CERTAIN C*-ALGEBRAS WITH NON-ZERO REAL RANK AND EXTREMAL RICHNESS

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Abstract

We know that real rank zero is not stable under the spatial C*-algebraic tensor product. Indeed, if H is a separable infinite dimensional Hilbert space, we show that $B(H) \otimes B(H)$ has not real rank zero, although $B(H)$ has real rank zero. We can also construct many interesting C*-algebras with non-zero real rank. To determine their real rank, we investigate the size of one-sided invertible elements $V(A)$ in a C*-algebra A, and we show that $\overline{V(A)} = A$ under some conditions. From this observation we know that these C*-algebras are extremally rich, which is defined by Brown and Pedersen ([3]). In particular, their real rank is at most one.

Introduction

Let A be a unital C*-algebra and A_{sa} be the set of self-adjoint elements. The concept of the real rank of a C*-algebra $A\,(= \mathrm{RR}(A))$ was defined by Brown and Pedersen ([2]) as follows: $\mathrm{RR}(A)$ is the least integer n such that $\{(a_0, a_1, \cdots, a_n) \in A_{\mathrm{sa}}^{n+1} : \sum_{i=0}^{n} A a_i = A\}$ is dense in A_{sa}^{n+1}. If A is non-unital, its real rank is defined by $\mathrm{RR}(\tilde{A})$, where $\tilde{A}$ is the C*-algebra obtained by adding a unit to A.

An important part of the motivation for introducing it is to have an analogue of the dimension concept for C*-algebras. In the commutative case, if X is a compact Hausdorff space, then $\dim X$ can be defined as a property of the algebra $C(X)$ of continuous functions on X. Thus, $\dim X \leq n$ if every continuous function $f : X \to \mathscr{R}^{n+1}$ can be approximated by another continuous function g, for which $0 \notin g(X)$ ([23]). If X is compact Hausdorff space, then $\mathrm{RR}(C(X)) = \dim X$.

We hope that the real rank would have the parallel property of the dimension theory such that $\dim(X \times Y) \leq \dim X + \dim Y$. However, Kodaka and the author ([14] [15]) showed that $\mathrm{RR}(A \otimes B) \leq \mathrm{RR}(A) + \mathrm{RR}(B)$ does not hold even if both C*-algebras are separable, nuclear and have real rank zero, where $\otimes$ means the spatial C*-algebraic tensor product. For example, $E_n \otimes O_n \otimes O_n (n \geq 3)$ does not have real rank zero, although E_n and $O_n \otimes O_n$

have real rank zero ([15]), where O_n is the Cuntz algebra generated by n-elements $S_1, S_2, \cdots, S_n$ such that $S_i^* S_j = \delta_{i,j} (1 \leq i, j \leq n)$ and $\sum_{i=1}^n S_i S_i^* = 1$, and E_n is the C*-subalgebra generated by $S_1, S_2, \cdots, S_n$ of O_{n+1} ([5]). We note that if $A \otimes B$ has not real rank zero, then any C*-algebraic tensor product of A and B has not real rank zero.

The motivation of this paper is to determine the real rank of $E_n \otimes O_n \otimes O_n$.

Before calculating its real rank, we must touch on recent works due to Pedersen ([25]) and Brown and Pedersen ([3] [4]). Pedersen tried to describe the geometry of the unit ball of a C*-algebra from the concept of the λ-property in operator algebras. Here, a C*-algebra A has the λ-property if any element T in the closed unit ball A_1 of A, $\lambda(T) = \sup\{\lambda | T = \lambda V + (1 - \lambda)B; V \in \mathrm{ex}(A_1), B \in A_1, \lambda \in [0, 1]\}$ is positive, where $\mathrm{ex}(A_1)$ means the set of extremal points in A_1. He showed that every von Neumann algebra has the λ-property and that a prime C*-algebra A has the λ-property if and only if the set of one-sided invertible elements $V(A)$ is dense in A. In particular, purely infinite simple C*-algebra(not necessarily unital) has λ-property. Related to the real rank, he proved that the real rank of C*-algebras with the λ-property is at most one. After that, he and Brown succeeded this work and clarified the structure of the gometry of the unit ball of a C*-algebra A with the λ-property, i.e., A has the λ-property if and only if the set of quasi-invertible elements $A^{-1} \mathrm{ex}(A_1) A^{-1}$ is dense in A if and only if for every T in A_1 and $0 < \epsilon < \frac{1}{2}$, there are extremal points U_1, U_2, U_3 in $\mathrm{ex}(A_1)$ such that $T = \frac{1}{2}(1 - \epsilon)U_1 + \frac{1}{2}(1 - \epsilon)U_2 + \epsilon U_3$ ([4] [25]). They say that a unital C*-algebra A is "extremally rich" if A satisfies the λ-property. A non-unital C*-algebra A then is called extremally rich if its unitaization $\tilde{A}$ is extremally rich.

We also note that Rørdam ([30]) investigated the size of $V(A)$ and his method is very useful to prove the extremal richness of some C*-algebras.

In section 1, we present examples of C*-algebras with non-zero real rank. In particular, we show that $B(H) \otimes B(H)$ has not real rank zero. After that we consider an extension E of A by B, and show that E is extremally rich if both A and B are purely infinite simple C*-algebras. From this observation we know that $E_n \otimes O_n \otimes O_n (n \geq 3)$ has real rank one. At the last section we note that $B(H) \otimes B(H)$ is not extremally rich. So, we don't know its real rank yet.

1. C*-algebras with non-zero real rank

As Zhang pointed out in [35] (cf. [2]), even if E is an extension of A by B and both A and B have real rank zero, we can not in general deduce that E has

real rank zero. This obstruction follows from the lifting problem of projections. He proved ([35, Proposition 2.3]) that if $K_1(B) = 0$, then any projection in A can be lifted to a projection in E, and E has real rank zero. From this observation we get the following result which is an extended version of the result in [14].

Note that by $A \otimes B$ we denote the spatial C*-algebraic tensor product of A and B.

PROPOSITION 1.1. *Let A be a unital C*-algebra with non-trivial K_1- group $(K_1(A) \neq 0)$ and let H be a separable infinite dimensional Hilbert space. Then, $\mathrm{RR}(A \otimes B(H)) \neq 0$.*

PROOF. Consider the following C*-exact sequence:

$$0 \longrightarrow A \otimes \mathscr{K} \longrightarrow A \otimes B(H) \overset{\pi}{\longrightarrow} A \otimes B(H)/A \otimes \mathscr{K} \longrightarrow 0,$$

where $\mathscr{K}$ denotes the C*-algebra of all compact operators on H.

Since $K_i(A \otimes B(H)) = 0$ $(i = 1, 2)$ ([33, Theorem 10.2]) we have

$$
\begin{array}{ccccc}
K_0(A) & \longrightarrow & 0 & \overset{\pi_*}{\longrightarrow} & K_0(A \otimes B(H)/A \otimes \mathscr{K}) \\
\uparrow & & & & \downarrow \\
K_1(A \otimes B(H)/A \otimes \mathscr{K}) & \longleftarrow & 0 & \longleftarrow & K_1(A)
\end{array}
$$

by the sixterm exact sequence ([33, Theorem 9.3.2]).

We know that π_* is not surjective. Hence, there is a projection in $(A \otimes B(H)/A \otimes \mathscr{K}) \otimes \mathscr{K}$ which can not be lifted to a projection in $A \otimes B(H) \otimes \mathscr{K}$. So, $\mathrm{RR}(A \otimes B(H) \otimes \mathscr{K}) \neq 0$, and $\mathrm{RR}(A \otimes B(H)) \neq 0$ by [2, Theorem 3.14] [35, Theorem 3.2].

COROLLARY 1.2. *Let H be a separable infinite dimensional Hilbert space. Then $\mathrm{RR}(B(H) \otimes B(H)) \neq 0$.*

PROOF. Let $Q(H)$ be the Calkin algebra $B(H)/\mathscr{K}$. Since $K_1(Q(H)) \cong Z$, we know that $\mathrm{RR}(Q(H) \otimes B(H)) \neq 0$, hence $\mathrm{RR}(B(H) \otimes B(H)) \neq 0$ from the property of real rank zero to pass to quotients.

REMARK 1.3. (1) We can also construct a separable C*-algebra $A \otimes B$ with non-zero real rank, although both A and B have real rank zero ([15]):

Let A be a unital simple C*-algebra. Suppose that $K_1(A)$ contains a finite group Z_n as a summand. Then, $\mathrm{RR}(A \otimes E_n) \neq 0$ if $n \geq 3$. For examples, take $O_n \otimes O_n$ and $C^*_\gamma(F_k) \otimes O_n$ for A, where $C^*_\gamma(F_k)$ is the reduced group C*-algebra of the free group with k generators.

(2) Let A be a σ-unital purely infinite simple C*-algebra. Then, Zhang ([38, Theorem 2.1]) proved that $M(A)$ has real rank zero if and only if

$K_1(A) = 0$. Moreover, we can easily construct purely infinite simple C*-algebras with non-trivial K_1-group from the fact that $A \otimes O_\infty$ is purely infinite simple C*-algebra if A is simple ([13]), where O_∞ is a C*-algebra generated by countably many orthogonal isometries on some Hilbert space ([5]).

2. Extremal richness and Real rank

In this section, we consider a C*-exact sequence $0 \to I \to A \to A/I \to 0$ in case that I is an essential closed two-sided ideal, and both I and A/I are purely infinite simple C*-algebras. Using the idea in [30] we can conclude that A is extremally rich.

At first we introduce Rørdam's work. Let A be a unital C*-algebra and $\mathrm{ZD}(A)$ be the set of elements in A for which $ax = 0 = xb$ for some non-zero elements a and b in A. In [30] he noted that the closure of $\mathrm{ZD}(A)(= \overline{\mathrm{ZD}(A)})$ is the set of all elements in A that are not one-sided invertible. Therefore, if we could prove that $\mathrm{ZD}(A)$ is contained in the closure of the set of all one-sided invertible elements in $A(= V(A))$, then we know that $V(A)$ is dense in A, and hence, A is extremally rich ([25, Theorem 8.3]). He proved that a purely infinite simple unital C*-algebra A satisfies $\mathrm{ZD}(A) \subseteq \overline{\mathrm{GL}(A)}$, where $\mathrm{GL}(A)$ denotes the set of all invertible elements in A.

Using this concept we get the main result which is an extended version of [30, Theorem 4.5]. The proof of the following theorem is omitted because the proof in [17, Corollary 3.7] is better than our original proof. We note that our result has the priority over [17].

THEOREM 2.1. *Let A be a unital C*-algebra and J be an essential closed two-sided ideal of A. Suppose that J and A/J are purely infinite simple C*-algebras. Then $\mathrm{ZD}(A) \subset \overline{V(A)}$. In particular, A is extremally rich and $\mathrm{RR}(A) \leq 1$.*

REMARK 2.2. Combining Theorem 10.4 of [25] with Theorem 3.7 of [4] we know that every extremally rich C*-algebra has real rank at most one. We note that its direct proof is appeared in Proposition 3.6 of [12].

COROLLARY 2.3. *Let A be a unital, exact, separable, and purely infinite simple C*-algebra. Then $A \otimes B(H)$ is extremally rich and*

$$\mathrm{RR}(A \otimes B(H)) = \begin{cases} 0 \text{ if } K_1(A) = 0 \\ 1 \text{ if } K_1(A) \neq 0. \end{cases}$$

PROOF. Consider the following C*-exact sequence:

$$0 \longrightarrow A \otimes \mathscr{K} \longrightarrow A \otimes B(H) \longrightarrow A \otimes Q(H) \longrightarrow 0.$$

Note that $A \otimes Q(H)$ is a purely infinite simple C*-algebra by [13].

The first assertion comes from the previous theorem.

The second assertion comes from Proposition 1.1, [2, Theorem 3.14], [35, Theorem 3.2], and the previous theorem.

COROLLARY 2.4. *Let A be a σ-unital, purely infinite simple C*-algebra. Then M(A) is extremally rich, and*

$$\mathrm{RR}(M(A)) = 1 \quad \text{if } K_1(A) \neq 0 \text{ and } 1 \notin A$$

PROOF. Consider the following C*-exact sequence:

$$0 \longrightarrow A \longrightarrow M(A) \longrightarrow M(A)/A \longrightarrow 0.$$

Since A is stable ([38, Theorem 1.2]) and $M(A)/A$ is a purely infinite simple C*-algebra ([29, Theorem 3.2][34, Theorem 3.3]) we can get the assertion through the same argument as in Corollary 2.3.

COROLLARY 2.5. *Let n be an integer more than or equal to 2 and A be a unital purely infinite simple C*-algebra. Then $A \otimes E_n$ is extremally rich. Moreover, $\mathrm{RR}(A \otimes E_n) \leq 1$.*

PROOF. Consider the following C*-exact sequence ([8]):

$$0 \longrightarrow A \otimes \mathscr{K} \longrightarrow A \otimes E_n \longrightarrow A \otimes O_n \longrightarrow 0.$$

Since $A \otimes O_n$ is a purely infinite simple C*-algebra ([11] [13] [26] [31]) we get the assertion from Theorem 2.1.

REMARK 2.6. (1) We can not replace a pure infiniteness of J by $\mathrm{RR}(J) = 0$. Indeed, when J is a finite matroid algebra, we know that $M(J)$ is not extremally rich ([3] [17])

(2) In Corollary 2.4, if A has no unit and has trivial K_1-group, Zhang proved ([38, Theorem 2.1]) that $\mathrm{RR}(M(A)) = 0$.

(3) From Corollary 2.5, we know that both C*-algebras $O_n \otimes O_n \otimes E_n$ and $C_\gamma^*(F_k) \otimes O_n \otimes E_n$ have real rank one if $n \geq 3$ (when $n = 2$ they have real rank zero).

3. Non-extremal richness of $B(H) \otimes B(H)$

Since $B(H) \otimes B(H)$ is a prime C*-algebra and $B(H)$ is extremally rich, we might hope that it is extremally rich. However, we know that it is false.

As the same in the case of real rank (see §1) an obstruction to have the extremal richness is a lifting problem of extremal points in the closed unit ball of a quotient C*-algebra of some C*-subalgebra of $B(H) \otimes B(H)$ as follows.

Let B be a closed two-sided ideal $B(H) \otimes \mathscr{K} + \mathscr{K} \otimes B(H)$ of $B(H) \otimes B(H)$, and let D be a C*-algebra $Q(H) \otimes \mathscr{K} \oplus \mathscr{K} \otimes Q(H)$, where

$Q(H)$ is the Calkin algebra $B(H)/\mathcal{K}$. Then, we have the following C*-exact sequence:

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{K} \longrightarrow B \longrightarrow D \longrightarrow 0.$$

Recall that $\tilde{A}$ denotes the unitaization of a C*-algebra A.

LEMMA 3.1. *Let H be a separable infinite dimensional Hilbert space and $\mathcal{K}$ be the C*-algebra generated by compact operators on H. Then, there is an extremal element in the closed unit ball of $\tilde{D}$ which can not be lifted to an extremal element in the closed unit ball of $\tilde{B}$.*

PROOF. Note that we have the following C*-exact sequence:

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{K} \longrightarrow \tilde{B} \longrightarrow \tilde{D} \longrightarrow 0.$$

Since $\tilde{B}$ is prime, any extremal element in the closed unit ball of $\tilde{B}$ is either an isometry or a co-isometry. Therefore, we have only to show that there is an extremal element in the closed unit ball of $\tilde{D}$ that is neither an isometry nor a co-isometry.

Since $Q(H) \otimes \mathcal{K}$ is purely infinite simple, there is a partial isometry $v \in Q(H) \otimes \mathcal{K}$ such that $v^*v = p$ and $vv^* < p$. Similarly, there is a partial isometry $u \in \mathcal{K} \otimes Q(H)$ such that $uu^* = q$ and $u^*u < q$. Set $w = (v, u) + 1 - (p, q)$. Then, $w^*w = 1 - (p, q) + (p, u^*u)$ and $ww^* = 1 - (p, q) + (vv^*, q)$. So, w is a partial isometry, but w is neither an isometry nor a co-isometry. Moreover, $(1 - w^*w)\tilde{D}(1 - ww^*) = \{0\}$. Hence w is an extremal element by [24, 1.4.7], and this is required one.

PROPOSITION 3.2. *Under the same assumption as in Lemma 3.1 $B(H) \otimes B(H)$ is not extremally rich.*

PROOF. From the previous lemma we know that $\tilde{B}$ is not extremally rich ([3, Theorem 6.1]). Hence, since B is a closed two-sided ideal in $B(H) \otimes B(H)$, $B(H) \otimes B(H)$ is not extremally rich by [3, Theorem 3.5].

QUESTION 3.3.
Is real rank of $B(H) \otimes B(H)$ one ?

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