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C^*-ALGEBRAS ASSOCIATED TO NON-HOMOGENEOUS MINIMAL SYSTEMS AND THEIR K-THEORY

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Introduction

It has been proved by Giordano, Putnam and Skau [4, Theorem 2.1] that a Krieger type theorem holds for the family of Cantor minimal systems, i.e. the (topological) orbit structure is related to the isomorphism class of the associated C^*-crossed products. It is natural to investigate whether a similar result is true for a larger class of minimal topological systems. Assuming the Elliott conjecture, namely that a complete isomorphism invariant for the associated (simple) C^*-crossed products is of (ordered) K-theoretic nature, we give a resounding counterexample by considering special non-homogeneous systems. The construction we present also has a purely dynamical aspect, which is of independent interest.

If we restrict attention to the very special family of equicontinuous minimal systems, we show that a Krieger type theorem is true, extending a previous result by Riedel [13].

Finally, we investigate the notion of strong orbit equivalence for minimal topological systems in general. We present a rather surprising example in this connection.

1. Some background from topological dynamics and C^*-algebras

As a general reference on dynamical systems and C^*-crossed products, we refer to the books by Walters [15] and Tomiyama [14].

By a *topological dynamical system*, we mean a pair (X, φ), where X is a compact metric space and $\varphi\colon X \to X$ is a homeomorphism. To avoid trivialities we assume that X is infinite. We will be exclusively interested in the case where φ is *minimal*. That is, if A is a closed subset of X and $TA = A$, then $A =$ or $A = X$. An equivalent formulation is that the *orbit of x under φ*, $\mathrm{orb}_\varphi(x) = \{\varphi^n(x) : n \in \mathsf{Z}\}$, is dense in X for each $x \in X$.

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Two topological dynamical systems (X, φ) and (Y, ψ) are said to be *conjugate* if there exists a homeomorphism $F: X \to Y$ such that $F \circ \varphi = \psi \circ F$. We say that (X, φ) and (Y, ψ) are *flip conjugate* if (X, φ) is conjugate to either (Y, ψ) or to (Y, ψ^{-1}). Considerably weaker than flip conjugacy is the notion of *orbit equivalence*. Two topological dynamical systems are orbit equivalent if there exists a homeomorphism $F: X \to Y$, called an *orbit map*, such that $F(\mathrm{orb}_\varphi(x)) = \mathrm{orb}_\psi(F(x))$ for each $x \in X$. To an orbit map F we associate two functions $m, n: X \to Z$, called the *orbit cocycles associated to F*, such that $F(\varphi(x)) = \psi^{m(x)}(F(x))$ and $F(\varphi^{n(x)}(x)) = \psi(F(x))$. If there exists an orbit map so that the associated orbit cocycles n and m have at most one point of discontinuity each, we say that (X, φ) and (Y, ψ) are *strong orbit equivalent*. (One can show that if n has a point of discontinuity, then m must have a point of discontinuity in the same orbit [4].) This notion is the mildest possible weakening of the notion of flip conjugacy, since Boyle has shown that if one (hence both) of the orbit cocycles is continuous everywhere, then we have flip conjugacy [2].

If (X, φ) is a topological dynamical system, we get an induced $*$-automorphism $U_\varphi: C(X) \to C(X)$ by $U_\varphi f = \varphi(f)(= f \circ \varphi^{-1})$. We say that $f \in C(X)$, where $f \neq 0$, is an *eigenfunction* for φ with *eigenvalue* λ if $U_\varphi f = \lambda f$ for some $\lambda \in S^1$ (the unit circle in C). We say that φ has *topological discrete spectrum* and that (X, φ) is a *discrete dynamical system* if the eigenfunctions of φ span a dense subspace of $C(X)$.

If X is a compact topological group and $a \in X$, then we may define a homeomorphism $\varphi_a: X \to X$ given by $\varphi_a(x) = ax$. Such a homeomorphism is called a *group rotation*, and (X, φ_a) is a topological dynamical system. The group rotation φ_a is minimal iff $\{a^n : n \in \mathsf{Z}\}$ is dense in X. In particular, X must be a monothetic abelian group.

A dynamical system (X, φ) is *equicontinuous* if $\{\varphi^n : n \in \mathsf{Z}\}$ is an equicontinuous family of maps.

We will need the following two well known theorems from topological dynamics [15, Theorem 5.18 and Theorem 5.19].

THEOREM 1. *Two minimal discrete dynamical systems (X, φ) and (Y, ψ) are conjugate iff they have the same eigenvalues.*

THEOREM 2. *Let (X, φ) be a minimal dynamical system. The following are equivalent:*
(i) *(X, φ) is discrete.*
(ii) *(X, φ) is conjugate to a minimal group rotation.*
(iii) *(X, φ) is an equicontinuous dynamical system.*

A minimal topological dynamical system, (X, φ), where X is a Cantor set is called a *Cantor minimal system*. Each Cantor minimal system is conjugate to a *Bratteli-Veršik system* on a simple ordered Bratteli diagram [7, Theorem 4.6].

Recall that any minimal dynamical system (Y, ψ) is a *factor* of a Cantor minimal system (X, φ), that is, there exists a continuous surjection (a *factor map*) $\pi: X \to Y$ such that $\pi \circ \varphi = \psi \circ \pi$. (We say that (X, φ) is an *extension* of (Y, ψ).) The family of Cantor minimal systems is vast–the "universal" property just referred to being an indication of this.

A topological space X is *homogeneous* if for any $x, y \in X$ there is a homeomorphism $F_{xy}: X \to X$ such that $F_{xy}(x) = y$. It is easy to see that a Cantor set is homogeneous. Obviously, homogeneity is an invariant property under homeomorphism. We say that the minimal dynamical system (X, φ) is *non-homogeneous* if X is non-homogeneous.

The C^*-*crossed product* $C(X) \rtimes_\varphi \mathbb{Z}$ is the universal C^*-algebra generated by $C(X)$ and a unitary u such that $u^* f u = \varphi(f) = f \circ \varphi^{-1}$ for all $f \in C(X)$. It is well known that $C(X) \rtimes_\varphi \mathbb{Z}$ is a simple C^*-algebra iff (X, φ) is a minimal dynamical system [14]. If (Y, ψ) is a factor of (X, φ), then there is a natural injection $\hat{\pi}$ of $C(Y) \rtimes_\psi \mathbb{Z}$ into $C(X) \rtimes_\varphi \mathbb{Z}$, where π is the factor map [9, Proposition 7.7.9].

We will study the *K-theoretic invariants* of a minimal topological dynamical system (X, φ), namely those that are derived from the associated simple C^*-crossed product $C(X) \rtimes_\varphi \mathbb{Z}$. They consist of the groups $K_0(C(X) \rtimes_\varphi \mathbb{Z})$ (with order and distinguished order unit) and $K_1(C(X) \rtimes_\varphi \mathbb{Z})$, denoted $K^0(X, \varphi)$ and $K^1(X, \varphi)$ respectively. Furthermore, the natural continuous affine map

$$r_{(X,\varphi)}: \mathrm{Tr}(C(X) \rtimes_\varphi \mathbb{Z})) \to S(K_0(C(X) \rtimes_\varphi \mathbb{Z}))$$

from the normalized traces on $C(X) \rtimes_\varphi \mathbb{Z}$ to the states on $K_0(C(X) \rtimes_\varphi \mathbb{Z})$ is part of the K-theoretic data. (The map $r_{(X,\varphi)}$ is onto by a result of Haagerup [6].) The triple $(K^0(X, \varphi), K^1(X, \varphi), r_{(X,\varphi)})$ is according to *Elliott's conjecture* a complete isomorphism invariant for $C(X) \rtimes_\varphi \mathbb{Z}$. For simplicity, we denote this triple by $K(X, \varphi)$.

Definition 3. The minimal systems (X, φ) and (Y, ψ) have *isomorphic K-theoretic invariants* (written $K(X, \varphi) \cong K(Y, \psi)$) if

(i) $K^1(X, \varphi)$ is isomorphic to $K^1(Y, \psi)$ as (abstract) groups.

(ii) there exists an order isomorphism $\Phi_0: K^0(X, \varphi) \to K^0(Y, \psi)$ preserving distinguished order units.

(iii) there exists an affine homeomorphism $\Phi_\tau: \mathrm{Tr}(C(Y) \rtimes_\psi \mathbb{Z}) \to \mathrm{Tr}(C(X) \rtimes_\varphi \mathbb{Z})$ so that the diagram

$$\begin{array}{ccc} \mathrm{Tr}(C(Y) \rtimes_\psi \mathbb{Z}) & \xrightarrow{\Phi_\tau} & \mathrm{Tr}(C(X) \rtimes_\varphi \mathbb{Z}) \\ {\scriptstyle r_{(Y,\psi)}} \downarrow & & \downarrow {\scriptstyle r_{(X,\varphi)}} \\ S(K^0(Y,\psi)) & \xrightarrow{(\Phi_0)_*} & S(K^0(X,\varphi)) \end{array}$$

commutes, where $(\Phi_0)_*$ is the dual of Φ_0.

REMARK. Clearly, if $r_{(X,\varphi)}$ and $r_{(Y,\psi)}$ are bijections, then a map Φ_τ exists satisfying condition (iii).

2. Main results

THEOREM 4. *Let (Y,ψ) be a Cantor minimal system. There exists a non-homogeneous minimal system (X,φ), which is an extension of (Y,ψ), such that $K(Y,\psi) \cong K(X,\varphi)$, where the isomorphism is induced by the factor map. In particular, $K^0(Y,\psi)$ is a simple dimension group and $K^1(Y,\psi) \cong \mathbb{Z}$.*

REMARK. It is known that for the family $\mathscr{C}$ consisting of Cantor minimal systems a (topological) "Krieger theorem" holds true [4, Theorem 2.1]. Specifically, if (X_1,φ_1) and (X_2,φ_2) are in $\mathscr{C}$, then (X_1,φ_1) is strong orbit equivalent to (X_2,φ_2) iff $C(X_1)\rtimes_{\varphi_1}\mathbb{Z} \cong C(X_2)\rtimes_{\varphi_2}\mathbb{Z}$. This is again equivalent to $K(X_1,\varphi_1) \cong K(X_2,\varphi_2)$. (Recall that the K^0-group of a Cantor minimal system is a simple dimension group and that all simple dimension groups arise in this manner; also that the K^1-group is $\mathbb{Z}$ [7].) Assuming a rather restricted form of Elliott's conjecture to be true, Theorem 4 renders a resounding counterexample for a Krieger theorem to hold for the family $\mathscr{M}$ consisting of all minimal topological systems–in fact the topological spaces themselves are not homeomorphic (in Theorem 4). (Putnam has produced another example of two non-orbit equivalent systems (Z,ψ_1) and (Z,ψ_2) on the same non-Cantor set Z, so that $K^0(Z,\psi_1) \cong K^0(Z,\psi_2)$ [12].) The above theorem also has a purely dynamical aspect, namely the existence of a non-homogeneous minimal extension of a Cantor minimal system, generalizing Floyd's construction, which is described in the next section.

For the special family $\mathscr{E}$ of (minimal) equicontinuous systems we do indeed have a very strong form of a Krieger theorem, which was shown by Riedel [13]. This can be strengthened by invoking the following result, which we shall prove.

PROPOSITION 5. *For the family $\mathscr{E}$ of equicontinuous minimal systems the notions of strong orbit equivalence and conjugacy coincide.*

Combining Proposition 5 with Riedel's result, we get the following theorem.

THEOREM 6. *Let* (X_1, φ_1) *and* (X_2, φ_2) *be in* $\mathscr{E}$. *Then the following are equivalent:*

(i) (X_1, φ_1) *and* (X_2, φ_2) *are strong orbit equivalent.*
(ii) (X_1, φ_1) *is conjugate to* (X_2, φ_2).
(iii) $K(X_1, \varphi_1) \cong K(X_2, \varphi_2)$.
(iv) $C(X_1) \rtimes_{\varphi_1} \mathsf{Z} \cong C(X_2) \rtimes_{\varphi_2} \mathsf{Z}$.

REMARK. In his paper Riedel actually shows that $K^0(X_1, \varphi_1) \cong K^0(X_2, \varphi_2)$ (as ordered groups with distinguished order units) implies that (X_1, φ_1) and (X_2, φ_2) have the same set of eigenvalues, hence are conjugate by Theorem 1. We conjecture that implication (i) $\Rightarrow$ (iv) is true in general, that is for any two minimal systems strong orbit equivalence implies isomorphic crossed products.

For the family consisting of minimal *continuum* systems (X, φ), i.e. X is connected, the notions of orbit equivalence and flip conjugacy coincide. This result is an immediate consequence of an old theorem by Sierpinski, which states that if X is a compact connected space and $X = \cup_{l=-\infty}^{\infty} A_l$ is a countable disjoint union of closed sets A_n, then $X = A_k$ for some k. In our situation, let $A_k = \{x \in X : n(x) = k\}$, where $n: X \to \mathsf{Z}$ is one of the orbit cocycles associated to the orbit map $F: X \to Y$ between (X, φ) and (Y, ψ). Then $X = \cup_{l=-\infty}^{\infty} A_l$ and it is easy to check that each A_l is closed even when n is not continuous. By Sierpinski's theorem, $X = A_k$ for some $k \in \mathsf{Z}$. Since we have orbit equivalence, we must have $k = \pm 1$, which yields flip conjugacy.

A corollary of Sierpinski's result is that if the two minimal systems (X_1, φ_1) and (X_2, φ_2) are orbit equivalent by an orbit map $F: X_1 \to X_2$, then the associated orbit cocycles $m, n: X_1 \to \mathsf{Z}$ are constant on each connected component of X_1. It seems therefore reasonable to conjecture that if n, say, has a discontinuity point $x \in X_1$, then n is discontinuous at each point of the connected component containing x. However, this is surprisingly not true, and an example to that effect is shown in Section 5.

3. The construction of non-homogeneous systems from Cantor systems

To facilitate the understanding of the general construction, we will first consider a special case. This special case was the first known example of a non-homogeneous minimal system. It was constructed by Floyd [3] in 1948 answering a question raised by Gottschalk. We follow Auslander [1, p. 24-27] in our description.

3.1. *Description of a non-homogeneous system which is an extension of a particular odometer system.* Let B denote the rectangle $B = [a, a+h] \times [b, b+k]$. Let $\lambda(B) = B_0 \cup B_1 \cup B_2$ where

$$B_0 = \left[a, a+\frac{h}{5}\right] \times \left[b, b+\frac{k}{2}\right]$$

$$B_1 = \left[a+\frac{2h}{5}, a+\frac{3h}{5}\right] \times [b, b+k]$$

$$B_2 = \left[a+\frac{4h}{5}, a+h\right] \times \left[b+\frac{k}{2}, b+k\right]$$

If K is a disjoint union of rectangles $K = \cup_{i=1}^m B_i$, we may extend λ by letting $\lambda(K) = \cup_{i=1}^m \lambda(B_i)$. We construct our space by induction. Let $B^{(0)} = [0,1]\times[0,1]$ and define $B^{(n+1)} = \lambda(B^{(n)})$. Then $B^{(n)}$ is a disjoint union of 3^n rectangles, and $\{B^{(n)}\}$ is a decreasing sequence of compact sets. Let $X = \cap_{n=0}^\infty B^{(n)}$, then X is a compact metric space. To get a more convenient description of X, let us label the rectangles in a systematic manner. Let $B^{(1)} = \lambda(B^{(0)}) = B_0 \cup B_1 \cup B_2$. Let $B^{(2)} = \lambda(B^{(1)}) = \lambda(B_0) \cup \lambda(B_1) \cup \lambda(B_2) = B_{00} \cup B_{01} \cup B_{02} \cup B_{10} \cup B_{11} \cup B_{12} \cup B_{20} \cup B_{21} \cup B_{22}$, i.e. $B_{i0} \cup B_{i1} \cup B_{i2} = \lambda(B_i)$ for $i = 0,1,2$. More generally, having $B_{a_1a_2\ldots a_n} \in B^{(n)}$ where $a_i \in \{0,1,2\}$ we get $\lambda(B_{a_1a_2\ldots a_n}) = B_{a_1a_2\ldots a_n0} \cup B_{a_1a_2\ldots a_n1} \cup B_{a_1a_2\ldots a_n2}$.

If $x \in X$, then we get a triadic number $a_1a_2\ldots a_n\ldots$ corresponding to the fact that $x \in B_{a_1a_2\ldots a_n}$ for all n. It is not hard to see that if $a_1a_2\ldots a_n\ldots$ is a triadic number such that there is a K such that $a_m = 1$ for all $m \geq K$, then $a_1a_2\ldots a_n\ldots$ corresponds to an interval in X. If this is not the case it corresponds to a single point. In Figure 1, we have drawn the rectangles of $B^{(0)}, B^{(1)}$, and $B^{(2)}$ together with some of the intervals in X. (The intervals in

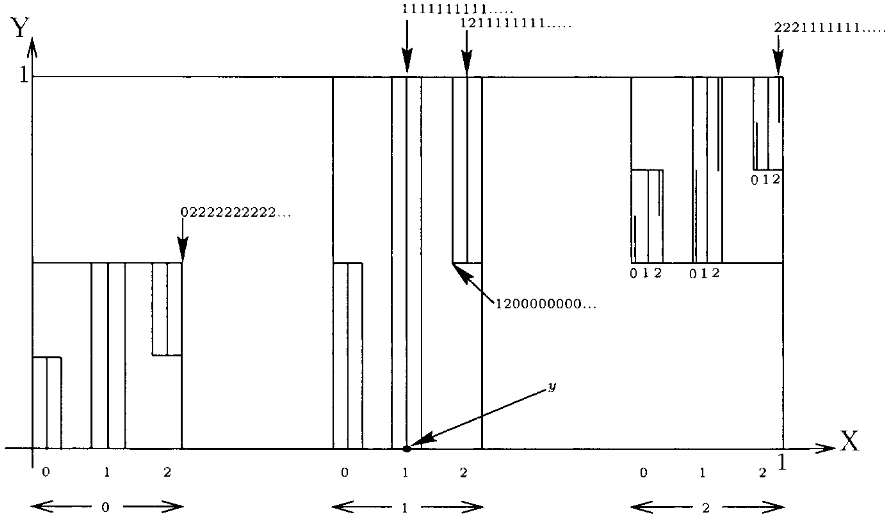

Figure 1

X correspond to the vertical mid-lines of each rectangle. The lower left and the upper right corner of each rectangle will be a point in X.) We have also labeled some of the intervals and points with their corresponding triadic numbers. (The point labeled y will be relevant in Section 5.)

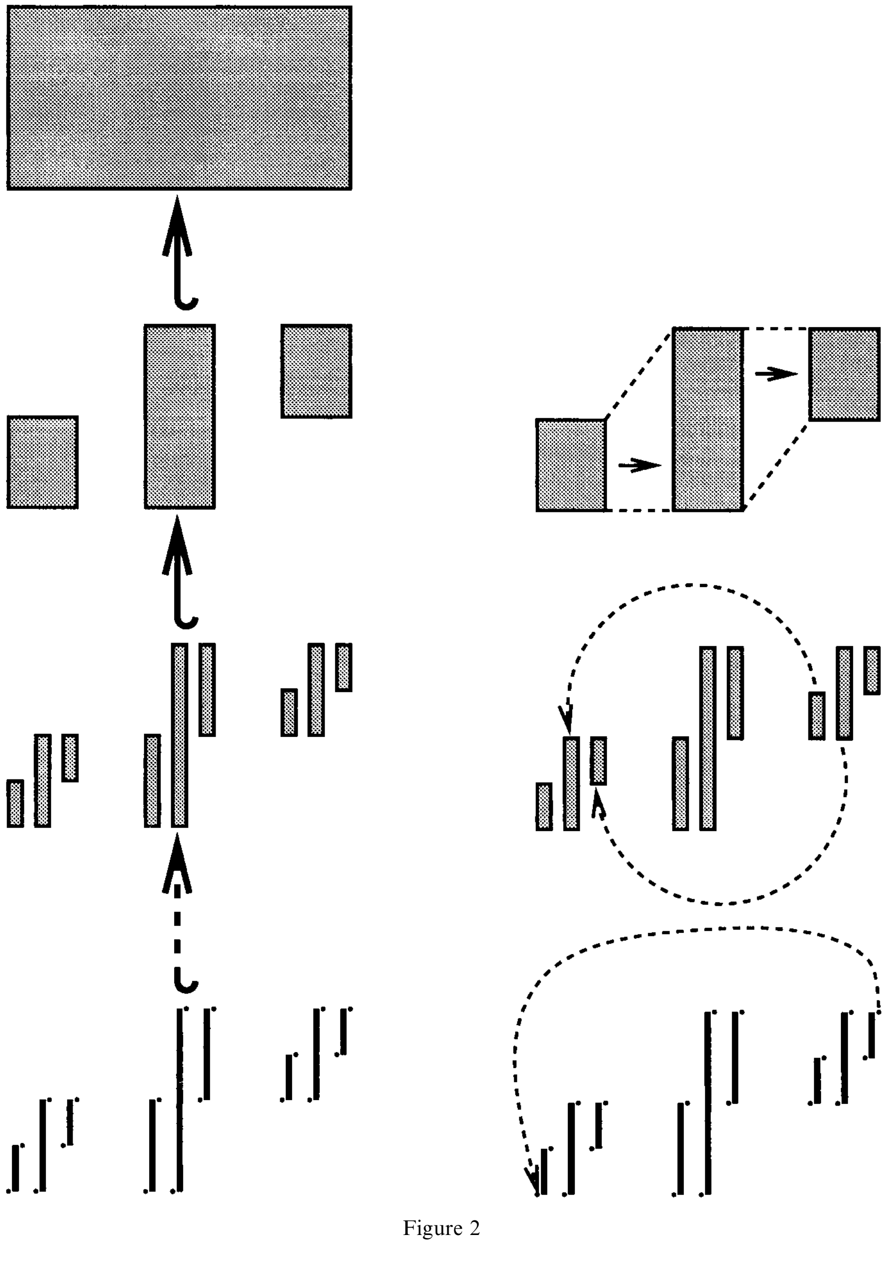

Figure 2

Since some components of X are intervals and some are points, X cannot be homogeneous.

We see that if we project X along the x-axis we get a Cantor set. We will now see that we may define a minimal homeomorphism on this Cantor set which in a natural way induces a minimal homeomorphism on X. Consider the ordered Bratteli diagram $(B, V, \geq)$, where $|V_n| = 1$ for all $n \geq 0$ and $|E_n| = 3$ for all $n \geq 1$. At each level the edges are ordered by $\{0, 1, 2\}$. Together with the Veršik map we get a Cantor minimal system (an odometer system) [7]. This may be associated to X in the following way: Each path in $(V, E, \geq)$ corresponds to a triadic number by reading off the labels of the (ordered) edges. This again corresponds to an interval or a point in X according to the description above. By this correspondence, the Veršik map induces a well-defined homeomorphism $\varphi: X \to X$. (Intervals are taken bijectively to intervals by appropriate scalings.) We will show that φ is minimal when considering the general construction.

In Figure 2 we have illustrated how the mapping can be viewed as coming from permutations of the rectangles of each $B^{(n)}$.

Note that if we have a point $x = (x_1, x_2) \in X$ corresponding to the sequence $a_1 a_2 \ldots a_n \ldots$ (where not all the a_k eventually are 1, since we then have an interval), then it is easy to find its vertical coordinate, x_2. In fact, let $b_1 \ldots b_n \ldots$ be the sequence obtained by first deleting all the 1's in $a_1 a_2 \ldots a_n \ldots$, and then replace all the 2's with 1's. We then have $x_2 = \sum_1^\infty \frac{b_i}{2^i}$.

3.2. *A generalization to arbitrary Cantor minimal systems*. Generalizing our construction from the last section, we will start off with any Cantor minimal system. This can be modeled by a simple OBD and its Veršik map [7]. We will show that by picking any path, p, in the Bratteli compactum which is neither cofinal with the unique maximal path nor with the unique minimal path, we may construct a minimal non-homogeneous topological dynamical system (X, φ), where the paths cofinal to p correspond to closed intervals in X and the rest of the paths correspond to points in X. Furthermore, φ is induced by the Veršik map in a natural way.

Specifically, let us suppose that we have a simple OBD, $(V, E, \geq)$, modeling our Cantor system. Let $(p_1, p_2, p_3, \ldots)$ be the distinguished path that is neither cofinal with the maximal nor the minimal path of the diagram. Let $V_n = \{v_1^n, \ldots, v_{k_n}^n\}$ denote the vertices at level n. By simplicity, we may assume (by contracting the OBD if necessary) that there are at least 3 edges from v_i^n to V_{n+1} for each $n \in \mathsf{N}$ and all $i \in \{1, \ldots, k_n\}$. As in the special case, we will start off with $B^{(0)} = [0, 1] \times [0, 1]$. This rectangle will be subdivided into smaller rectangles which in their turn will be subdivided into smaller rectangles, and so on.

More precisely, assume that we have done the process n times, and we have obtained a finite union of rectangles $B^{(n)}$. Assume that each of these rectangles corresponds to a particular path from $v_0 \in V_0$ to some v_i^n for $i \in \{1, \ldots, k_n\}$. We want to obtain $B^{(n+1)}$ from $B^{(n)}$.

Let A be some rectangle in $B^{(n)}$, and let $(q_1, \ldots, q_n)$ be the path from v_0 to v_i^n corresponding to A. There are two possibilities. Either p_{n+1} is an edge from v_i^n to V_{n+1} or it is not. Assume first that it is not. Let t be the total number of edges from v_i^n to level $n+1$. If $A = [a, a+r] \times [b, b+s]$ then let the rectangles in $B^{(n+1)}$ contained in A be the rectangle $[a, a + \frac{1}{2t-1} r] \times [b, b + \frac{s}{2}]$ together with the $t-1$ rectangles of the form $[a + \frac{l}{2t-1} r, a + \frac{l+1}{2t-1} r] \times [b + \frac{s}{2}, b+s]$ where $l = 2, 4, \ldots, 2t-2$. Identify these rectangles with the paths from v_0 to V_{n+1} of the form $(q_1, \ldots, q_n, q)$ where q is an edge from v_i^n to V_{n+1} in a manner consistent for all paths through v_i^n, but otherwise arbitrary.

Now assume that one of the edges from v_i^n to V_{n+1} is p_{n+1}. As before $t \geq 3$. Let the rectangles in $B^{(n+1)}$ contained in A be described as follows

$$\left[a, a + \frac{1}{2t-1} r\right] \times \left[b, b + \frac{s}{2}\right]$$

$$\left[a + \frac{2}{2t-1} r, a + \frac{3}{2t-1} r\right] \times [b, b+s]$$

$$\left[a + \frac{l}{2t-1} r, a + \frac{l+1}{2t-1} r\right] \times \left[b + \frac{s}{2}, b+s\right]; \; l = 4, 6, \ldots, 2t-2$$

Identify the second rectangle with the path $(q_1, \ldots, q_n, p_{n+1})$. The other rectangles are identified with the rest of the paths of the form $(q_1, \ldots, q_n, q)$ where q is an edge from v_i^n to V_{n+1} in a manner consistent for all paths through v_i^n, but otherwise arbitrary.

To get a better understanding of the process, let us consider the example shown in Figure 3.

It shows how the construction goes for the two first levels of a particular OBD. We have labeled the first two edges of the distinguished path p_1 and p_2. In step 1, we see how $B^{(0)}$ is subdivided. Since there are 5 edges from v_0 to V_1, we construct 5 rectangles in $B^{(0)}$ as described above. We identify the rectangle which is not vertically divided with the edge corresponding to p_1. The other 4 rectangles are identified with the remaining 4 edges in an arbitrary way. All of these rectangles have been obtained by vertically halving $B^{(0)}$. Step 2 shows how the two first rectangles from step 1 are further subdivided in the same fashion. We have assumed that the leftmost rectangle from step 1 is identified with one of the edges from v_0 to v_2^1. Since the set of

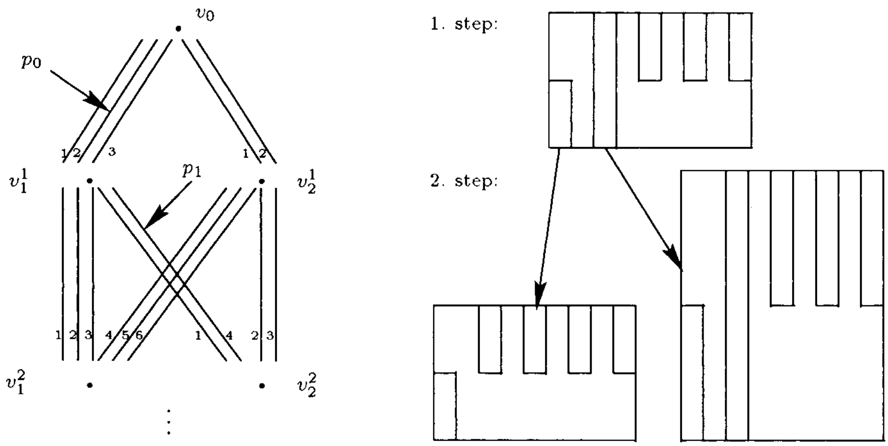

Figure 3. The first two steps of the construction for a concrete OBD.

edges from v_2^1 to V_2 does not contain p_2, all the rectangles inside this rectangle in step 2 are constructed by vertical halving.

It should be clear that by the procedure described we get a sequence of finite unions of rectangles $\{B^{(n)}\}_{n=0}^{\infty}$, where $B^{(n)} \supset B^{(n+1)}$ for each n. Let $X = \bigcap_{n=0}^{\infty} B^{(n)}$. This will, as in the special case, be a space where each component (in the relative topology from $B^{(0)}$) corresponds to a path in $(V, E, \geq)$. Furthermore, all paths not cofinal to $(p_1, p_2, \ldots)$ correspond to points in X, while paths cofinal to $(p_1, p_2, \ldots)$ correspond to intervals.

A homeomorphism $\varphi\colon X \to X$ is induced by the Veršik map associated to $(V, E, \geq)$ by the analogous procedure as exemplified above with the odometer: simply let φ be determined by the underlying Veršik map. (Here lies the reason for requiring that the distinguished path should not be cofinal with the maximal or the minimal path. If it was, one of these paths would correspond to an interval and the other would correspond to a point. But then the induced map could not become a homeomorphism since the Veršik map takes the maximal path to the minimal path.) We get the following proposition.

Proposition 7. *Let (Y, ψ) be a Cantor minimal dynamical system. Carrying out the construction above, we obtain a minimal dynamical system (X, φ), where X is a non-homogenous space, which is an extension of (Y, ψ).*

Proof. The factor map, $\pi\colon (X, \varphi) \to (Y, \psi)$, is the obvious one, namely to a connected component in X, i.e. a point set or a closed interval, we associate the path it corresponds to in the Veršik model for (Y, ψ).

The only thing that remains to show is that φ is minimal. It is enough to show that if l is the unique interval of length 1 in X corresponding to the distinguished path $(p_1, p_2, \ldots)$ and X_0 is a minimal subset of X, then $l \subset X_0$. This is because l is mapped onto all intervals in X, since intervals correspond to cofinal paths and the subset of intervals is dense in X. (The factor map π maps l to p. The orbit of p is dense in the Cantor set. It follows that the closure of the orbit of l must contain at least one point from each component of X. Since we know that it contains all the intervals, it follows that it must be all of X.) We claim that the orbit of every $x \in X$ intersects every rectangle in $B^{(n)}$ for each n. The reason is that the simplicity of the OBD implies that the paths of the elements in the orbit of x "sweeps through" all paths down to level n for each n. But if $x' \in l$ and V is a neighborhood of x', it follows by our construction that there is some rectangle $B_j^{(n)} \subset V$ for some n. If $x \in X_0$, we therefore have $\mathrm{orb}_\varphi(x) \cap V \neq \emptyset$. We conclude that $x' \in X_0$, so $l \subset X_0$ and (X, φ) is minimal.

Remark. It is easily observed that the factor map $\pi : (X, \varphi) \to (Y, \psi)$ induces a bijection between the φ-invariant probability measures on X and the ψ-invariant probability measures on Y.

4. Proofs of the results

Before we start proving the results of Section 2, we will say a few words about the component system of a topological system. Assume that (X, φ) is a minimal topological dynamical system. Let $\sim$ be the equivalence relation on X defined by $x \sim y$ iff x and y lie in the same connected component of X and denote by $[x]$ the equivalence class that x belongs to. Let $\tilde{X} = X/\sim$ be the quotient space. We know that X is a separable metric space. It follows from general topology that $\tilde{X}$ also is separable and metric. Furthermore, $\tilde{X}$ is clearly compact and totally disconnected. Since φ is a homeomorphism it takes components to components, and we therefore get an induced map $\tilde{\varphi}: \tilde{X} \to \tilde{X}$ given by $\tilde{\varphi}([x]) = [\varphi(x)]$. Now $\tilde{\varphi}$ is also a minimal homeomorphism, so we get a new minimal dynamical system $(\tilde{X}, \tilde{\varphi})$, which we call *the component system associated to* (X, φ).

If $\tilde{X}$ has one isolated point, then all its points are isolated by minimality and $\tilde{X}$ must be finite by compactness. In this case, one may easily show that we have the same situation as for continuum systems, i.e. the notions of orbit equivalence and flip conjugacy coincide.

If $\tilde{X}$ has no isolated points, then it is a perfect set (i.e. $\tilde{X}$ is dense in itself) and it follows from general topology that $\tilde{X}$ is uncountable. In this case, $\tilde{X}$ is a compact totally disconnected metric space with no isolated points, so $\tilde{X}$ is a Cantor set. If (X_1, φ) and (X_2, ψ) are strong orbit equivalent, the same holds for $(\tilde{X}_1, \tilde{\varphi})$ and $(\tilde{X}_2, \tilde{\psi})$ (since the orbit cocycles are constant on each connected component by Sierpinski's result referred to above). Let $\tilde{x} \in \tilde{X}_1$ be the point of discontinuity of one of the orbit cocycles. Intuitively, it seems plausible that the corresponding orbit cocycle on X_1 should be discontinuous for every x in the component corresponding to $\tilde{x}$ and that this component therefore should consist of only one point. At the end of this section, we will use our construction to show that this is not the case. More precisely, we will construct two strong orbit equivalent systems on the non-homogeneous system from Section 3.1, where the one discontinuity point of each of the two orbit cocycles lies in one of the interval components.

4.1. *Proof of Theorem 4.* Let (Y, ψ) be a Cantor minimal system and (X, φ) be the minimal non-homogeneous system constructed from (Y, ψ) as described in Section 3.2. We will show that Theorem 4 holds for these systems.

We start by computing $K_0(C(X)), K_1(C(X)), K_0(C(Y))$, and $K_1(C(Y))$. Let $B^{(0)}, B^{(1)}, B^{(2)}, \ldots$ be the sets of finite unions of rectangles from the construction of X. Recall that $X = \cap_{n=0}^{\infty} B^{(n)}$. Let $\tilde{B}^{(0)}, \tilde{B}^{(1)}, \tilde{B}^{(2)}, \ldots$ be the projections of $B^{(0)}, B^{(1)}, B^{(2)}, \ldots$, respectively, to the x-axis. Clearly, $Y = \cap_{n=0}^{\infty} \tilde{B}^{(n)}$. Let $f_{ji}: B^{(j)} \to B^{(i)}$ and $\tilde{f}_{ji}: \tilde{B}^{(j)} \to \tilde{B}^{(i)}$ be inclusion maps when $j \geq i$. Then $\{B^{(j)}, f_{ji}\}$ and $\{\tilde{B}^{(j)}, \tilde{f}_{ji}\}$ are inverse systems of topological spaces and $X \cong \varprojlim B^{(i)}$, $Y \cong \varprojlim \tilde{B}^{(i)}$.

We have a contravariant functor, F, from the category of compact Hausdorff spaces to the category of abelian C^*-algebras: if Z and Z' are compact Hausdorff spaces and if $\alpha: Z \to Z'$ is a continuous map, then $F(Z) = C(Z)$, $F(Z') = C(Z')$, and $F(\alpha): C(Z') \to C(Z)$ is defined by $f \to f \circ \alpha$. It is elementary that this functor turns inverse systems into direct systems so that $F(\varprojlim Z_i) \cong \varinjlim F(Z_i)$. The K-functors preserve direct limits, so $K_0(C(X)) \cong \varinjlim K_0(C(B^{(i)}))$ and $K_0(C(Y)) \cong \varinjlim K_0(C(\tilde{B}^{(i)}))$. Since $B^{(i)}$ is a disjoint union of contractible sets and $\tilde{B}^{(i)}$ is a disjoint union of corresponding contractible sets, by the homotopy invariance of the K-functors we get that $K_0(C(B^{(i)})) \cong K_0(C(\tilde{B}^{(i)})) \cong C(\tilde{B}^{(i)}, \mathsf{Z})$ and $K_1(C(B^{(i)})) \cong K_1(C(\tilde{B}^{(i)})) = 0$ for each i. The induced maps $K_0(C(B^{(i)})) \to K_0(C(B^{(j)}))$ and $K_0(C(\tilde{B}^{(i)})) \to K_0(C(\tilde{B}^{(j)})), j \geq i$ are equal by this identification, and the same holds for K_1. We conclude that $K_0(C(X)) \cong K_0(C(Y)) \cong C(Y, \mathsf{Z})$ and $K_1(C(X)) \cong K_1(C(Y)) = 0$.

Let $\pi: X \to Y$ be the projection map. It is easily verified that the induced map $F(\pi)_*: K_0(C(Y)) \to K(C(X))$ is the map yielding this isomorphism. Furthermore, π induces the order-preserving map

$$\hat{\pi}_*: K_0(C(Y) \rtimes_\psi \mathsf{Z}) \to K_0(C(X) \rtimes_\varphi \mathsf{Z})$$

since $C(Y) \rtimes_\psi \mathsf{Z}$ embeds canonically by $\hat{\pi}$ into $C(X) \rtimes_\varphi \mathsf{Z}$. Obviously, $\hat{\pi}_*$ preserves the canonical order units. Using that $K_1(C(Y)) = K_1(C(X)) = 0$, the Pimsner-Voiculescu six-term exact sequence and the naturality of its construction it is routine to verify that we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
K_1(C(Y)\rtimes_\varphi \mathsf{Z}) & \xrightarrow{\delta_1} & K_0(C(Y)) & \xrightarrow{1-F(\varphi)_*} & K_0(C(Y)) & \xrightarrow{i_{1*}} & K_0(C(Y)\rtimes_\varphi \mathsf{Z}) \\
{\scriptstyle \hat{\pi}_*}\big\downarrow & & {\scriptstyle F(\pi)_*}\big\downarrow & & {\scriptstyle F(\pi)_*}\big\downarrow & & {\scriptstyle \hat{\pi}_*}\big\downarrow \\
K_1(C(X)\rtimes_\varphi \mathsf{Z}) & \xrightarrow{\delta_1} & K_0(C(X)) & \xrightarrow{1-F(\varphi)_*} & K_0(C(X)) & \xrightarrow{i_{1*}} & K_0(C(X)\rtimes_\varphi \mathsf{Z})
\end{array}$$

Here i_{1*} and i_{2*} are surjective (where i_1 and i_2 are the canonical inclusion maps), while δ_1 and δ_2 are injective. Using the five lemma we get that the $\hat{\pi}_*$ are isomorphisms and

$$K^i(X,\varphi) \stackrel{\text{def}}{=} K_i(C(X)\rtimes_\varphi \mathsf{Z}) \cong K_i(C(Y)\rtimes_\psi \mathsf{Z}) \stackrel{\text{def}}{=} K^i(Y,\psi); i=0,1$$

(In fact, since $\ker(1-F(\psi)_*) = \mathsf{Z}$ by minimality of φ, we get that the K^1-groups are isomorphic to Z.)

We have shown that $K^0(X,\varphi)$ and $K^0(Y,\psi)$ are isomorphic as abstract groups. It remains to show that they are order isomorphic.

Let $\tilde{X},\alpha)$ be a Cantor extension of (X,φ) (cf. Section 1). We then have the following commutative diagram

$$\begin{array}{ccc}
\tilde{X} & \longrightarrow & \tilde{X} \\
{\scriptstyle \rho}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\
X & \xrightarrow{\varphi} & X \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
Y & \xrightarrow{\psi} & Y
\end{array}$$

Now $K^0(Y,\psi)$ is order isomorphic to $C(Y,\mathsf{Z})/\partial_\psi C(Y,\mathsf{Z})$ by a map preserving the canonical order units, and $K^0(\tilde{X},\alpha)$ is order isomorphic to $C(\tilde{X},\mathsf{Z})/\partial_\alpha C(\tilde{X},\mathsf{Z})$, where the quotients are given the induced orderings [11, Theorem 4.1]. (Recall that $\partial_\psi f = f - f\circ\psi^{-1}$ and ∂_α is defined similarly.)

Let $\hat{\pi}_*: K^0(Y,\psi)\to K^0(X,\varphi)$ be the map induced by π and let $\hat{\rho}_*: K^0(X,\varphi)\to K^0(\tilde{X},\alpha)$ be the corresponding map induced by ρ. We know that $\hat{\pi}_*$ is a group isomorphism and order preserving. We need to show that if $\hat{\pi}_*(a)\in K^0(X,\varphi)^+$ then $a\in K^0(Y,\psi)^+$. We have that

$$(\hat{\rho}\circ\hat{\pi})_*(a) = \hat{\rho}_*(\hat{\pi}_*(a)) \in K^0(\tilde{X},\alpha)^+$$

Using the order isomorphism of $K^0(Y,\psi)$ and $C(Y,\mathsf{Z})/\partial_\psi C(Y,\mathsf{Z})$, $a\in K^0(Y,\psi)$ is of the form $[g]$ where $[g]$ denotes the equivalence class in $C(Y,\mathsf{Z})/\partial_\psi C(Y,\mathsf{Z})$ of $g\in C(Y,\mathsf{Z})$, and we get $(\widehat{\pi\circ\rho})_*([g])\in K^0(\hat{X},\alpha)^+$. We conclude that $[g] \in K^0(Y,\psi)^+$ by a result of Glasner and Weiss [5, Proposition 3.1]. This finishes the proof of the order isomorphism.

There is a bijective correspondence between ψ-invariant probability measures on Y and normalized traces on $C(Y) \rtimes_\psi \mathbf{Z}$. Similarly, there is a bijection between φ-invariant probability measures on X and normalized traces on $C(X) \rtimes_\varphi \mathbf{Z}$. As mentioned earlier, the ψ-invariant probability measures on Y may be identified with the φ-invariant probability measures on X. Hence $\mathrm{Tr}(C(X) \rtimes_\varphi \mathbf{Z})$ may be identified with $\mathrm{Tr}(C(Y) \rtimes_\psi \mathbf{Z})$. Now $r_{(Y,\psi)}$ is a bijection [11, Corollary 5.7]. The way we identified $K^0(X,\varphi)$ with $K^0(Y,\psi)$ implies that $r_{(X,\varphi)}$ is also a bijection. By the remark to Definition 3, we may conclude that $K(Y,\psi) \cong K(X,\varphi)$.

4.2. *Proof of Proposition 5.* Let (X_1,φ_1) and (X_2,φ_2) be two topological dynamical systems where X_1 and X_2 are compact metric topological groups and where φ_1 and φ_2 are minimal group rotations. We need to show that if these systems are strongly orbit equivalent then they are conjugate. Lemma 8 will show that X_1 (and thus X_2) has to be a Cantor set, and the result follows. (In this case (X_1,φ_1) and (X_2,φ_2) are odometer systems and $K^0(X_1\varphi_1) \cong$
$K^0(X_2,\varphi_2)$ [4, Theorem 2.1]. Hence their corresponding ordered Bratteli diagrams are equivalent and conjugacy follows [7]. In fact, it is enough to assume orbit equivalence for minimal group rotations on Cantor sets to conclude conjugacy because the K^0-groups of such systems have no non-zero infinitesimal elements. Hence the K^0-groups are isomorphic, and they are actually strong orbit equivalent [4, Theorem 2.2].)

Lemma 8. *Let X be an (infinite) compact topological group. The existence of a map $n\colon X \to \mathbf{Z}$ with exactly one point of discontinuity implies that X is a Cantor set.*

Proof. By Sierpinski's theorem, which we have already mentioned, n must be constant on each component of X. For each $x \in X$, we denote the component containing x by A_x. Let a be the point of discontinuity of X and let $y \in A_a$. Then $n(a) = n(y)$ and, moreover, $z = a^{-1}y \in A_e$ where e is the identity of X. Define $T\colon X \to X$ by $x \to xz$. Then T preserves components of X, so $n(T(x)) = n(x)$ for all $x \in X$. Since T is a homeomorphism, it defines a bijection between the neighborhoods of a and the neighborhoods of y. If U_a is a neighborhood of a and $x \in U_a$ then $T(x) \in T(U_a)$. This is a neighborhood of y and it follows since T preserves n that y is also a point of discontinuity for n. Since we assume strong orbit equivalence we conclude that $y = a$. Hence $\{a\}$ is a connected component. Since X is a topological group, we conclude that the components consist of one-point sets, so X is totally disconnected. Thus X is a Cantor set.

REMARK. It is a well-known topological fact that a non-trivial connected metric space must be uncountable. Hence the connected component of the identity, e, of a metric group is either uncountable or is just $\{e\}$. Since all components are homeomorphic, they are either all uncountable or all consist of one point. It follows that Lemma 8 and thereby Proposition 5 also would hold if we instead of strong orbit equivalence demanded orbit equivalence where the cocycles have countably many discontinuities.

We have seen that strong orbit equivalence between minimal rotations on compact metric groups implies conjugacy, and that if the group is a Cantor set, orbit equivalence is enough to conclude conjugacy. It is tempting to conjecture that orbit equivalence implies conjugacy for minimal group rotations on a compact, metric group in general.

Let (X, φ) be a topological dynamical system where X is a compact, metric group, and φ is a minimal rotation. Then the quotient system $(\tilde{X}, \tilde{\varphi})$ is also a minimal group rotation system. If $\tilde{X}$ is not finite, then $\tilde{X}$ has to be a Cantor set. This follows from the arguments given above.

If we assume that (X, φ) and (Y, ψ) are orbit equivalent minimal group rotation systems, then $(\tilde{X}, \tilde{\varphi})$ and $(\tilde{Y}, \tilde{\psi})$ are also orbit equivalent. Hence they are conjugate since $\tilde{X}$, $\tilde{Y}$ are Cantor sets. Our general problem would have been solved if we in some way could lift this conjugacy to a conjugacy between (X, φ) and (Y, ψ). We have not been able to find such a lifting. This means that the question of whether orbit equivalence between minimal group rotations on compact, metric groups implies conjugacy is still open.

4.3. *A remark on Theorem 6*. Riedel [13] showed the equivalence of (ii), (iii) and (iv). We have shown the equivalence of (i) and (ii), so the theorem follows.

5. A surprising example of strong orbit equivalence on a non-homogeneous space

We will use the non-homogeneous system (X, φ) from Section 3.1 to produce an example of strong orbit equivalence, where the discontinuity point of one orbit cocycle lies in a nontrivial component. Incidentally, this example will also show that Lemma 8 does not hold in general for topological dynamical systems. In other words, strong orbit equivalence does not in general imply that X has to be a Cantor set, as we saw was the case for group rotations.

Each point of X is, as mentioned before, assigned a sequence $a_1 a_2 a_3 \dots$ where $a_i \in \{0, 1, 2\}$ for all i. All points in an interval of X correspond to the same sequence cofinal with the sequence of only 1's.

If $x \in X$, let $x_1 x_2 x_3 \dots$ be the corresponding sequence. The sequence corresponding to $\varphi(x)$ is the one we get by adding triadically $z_1z_2z_3\ldots$ to $x_1x_2x_3\ldots$, where $z_1 = 1$ and $z_i = 0, \forall i > 1$. Let

$$A_n = \{x \in X : x_1 = \cdots = x_n = 1; x_{n+1} = \cdots = x_{2n} = 0\}$$

$$A_n^{ij} = \{x \in X : x \in A_n, x_{2n+1} = i, x_{2n+2} = j\}$$

Then $\{A_n^{ij} : i,j \in \{0,1,2\}\}$ is a clopen partition of A_n. Given an element y of A_n^{02} it is elementary that $\varphi^{3^{2n}}(y) \in A_n^{12}$. Furthermore, at position $2n+2$ each $\varphi^i(y)$ where $i \in \{0,1,\ldots,3^{2n}-1\}$ will have a 2 which implies that $\varphi^i(y) \notin A_m$ for any $m > n$. Similarly, if $y \in A_n^{12}$ then $\varphi^{3^{2n}}(y) \in A_n^{22}$ and $\varphi^i(y) \notin A_m$ for every $i \in \{0,1,\ldots,3^{2n}-1\}$ and $m > n$. We define $\psi\colon X \to X$ by

$$\psi(x) = \begin{cases} \varphi^{3^{2n}+1}(x) & \text{if } x \in A_n^{02} \text{ or } x \in A_n^{12} \\ \varphi^{-2\cdot 3^{2n}+1}(x) & \text{if } x \in A_n^{22} \\ \varphi(x) & \text{otherwise} \end{cases}$$

Let $C_n = A_n^{02} \cup A_n^{12} \cup A_n^{22}$ for each $n \in \mathsf{N}$. We see that $\psi(A_n^{02}) = \varphi(A_n^{12}), \psi(A_n^{12}) = \varphi(A_n^{22})$, and $\psi(A_n^{22}) = \varphi(A_n^{02})$ for each n. It follows that $\psi(C_n) = \varphi(C_n)$ for each n. Moreover, it is obvious that $\psi(X \setminus \cup_{n=1}^{\infty} C_n) = \varphi(X \setminus \cup_{n=1}^{\infty} C_n)$. It follows directly that ψ is injective and surjective.

The fact that φ^k is continuous for any $k \in \mathsf{Z}$ implies that ψ is continuous on each C_n and on $X \setminus \cup_{n=1}^{\infty} C_n$. This means that ψ is continuous on each C_n and on all points of $X \setminus \cup_{n=1}^{\infty} C_n$, except possibly the unique point y in Figure 1 for which any neighborhood U_y intersects some C_n. But as $\operatorname{diam}(C_n) \to 0$ and $\psi(C_n) = C_n$ continuity at y also follows. Since X is a compact Hausdorff space, we conclude that ψ is a homeomorphism.

We want to show that ψ is orbit equivalent to φ. Let $x \in X$. It is enough to show that $\varphi(x) \in \operatorname{orb}_\psi(x)$. If $x \in X \setminus \cup_{n=1}^{\infty} C_n$, then by definition $\psi(x) = \varphi(x)$. Assume now that $x \in C_n$ for some $n \in \mathsf{N}$. For simplicity, we may just consider the case $x \in A_n^{02}$ as similar arguments work also for the two other possibilities. We will do an induction on n. Suppose that $\varphi(x) \in \operatorname{orb}_\psi(x)$ whenever $x \in C_m$ and $m \in \{1,2,\ldots,n-1\}$. We will show that $\varphi(x) \in \operatorname{orb}_\psi(x)$ for $x \in A_n^{02}$. We define an equivalence relation on X by $x \sim y$ if and only if $x \in \operatorname{orb}_\psi(y)$. So we need to show that $x \sim \varphi(x)$ for $x \in A_n^{02}$. Now $x \sim \psi(x)$ and $\psi(x) = \varphi^{3^{2n}+1}(x)$. By the induction hypothesis $\varphi^i(x) \sim \varphi^{i+1}(x)$ for $i \in \{3^{2n}+1,\ldots,2\cdot 3^{2n}-1\}$ since as already mentioned $\{\varphi^{3^{2n}+1}(x), \varphi^{3^{2n}+2}(x),\ldots,\varphi^{2\cdot 3^{2n}-1}(x)\} \cap C_m = \emptyset$ for all $m \geq n$. Moreover, $\varphi^{2\cdot 3^{2n}}(x) \sim \psi(\varphi^{2\cdot 3^{2n}}(x))$ and $\psi(\varphi^{2\cdot 3^{2n}}(x)) = \varphi(x)$ by definition. By transitivity, we conclude that $\psi(x) \sim \varphi(x)$ when $x \in A_n^{02}$. Thus we have shown that φ and ψ are orbit equivalent, and it follows automatically that ψ is minimal.

To see that φ and ψ are in fact strongly orbit equivalent let $n\colon X \to \mathsf{Z}$ be the orbit cocycle such that $\psi(x) = \varphi^{n(x)}(x)$ for each $x \in X$. Discontinuity of n

at x is equivalent to n being unbounded in the neighborhood base of x. From the construction, we see that for this to happen each neighborhood U_x of x must intersect C_n for each $n \in \mathsf{N}$. It is easy to see that this only holds for $x = y$ where y is the point marked in Figure 1. So n has exactly one point of discontinuity, and strong orbit equivalence follows.

REMARK. Our example is a bit artificial because the two systems (X, φ) and (X, ψ) are in fact conjugate. To see this we can just pass to the component systems. It follows by construction that $(\tilde{X}, \tilde{\psi})$ is an equicontinuous system since $(\tilde{X}, \tilde{\varphi})$ is. By Proposition 5 they are therefore conjugate. We can choose the conjugacy such that it fixes some point corresponding to an interval. We can then lift the conjugacy to a conjugacy between (X, φ) and (X, ψ). It may be possible to find two homeomorphisms on X which are not conjugate but are strong orbit equivalent with the discontinuity in an interval, but we do not know how to do this.

We may, however, construct two homeomorphisms on X that are strong orbit equivalent, but not flip conjugate. The way to do this is to consider two non flip conjugate homeomorphisms on the component system which are strong orbit equivalent and where the point of discontinuity corresponds to a one-point component. (Such homeomorphisms do exist [4, Remark p. 65].) By lifting these homeomorphisms to X we get strong orbit equivalent systems which are not flip conjugate.

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