q_\alpha$ and $Q_{\alpha'}^1 = Q_{\alpha'}$ are of first generation. The cubes $Q_\alpha^2 = g_\alpha(Q_\alpha^1)$ and $Q_{\alpha'}^2 = g_\alpha^{-1}(Q_{\alpha'}^1)$ are of second generation and are of side length $2(1-\delta)^2/m$. By repeated application of g_α and g_α^{-1} one obtains the cubes of all generations. In particular,

$$Q_\alpha^k = g_\alpha^k(Q_\alpha^1) \text{ and } Q_{\alpha'}^k = g_{\alpha'}^{-k}(Q_{\alpha'}^{-1})$$

are cubes of k-th generation or shortly k-cubes. Here g_α^k and $g_{\alpha'}^{-k}$ denote the k-th iteration of g_α and $g_{\alpha'}^{-1}$, respectively. Each k-cube is of side length $(2 - 2\delta)(2m)^{1-k}$ and contains $(2m)^{n-1} - 1$ $(k+1)$-cubes, see Figure 3. Now the limit set Λ of G, which is the intersection of all cubes, is a self similar fractal set generated by $(2m)^{n-1}$ similarities each of which has $(1-\delta)/2m$ as a scaling factor. A standard computation, see [3, Theorem 8.6], shows that Λ is of Hausdorff dimension

$$\dim \Lambda = s_m \tag{5.15}$$

where

$$s_m = \frac{(n-1)\log(2m)}{\log(2m/(1-\delta))}. \tag{5.16}$$

Note that $s_m \to n-1$ as $m \to \infty$.

The map $f : H^n \to B^n$ is as was described in the proof of Theorem 4.1 for $n = 3$. In particular, f is locally injective, quasiregular and G-invariant, and $E(f) = \Lambda$. Since the Hausdorff dimension is invariant under a Möbius transformation, composing f with a Möbius transformation $T : B^n \to H^n$ yields the map $F = f \circ T$ which satisfies

$$\dim E(F) = \dim E(f) = \dim \Lambda = s_m, \tag{5.17}$$

where s_m is given in (5.16).

We now estimate the multiplicity function $N(r) = N(r, F)$. For $t > 0$, let $H_t = \{x \in H^n : x_n > t\}$ and

$$N_1(t) = N_1(t,f) = \sup\{\# f^{-1}(y) \bigcap H_t : y \in R^n\}.$$

The domain $D = H^n \setminus \cup\, (Q_\alpha \cup Q_{\alpha'})$ is a fundamental domain for the action of f on H^n, and the domains

$$D_\alpha^k = H^n \bigcap \left[\text{int } Q_\alpha^k \setminus \bigcup_\beta Q_\beta^{k+1}\right]$$

together with D are the tiles in the tessellation of H^n.

For $k = 1, 2, \ldots$, let

$$t_k = \frac{(1-\delta)^k}{(2m)^{k-1}}. \tag{5.18}$$

Then for $t \in (t_k, t_{k-1})$, ∂H_t meets all l-cubes, $l = 1, 2, \ldots, k$, but none of the $(k+1)$-cubes, and thus

$$N_1(t) = 1 + (2m)^{n-1} + (2m)^{n-1}[(2m)^{n-1} - 1] + \cdots + (2m)^{n-1}[(2m)^{n-1} - 1]^{k-1},$$

by the injectivity of f on each tile. Hence

$$N_1(t) = 1 + \frac{(2m)^{n-1}}{(2m)^{n-1} - 2}\{[(2m)^{n-1} - 1]^{k-1}\}, \; t_k < t < t_{k-1}. \tag{5.19}$$

Now let

$$r_k = \frac{1 - t_k}{1 + t_k}. \tag{5.20}$$

By considering the image of the circle $|z| = r_k$ under the transformation $w = \frac{1-z}{1+z}$, and recalling (5.18), one can verify that $T(S^{n-1}(r_k))$ meets all l-

cubes, $l = 1, \ldots, k$, but none of the $(k+1)$-cubes. Hence $N(r_k) = N_1(t)$, for $t_k < t < t_{k-1}$ and thus, by (5,19), we have

$$N(r_k) = 1 + \frac{(2m)^{n-1}}{(2m)^{n-1} - 2}\{[(2m)^{n-1} - 1]^{k-1}\} < c[(2m)^{n-1} - 1]^k. \tag{5.21}$$

Now, let $r_k \leq r \leq r_{k+1}$. Then

$$N(r)(1-r)^{s_m} \leq N(r_{k+1})(1-r_k)^{s_m},$$

and by (5.21), (5.20), (5.18) and (5.16), one can show that $N(r)(1-r)^{s_m}$ is bounded for all $0 < r < 1$. Consequently, $F \in F(C, s_m)$ for some constant $C = C(m)$, which together with (5.17) shows that Theorem 5.1 is sharp.

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