CARTAN SUBALGEBRAS OF FINITE VON NEUMANN ALGEBRAS

ALLAN M. SINCLAIR and ROGER R. SMITH

1. Introduction

Sorin Popa [Po 1] [Po 2] has proved several results on the existence and properties of hyperfinite subfactors R of a type II_1 factor M with the relative commutant $R' \cap M$ of R in M equal to $C1$. These theorems have been used in various cohomology calculations [CES, CS, PS, CPSS], as averaging over an amenable subgroup that generates the hyperfinite subfactor is a major step in showing that the continuous and completely bounded Hochschild cohomology groups are equal. It has seemed reasonable that Popa's results could be extended from factors to general type II_1 von Neumann algebras by direct integral theory [KR, Chapter 4]. However, we do not know of such an attempt. Direct integral theory can be used directly to prove cohomology is zero and deduce results like [CPSS, Theorems 5.4 and 5.5] however these theorems on the continuous Hochschild cohomology for a von Neumann algebra with Cartan subalgebras are deduced from the theorems in this paper. This paper provides direct proofs of Popa's main two results in [Po 1] by modifying his proofs using an interpolation type result for projections in a maximal abelian selfadjoint sub-algebra (masa) of the type II_1 algebra.

This introduction contains a more detailed description of how our results extend Popa's, the basic definitions, and a brief reference to their use in the calculation of Hochschild cohomology groups in von Neumann algebras. Though averaging plays an important role in calculating the Hochschild cohomology of von Neumann algebras for all types I, II_1, II_∞, III of von Neumann algebras (see [Ri]), the results proved here are only used in the type II_1 situation. The reason is that the type I's are already trivially hyperfinite, and the type II_∞ and III von Neumann algebras may be handled by their stability under tensoring with $B(H)$. This tensor factor $B(H)$ of M in the II_∞ and III cases gives a suitable hyperfinite algebra over which to average. Popa [Po 1] restricts his attention to type II (II_1 and II_∞) factors.

Throughout this paper we restrict attention to type II_1 von Neumann algebras. A maximal abelian selfadjoint subalgebra A of a von Neumann algebra M is a Cartan subalgebra if the von Neumann algebra generated by the unitary normaliser $\{u \in \mathscr{U}(M) : u\,A\,u^* = A\}$ of A in the unitary group $\mathscr{U}(M)$ of M is equal to M [Po 1]. A von Neumann subalgebra N of a von Neumann algebra M with centre Z is said to have trivial relative commutant if $N' \cap M = Z$.

The two results of [Po 1] that we generalise (Theorems 8 and 9) from type II_1 factors to general type II_1 von Neumann algebras with separable predual are the following.

Each type II_1 factor M with separable predual contains a hyperfinite subfactor N such that N has trivial relative commutant in M.

For each Cartan subalgebra A in a type II_1 factor M with separable predual there is a hyperfinite subfactor N with trivial relative commutant such that A is a Cartan subalgebra of N.

Note that the counter example in [Po 1, Section 4] shows that the hypothesis "with separable predual" is necessary.

The technical lemmas on the comparison of projections in a masa are stated and proved in Section 2. The proofs of these lemmas are modifications of standard results on the comparison of projections in von Neumann algebras. Observe that though all the projections are in an abelian subalgebra, the techniques are highly non-commutative as the equivalence is calculated in the whole algebra. Throughout M is taken to be a type II_1 von Neumann algebra with centre Z and centre valued trace T [Ta 2]. The separable predual condition implies that there is a faithful normal state ρ on Z, and $\tau = \rho \circ T$ is a faithful normal tracial state on M. Let $\|x\|_2 = \tau(x^*x)^{1/2}$ for all $x \in M$, and let $L^2(M)$ be the completion of M in this norm. Since τ is a normal tracial state, a result of Takesaki [Ta 1] [St] implies that for each unital von Neumann subalgebra N of M there is a τ preserving conditional expectation E_N from M onto N.

Section 3 contains the statement and proof of Theorem 8. This result shows that for each type II_1 von Neumann algebra there is a hyperfinite subalgebra N with trivial relative commutant. This conclusion is the input required into the averaging arguments used in the applications of Grothendiecks's inequality to cohomology computations (see [ES], [PS], [CPSS]).

Theorem 9, and a lemma on the construction of matrix units associated with a Cartan subalgebra, are stated and proved in Section 4. The result enables one to avoid the use of direct integral theory in the proof that $H_c^3(M, M) = 0$ for a type II_1 von Neumann algebra M with Cartan subalgebra [CPSS, Theorem 6.4]. One just needs to observe that the conclusion

of Theorem 9 can be used in place of Popa's result [Po 1] in the proof of [CPSS, Theorem 5.5].

The proofs of Theorems 8 and 9 are rather similar with one important difference which occurs in Popa's arguments [Po 1]. As the $k_n \times k_n$ matrix units are constructed at the nth stage, they perturb the fine structure of the initial masa. In Theorem 8 this means the masa is changing from the nth to the $(n+1)^{th}$ stage of the construction. There is little control over the final masa. This difficulty is avoided in the Cartan algebra situation in Theorem 9 by ensuring that the partial isometries chosen leave the masa invariant (Lemma 10).

2. Finite dimensional subalgebras of masas

This section contains the basic lemmas on constructing nice finite dimensional subalgebras of masas in type II_1 von Neumann algebras. Throughout this section we shall assume that M is a type II_1 von Neumann algebra with separable predual and with centre Z, and that A is a masa in M. The centre valued tracial conditional expectation will be denoted by T (see [Ta 2], [Di], [SZ]). Since M has separable predual, so does Z, and there is a faithful normal state ρ on Z. Let $\tau = \rho \circ T$ be the resulting faithful tracial state on M, and let $||x||_2 = (\tau(x^*x))^{1/2}$ for all $x \in M$. If C is a von Neumann subalgebra of M, let E_C denote the τ-preserving conditional expectation from M onto C [Ta2] [St], and let $P(C)$ denote the set of projections in C. Recall that the $|| \cdot ||_2$ topology on the closed unit sphere $M_1 = \{x \in M : ||x|| \leq 1\}$ in M is equivalent to the strong and ultrastrong topology on M_1 [Ta 1] [Di].

Throughout equivalence of projections $e \sim f$ is equivalence within M, i.e. there is a v (a partial isometry) in M such that $v^*v = e$ and $vv^* = f$. Recall that the order $\preccurlyeq$ on projections is defined by $e \preccurlyeq f$ if and only if there is a projection g with $e \sim g \leq f$.

Lemmas 1 to 4 are modifications of the standard comparison theory lemmas for projections in von Neumann algebras but taking into account that the projections lie in the masa A. Lemma 3 provides the crucial interpolation

step that enables one to choose a projection in the masa between two other projections in a suitable way to continue the inductive construction of the increasing sequence of finite dimensional algebras.

LEMMA 1 (SZ, 4.5). *If $h(\neq 0) \in P(A)$, then there are f_1, f_2 in $P(A)$ such that $f_1 \leq h$, $f_2 \leq h$, $f_1 f_2 = 0$ and $f_1 \sim f_2$.*

PROOF. Choose a $g \in P(A)$ such that $g \leq h$, $g \neq 0$, $g \neq h$ and $g \notin Zh$. This choice of g is possible, because hA is a masa in hMh, which is a type II_1 von Neumann algebra so that

$$hA \supsetneqq Z(hMh) = \text{the centre of } hMh \supseteq hZ.$$

By the Comparison Theorem for projections in von Neumann algebras [SZ, Theorem 4.6] applied to M, there is a $p \in P(Z)$ such that

$$g\,p \precsim (h-g)p \text{ and } g(1-p) \succsim (h-g)\,(1-p)$$

Now either $gp \neq 0$ or $(h-g)(1-p) \neq 0$; because if $gp = 0$ and $h - g - hp + gp = 0$, then $g = h(1-p) \in Z(hMh)$ contrary to the assumption g is not in $Z(hMh)$.

Let f_1 be either gp or $(h-g)(1-p)$, whichever is non zero. Let
$f_2 \sim gp$ with $f_2 \leq (h-g)p$ if $f_1 = gp$, or
$f_2 \sim (h-g)(1-p)$ with $f_2 \leq g(1-p)$ if $f_1 = (h-g)(1-p)$.
In the first case $f_1 f_2 \leq gp \cdot (h-g)p = 0$ and in the second case $f_1 f_2 \leq (h-g)(1-p)g(1-p) = 0$. Further $f_1 \leq h$ and $f_2 \leq h$. (Recall in these last calculations that all our projections are in A and $hg = g$.)

LEMMA 2 (SZ, 4.11). *If $e \in P(A)$, then there are e_1, $e_2 \in P(A)$ such that $e = e_1 + e_2$, $e_1 \sim e_2$, $e_1 e_2 = 0$.*

PROOF. Let $\mathscr{F} = \{(e_{1i}, e_{2i}) : i \in I\}$ be a maximal family of pairs of projections in A such that $e_{1i} \leq e$, $e_{2i} \leq e$ and $e_{1i} \sim e_{2i}$ for all $i \in I$, and $e_{1i}.e_{2j} = 0$ for all $i, j \in I$; where I is a suitable index set. Let $e_1 = \vee e_{1i}$ and $e_2 = \vee e_{2i}$. Then $e_1 \sim e_2$ and $e_1 e_2 = 0$ by standard von Neumann algebra projection theorems [SZ, Theorem 4.2]. If $h = e - e_1 - e_2 \neq 0$, then by Lemma 1 there exists $f_1, f_2 \in P(A)$ with $0 \neq f_1 \leq h$, $f_2 \leq h$, $f_1 \sim f_2$ and $f_1 \cdot f_2 = 0$. Further $f_1 \cdot e_{2i}$ and $f_2 \cdot e_{1i} = 0$ for all $i \in I$. Thus the pair (f_1, f_2) may be adjoined to the family $\mathscr{F}$ contradicting the maximality of $\mathscr{F}$. Hence $e = e_1 + e_2$ as required.

The interpolation in A used is that of the following lemma. For a factor the following lemma is just the fact that given two projections $e \leq f$ in the masa A and a real number λ with $\text{tr}(e) \leq \lambda \leq \text{tr}(f)$ there is a projection g in A with $e \leq g \leq f$ and $\text{tr}(g) = \lambda$. If the hypothesis "in A" is dropped, then this fact is well known dating back to Murray and von Neumann.

LEMMA 3 (SZ, 7.17). *Let* $e \leq f \in P(A)$. *If* $z \in Z$ *with* $Te \leq z \leq Tf$, *then there is a* $g \in P(A)$ *with* $e \leq g \leq f$ *and* $Tg = z$.

PROOF. Let $\mathscr{F}$ be the family of all $h \in P(A)$ such that $e \leq h \leq f$ and $Th \leq z$, and let $\mathscr{F}$ have the order induced on it from the projection lattice $P(A)$. Let $\mathscr{F}_o$ be a maximal totally ordered subfamily of $\mathscr{F}$ and let $g = \vee\{h : h \in \mathscr{F}_o\}$. By the total order of $\mathscr{F}_o$ and the positivity and the ultra-weak continuity of T, $e \leq g \leq f$ and $Tg \leq z$.

Thus $g \; \varepsilon \; \mathscr{F}_o$. Suppose that $z - Tg \neq 0$. By standard von Neumann algebra techniques (the comparison theorem et al) there is an $\varepsilon > 0$ and a non-zero $p \in P(Z)$ such that $(z - T(g))p \geq \varepsilon \, p$ (if ε and p do not exist, then $z - Tg = 0$). Now $(f - g)p \; \varepsilon \; P(A)$, $(f - g)p$. $g = 0$ and $T((f - g)p) = (Tf - Tg)p \geq (z - Tg)p \geq \varepsilon p$, since $p \in Z$. Hence $(f - g)p \neq 0$. Let $n \in \mathsf{N}$ with $2^{-n} \leq \varepsilon$.

Repeated application of Lemma 2 to the projection $(f - g)p$ in A shows that there are mutually orthogonal equivalent projections $e_1, e_2, ..., e_{2^n}$ in A such that $(f - g)p = \Sigma \, e_j$. The equivalence implies that $T(e_j) = T(e_k)$ for all j, k (by [SZ, Theorem 7.11], [Ta, Theorem 2.6]). Hence

$$T(e_1) = 2^{-n} T((f - g)p) \leq 2^{-n} p \leq \varepsilon \, p.$$

since $T(f - g) \leq 1$. Note that $e_1 g = 0$ so $g + e_1 \in A$. Further

$$e \leq g + e_1 \leq g + (f - g)p = g(1 - p) + fp$$

$$\leq f(1 - p) + fp = f,$$

and

$$T(g + e_1) \leq T(g) + \varepsilon p \leq T(g) + (z - T(g))p$$

$$\leq zp + T(g)(1 - p) \leq z.$$

Thus $g + e_1 \in \mathscr{F}_o$ and this contradicts the maximality of g in $\mathscr{F}_o$, so $z = Tg$ as required.

LEMMA 4. *Let* $f \in P(A)$, $p \in P(Z)$ *and* $0 < r < m$ *be integers. If* $Tf = rm^{-1}p$, *then there are orthogonal equivalent projections* $e_1, ..., e_r$ *in* A *with* $f = \Sigma_1^r e_j$ (*and* $Te_j = m^{-1}p$).

PROOF. The projections $e_1, ..., e_r$ are constructed inductively. By Lemma 3 there is $e_1 \leq f$ with $Te_1 = m^{-1}p$. If orthogonal projections $e_1, ..., e_t (t < r)$ have been constructed with $Te_j = m^{-1}p$, then $T(f - \Sigma_1^t e_j) = (r - t)m^{-1}p$ so there is a projection e_{t+1} with $Te_{t+1} = m^{-1}p$ and $e_{t+1} \leq f - \Sigma_1^t e_j$. This gives orthogonal projections $e_1, ..., e_r$ with $f = \Sigma_1^r e_j$ and $Te_j = m^{-1}p$; the equivalence of e_j follows from them having equal central trace [SZ],[Ta1].

Note that although the next lemma involves approximation in a masa A in M it is not a commutative result, because the equivalence is that in M.

LEMMA 5, *Let B be a finite dimensional subalgebra of A such that there is an integer k with the property that if q is a minimal projection in $B \cap Z$, then there are exactly k minimal projections in qB and they are all equivalent (in M). Let $\varepsilon > 0$ and let $g_1, ..., g_n$ be projections in A. Then there is a finite dimensional subalgebra A_o of A that contains B and has the property that there is a positive integer m so that*

1. *k divides m,*
2. *if p is a minimal projection in $A_o \cap Z$, then there are exactly m minimal projections in pA_o and they are all equivalent (in M), and*
3. *$\|g_j - E_{A_o} g_j\|_2 \leq \varepsilon$ for $1 \leq j \leq n$.*

Note that standard lemmas on approximating projections [Co] and [Ch] imply that the y_j could be chosen to be projections in A_o, but we do not require this subsequently so do not follow this up.

The condition on the minimal projections in the centre giving rise to equivalent minimal projections below may be written symbolically for B: if q is a minimal projection in $Z \cap B$, then there are equivalent minimal projections $e_1, ..., e_k$, in qB with $e_1 \sim e_j$ for $1 \leq j \leq k$. In the proof below no effort is made to control the values of k. However, a little more care shows that if k is a power of 2, then m can be chosen to be a power of 2.

PROOF. The proof splits into a particular preliminary case with $B \cap Z = \mathbb{C}1$, which is the main part of the proof, and the general case, which is just a finite sum of the particular ones. For the first part of the proof (most of it) assume that $B \cap Z = \mathbb{C}1$ and that B is spanned by its equivalent minimal projections u_j ($1 \leq j \leq k$); they are equivalent by hypothesis. Thus the centre valued trace T has $Tu_j = k^{-1}.1$, since Tu_j is a multiple of 1.

Let C_0 be the finite dimensional *-subalgebra of A generated by $g_1, ..., g_n$ and B. Let m_0 be a positive integer such that $m_0 \geq 2^{n+1} \varepsilon^{-2}$ and let $m = m_0 k$.

If F is a finite set of projections, let $\mathsf{N} \operatorname{span}(F)$ denote the additive semigroup generated by $F \cup \{0\}$, and if r is a positive integer, let $r^{-1} \mathsf{N} \operatorname{span}(F)$ denote this semigroup times r^{-1}. The abelian von Neumann algebras with separable preduals are isomorphic to $L^\infty[0,1]$, $\ell^\infty(\mathsf{N})$, $\ell^\infty(r)$ for $r \in \mathsf{N}$, or to the direct sum of the first with one of the other two [Ta1, p.112]. Hence the real linear span of the set of projections in Z is dense in the self-adjoint part of Z in the norm topology, and the set $\{x = x^* \in Z : x \geq 0\}$ is covered by the open sets

$$\{x = x^* \in Z : x \geq 0, \|x - y\| < m^{-1} \text{ for some } y \in m^{-1} \mathsf{N} \operatorname{span}(F)\}$$

as F runs over all finite sets of pairwise orthogonal projections in Z that add to 1. The set $\{w \in TC_0 : 0 \leq w \leq 1\}$ is compact in the norm topology, since C_0 is finite dimensional. Hence there is a finite set F_0 of projections in Z such

that for each $w \in TC_0$ with $0 \le w \le 1$ there is a v in $m^{-1}\mathsf{N}\operatorname{span}(F_0)$ with $||w - v|| < m^{-1}$. If the projections in F_0 are multiplied by the minimal projections in TC_0, we obtain a finite set F of orthogonal projections in Z adding to 1 with the properties that

(1) if $w \in TC_0$ with $0 \le w \le 1$, then there is a v in $m^{-1}\mathsf{N}\operatorname{span}(F)$ with $||v - w|| < m^{-1}$, and

(2) each projection in TC_0 is a sum of projections in F.

Let C be the finite dimensional subalgebra of A generated by C_0 and F. Observe that each central projection in TC_0 is a sum of orthogonal projections in F, so F is the set of minimal projections in $Z \cap C$. Fix a projection p in F and a minimal projection $u (= u_j)$ in B. Since pu is a projection in the commutative finite dimensional algebra pC, there are minimal projections $f_1, ..., f_\ell$ in pC so that $pu = \sum_1^\ell f_j$. The dependence of ℓ and f_j on u and p is omitted here to simplify notation. Note that pu is not zero, since $T(pu) = pT(u) = k^{-1}p$. Note that the number of minimal projections in C that add to each minimal central projection in C (i.e. p in F) is no more than the corresonding maximal number adding to a central projection in C_0, because C is generated by C_0 and $F \subset Z$. This is because multiplying by a central projection does not increase this number. The projections g_j may not be in Z so each pair $(g_j, 1 - g_j)$ can split a minimal projection in two. Hence the number of minimal projections in C below pu is no more than 2^n, because u is minimal in B and p is minimal central in C. Thus $\ell \le 2^n$.

Now $T(pu) = pT(u) = k^{-1}p$. Also $puf_j = pf_j = f_j$ implies that $0 \le f_j \le pu \le 1$ so that

$$0 \le Tf_j = pTf_j \le T(pu) = k^{-1}p$$

for $j = 1, ..., \ell$. Thus Tf_j can be approximated in norm by a non-negative rational multiple of p with denominator m by (1), the choice of F and F_0, and the minimality of p in $C \cap Z$. Hence there is an integer r_j with $0 \le r_j \le m$ so that $||Tf_j - m^{-1}r_jp|| < m^{-1}$. As $Tf_j \ge 0$, this is equivalent to

$$0 \le m^{-1}\max\{0, (r_j - 1)\}p \le Tf_j \le m^{-1}(r_j + 1)p.$$

If $Tf_j = m^{-1}r_jp$, choose $e_j = f_j$. If $Tf_j \ne m^{-1}r_jp$, then there is a projection e_j in A with $e_j \le f_j$ and $Te_j = m^{-1}\max\{0, (r_j - 1)\}p$ by Lemma 3. Thus $0 \le T(f_j - e_j) \le 2m^{-1}p$ for each j. Let $e_0 = pu - \sum_1^\ell e_j$. Note that $e_0, ..., e_\ell$ are orthogonal projections, because $f_1, ..., f_\ell$ are orthogonal projections with $e_j \le f_j$ $(1 \le j \le \ell)$, and that any of the e_j could be zero. The central trace of e_0 satisfies

$$Te_0 = k^{-1}p - \sum_1^\ell T(e_j) = r_0 m^{-1}p$$

for some non-negative integer r_0, since $T(pu) = k^{-1}p = m_0 m^{-1}p$. By Lemma 4 each projection e_j ($0 \le j \le \ell$) may be subdivided into orthogonal equivalent projections e_{ji} with $T(e_{ji}) = m^{-1}p$ for all j and i. The projections e_{ji} are orthogonal equivalent and add up to pu. There are m_0 of these projections e_{ji} for each pu since $T(pu) = k^{-1}p = m_0 m^{-1}p$.

Note that $p = pu_1 + ... + pu_k$, so that there are $m = km_0$ orthogonal equivalent projections $e_{ji}^{p,1}, ..., e_{ji}^{p,k}$ that add to p; denote these projections by h_s^p for $1 \le s \le m$.

Let A_0 be the linear span of $\{h_s^p : p \in F, 1 \le s \le m\}$ so that A_0 is a commutative finite dimensional $*$-subalgebra of A. For each pu above, each e_j is in A_0 by construction of h_s^p. Hence for each u_j in B ($1 \le j \le k$) and each p in F, pu_j is in A_0 as it is a sum of e_i by construction of e_0. Hence u is in A_0 as $u = \sum\{pu : p \in F\}$. When constructed F was observed to be the set of minimal projections in $C \cap Z$ and each p in F was shown to be the sum of m orthogonal equivalent projections $h_s^p (1 \le s \le m)$. The equivalence of h_s^p and h_t^p for $1 \le s,\ t \le m$ imply that a sum $\sum h_j^p$ for fixed p over some j is in the centre Z if and only if it is over all $1 \le j \le m$. This implies that F is the set of minimal projections in $A_0 \cap Z$ and that each p in F is the sum of m equivalent (orthogonal) minimal projections in pA_0. This proves properties 1 and 2.

We now turn to proving property 3. Let g be a projection in C. Since g is a sum of minimal projections in C, there is a subset W of

$$\{(p,f_j^p) : p \in F, f_j^p \text{ is a minimal projection in } pC\};$$

such that

$$g = \sum\{f_j^p : (p,f_j^p) \in W\}.$$

Let $h = \sum\{e_j^p : (p,f_j^p) \in W\}$ where e_j^p are the projections constructed corresponding to the projections f_j^p. Note that the e_j^p are not minimal projections in A_0 but are each sums of equivalent minimal projections in A_0. By construction $e_j^p \le f_j^p$ for all j and p, so h is a projection in A_0 with $h \le g$. Hence

$$\|g - h\|_2^2 = \tau(g - h) = \sum \tau(p(f_j^p - e_j^p)) = \sum \rho(p \cdot T(f_j^p - e_j^p))$$

by the properties of T and $\tau = \rho T$ with the above sums over $(p,f_j^p) \in W$. For each j, $0 \le pT(f_j^p - e_j^p) \le 2m^{-1}p$ and for each p there are at most $2^n k$ such elements because this is the maximal number of minimal projections in C below a minimal p in $C \cap Z$. Hence using the minimality properties of the fact that E_{A_0} is the orthogonal projection onto A in the Hilbert space $(M, \|\cdot\|_2)$

$$\|g - E_{A_0}g\|_2^2 \leq \|g - h\|_2^2 \leq \sum \rho(p) \cdot 2 \cdot m^{-1} \cdot 2^n k$$

where the sum is over p in F.

Thus

$$\|g - E_{A_0}g\|_2^2 \leq m^{-1} \cdot 2^{n+1}k = m_0^{-1} \cdot 2^{n+1} \leq \varepsilon^2$$

by the choice of m_0 and $m = m_0 k$.

This proves the case where $B \cap Z = \mathsf{C}1$.

In general $B \cap Z$ is a finite dimensional commutative C^*-algebra with minimal projections $q_1, ..., q_N$. Cutting the projections $g_1, ..., g_n$ by a central projection q_t gives n projections $g_1 q_t, ..., g_n q_t$ in the algebra $A q_t$ with $B q_t \cap Z q_t = \mathsf{C} q_t$ and q_t the identity of the von Neumann algebra $q_t M$. Since $\rho(1) = 1 = \sum \rho(q_t)$, the approximations obtained in the various factors add to the whole algebra provided ρ is replaced on $M q_t$ by $\rho(q_t)^{-1} \rho(\cdot)$. Properties 1 and 2 follows directly as they concern elements below minimal projections in $B \cap Z$. Property 3 follows because of orthogonality of $\|\cdot\|_2$-norm on the direct sum $M = \bigoplus M q_t$ and the choice of state $\rho(q_t)^{-1} \rho(\cdot)$ on $M q_t$. This completes the proof.

The following lemma is just the inductive version of Lemma 5; however, in the form given below it fits in better with the Popa's characterization of masa's in von Neumann algebras [Po1, Lemma 1.2].

LEMMA 6. *Let A_o be a finite dimensional subalgebra of A so that there is a positive integer k_o with the property that if p is a minimal projection in $A_o \cap Z$, then there are exactly k_o minimal projections in $P A_o$ and they are all equivalent (in M). Then there is a sequence A_n of finite dimensional subalgebras of A for $n \in \mathsf{N}$ such that*

1. *$A_{n-1} \subseteq A_n$ for $n \geq 1$,*
2. *k_{n-1} divides k_n for $n \geq 1$,*
3. *if q is a minimal projection in $A_n \cap Z$, then there are exactly k_n minimal projections in $q A_n$ and they are all equivalent, and*
4. *the weak closure of the union of the A_n is A, i.e. $(\cup A_n)^{-w} = A$.*

PROOF. The von Neumann algebra A has a separable predual, since M has a separable predual, so there is a sequence $\{g_n : n \in \mathsf{N}\}$ of projections in A such that the linear span of $\{g_n : n \in \mathsf{N}\}$ is weakly dense in A.

By induction we choose an increasing sequence A_n of finite dimensional subalgebras of A satisfying (1), (2) and (3) of our conclusions above, and such that

5. *$\|g_j - E_{A_n}g_j\|_2 \leq 2^{-n}$*

for $1 \leq j \leq n$. Observe that Lemma 5 gives the step from $n-1$ to n in the induction. Property (4) follows from inequality (5).

The next lemma is used in the proofs of Theorems 8 and 9 to show that the hyperfinite algebras constructed there have trivial relative commutants.

LEMMA 7. *Let* $\mathsf{C} 1 = N_o \subseteq N_2 \subseteq ...$ *be an increasing sequence of finite dimensional *-subalgebras of M such that the centre of N_n is equal to $N_n \cap Z$ for all n. Let N be the weak closure of the union of N_n, i.e. $N = (\cup N_n)^{-w}$. If N contains Z, then $N \cap N' = Z$.*

PROOF. Let $x = x^* \in N \cap N'$ with $\|x\| \leq 1$. The strong topology and the $\|.\|_2$-topology coincide on the closed unit ball M_1 of M. Hence, by the Kaplansky density theorem [Ta 1], there is a $n \in \mathsf{N}$ and $y \in N_n$ such that $\|y\| \leq 1$ and $\|x - y\|_2 < \varepsilon$. Since $x \in N' \subseteq N_n'$, $\|yu - uy\|_2 < 2\varepsilon$ for all $u \in N_n$ with $\|u\| \leq 1$. Thus $\|y - uyu^*\|_2 \leq 2\,\varepsilon$ for all u in the unitary group $\mathscr{U}(N_n)$ of N_n. Let μ denote normalised Haar measure on the (compact) unitary group $\mathscr{U}(N_n)$. Let $z = \int (y - uyu^*)d\mu(u)$, with integration over the whole of the unitary group $\mathscr{U}(N_n)$. Then $\|z\|_2 \leq 2\varepsilon$ and $w(y - z)w^* = \int wuyu^*w^* d\mu(u) = \int vyv^* d\mu(v) = y - z$ for all $w \in \mathscr{U}(N_n)$. Hence $y - z$ is in the centre of N_n, which is equal to $Z \cap N_n$. Further $\|x - (yz)\|_2 \leq 3\varepsilon$; this shows that x is in $\| \cdot \|_2$ closure of Z, so is in Z.

3. Injective subalgebras with trivial relative commutant

THEOREM 8. *Let M be a type II_1 von Neumann algebra with separable predual and centre Z. Then there is a masa A in M and an injective von Neumann subalgebra N of M containing A such that A is a Cartan subalgebra of N and N has trivial relative commutant in M.*

PROOF. Let $\{x_j : j \in \mathsf{N} \cup \{0\}\}$ be a subsequence of the closed unit ball M_1 of M that is dense in this unit ball in the $\| \cdot \|_2$ norm; such a sequence exists by the separability of the predual of M and the equivalence of the $\| \cdot \|_2$-topology and the strong topology on M_1. We shall assume $x_o = 1$. By induction on n we shall construct sequences

$A_o = \mathsf{C}1 \subseteq A_1 \subseteq A_2 \subseteq ...$

of finite dimensional abelian subalgebras of M,

$N_o = \mathsf{C}1 \subseteq N_1 \subseteq N_2 \subseteq ...$

of finite dimensional *-subalgebras of N, and

$k_o \leq k_1 \leq k_2 \leq ...$

of positive integers such that

1. $A_n \subseteq N_n$,
2. the centre of N_n is $N_n \cap Z$, which is contained in A,
3. k_{n-1} divides k_n,

4. for each minimal projection q in $Z \cap A_n$, there are exactly k_n minimal projections in qA_n and they are all equivalent,

5. for each minimal projection q in $Z \cap A_n$, the algebra qN_n is isomorphic to $\mathsf{M}_{k_n}(\mathbb{C})$ and has matrix units e_{ij}^q so that $e_{ii}^q(1 \leq i \leq k_n)$ are the minimal projections of qA_n,

6. for each $1 \leq j \leq n-1$ and each minimal projection p in $A_j \cap Z$, the natural unitary in N_j that interchanges e_{11}^p and e_{ii}^p normalises A_n, and

7. $\|(E_{A_n' \cap M} - E_{A_n})(x_i)\|_2 \leq 2^{-n}$ for $1 \leq i \leq n$.

The induction starts with $A_o = N_o = \mathbb{C}1$ and $k_o = 1$. Suppose that A_n, N_n and k_n have been constructed as above. Let B be a maximal abelian subalgebra of M containing A_n. By Lemma 2.6 with A_o there equal to A_n choose an increasing sequence B_ℓ of finite dimensional subalgebras of B containing A_n with the properties of that lemma. As the algebra B is a masa in M and $B = (\cup B)^{-w}$, [Po 1, Lemma 1.2] implies that

$$\|E_{B_\ell' \cap M}(x) - E_{B_\ell}(x)\|_2 \to 0$$

as $\ell \to \infty$ for all $x \in M$. For each minimal projection p in $Z \cap A_n$, let $e_{ij}^p(1 \leq i,j \leq k_n)$ be the matrix units in the algebra pN_n. Let δ_n be the number of minimal projections in $Z \cap A_n$. Choose ℓ so large that

8. $\|(E_{B_\ell' \cap M} - E_{B_\ell})(e_{ij}^p x_i e_{ji}^p)\|_2 < 2^{-(n+1)}\delta_n^{-\frac{1}{2}}k_n^{-\frac{1}{2}}$

for $1 \leq i \leq n+1$, $1 \leq j \leq k_n$ and all minimal projections p in $Z \cap A_n$. Let Q_{n+1} be the set of all minimal projections in this $B_\ell \cap Z$. Now for each $q \in Q_{n+1}$ the projection qe_{11}^p is a sum of $m_n = k_{n+1}/k_n$ minimal projections in qB_ℓ by Lemma 6, where k_{n+1} is the integer corresponding to B_ℓ of that lemma. Number these m_n minimal projections in qB_ℓ as e_{ii}^q for $I \leq i \leq m_n$. For each q, the projections $e_{ii}^q(1 \leq i \leq m_n)$ are equivalent in M so form the diagonal of a set of matrix units $e_{ij}^q(1 \leq i,j \leq m_n)$ in M.

Let A_{n+1} be the linear subspace of M spanned by the set $\{e_{j1}^p e_{tt}^q e_{1j}^p : 1 \leq t \leq m_n, \ 1 \leq j \leq k_n, \ q \in Q_{n+1}, \ p \text{ the minimal projection in } Z \cap A_n \text{ with } q = qp\}$.

Let N_{n+1} be the linear subspace of M spanned by the set $\{e_{i1}^p e_{st}^q e_{1j}^p : 1 \leq s,t \leq m_n, 1 \leq i,j \leq k_n, q \in Q_{n+1}, p \text{ the minimal projection in } Z \cap A_n \text{ with } q = qp\}$.

Then A_{n+1} is an abelian subalgebra of M with minimal projections elements of its spanning set, $A_n \subseteq A_{n+1}$, and $A_{n+1} \subseteq N_{n+1}$. Further N_{n+1} is a finite dimensional subalgebra of M which is a direct sum over $q \in Q_{n+1}$ of algebras each of which is isomorphic to $\mathsf{M}_{k_{n+1}}(\mathbb{C})$ with matrix units $e_{i1}^p e_{st}^q e_{1j}^p$, where $1 \leq s,t \leq m_n$, $1 \leq i,j \leq k_n$ and p is the unique minimal projection in $Z \cap A_n$ such that $q = qp$. This shows that the centre of N_{n+1} is the linear span

of Q_{n+1} so equals $N_{n+1} \cap Z$. Let $1 \le s \le n-1$ and let r be a minimal projection in $Z \cap A_s$. Let u be the natural unitary in N_s that interchanges e_{11}^r and e_{ii}^r in N_s. Then u normalises A_n by (6), and so normalises the minimal projections $e_{j1}^p e_{tt}^q e_{1j}^p$ spanning A_{n+1}. If u is the natural unitary in N_n that interchanges e_{11}^p and e_{ii}^p, then u normalises A_{n+1} by definition of the minimal projections in A_{n+1}. We have checked conditions (1) to (6) of the induction, and only condition (7) remains to be proved from (8).

Firstly observe that for all $x \in M$,

$$||(E_{A'_{n+1} \cap M} - E_{A_{n+1}})(e_{jj}^p x \, e_{jj}^p)||_2 = ||(E_{A'_{n+1} \cap M} - E_{A_{n+1}})(e_{1j}^p x \, e_{j1}^p)||_2$$

because the map $e_{jj}^p x \, e_{jj}^p \to e_{1j}^p x \, e_{j1}^p$ is an isometric isomorphism from $e_{jj}^p M \, e_{jj}^p$ onto $e_{1j}^p M \, e_{j1}^p$ that carries $e_{jj}^p A'_{n+1} \cap M \, e_{jj}^p$ onto $e_{1j}^p A'_{n+1} \cap M \, e_{j1}^p$ and $e_{jj}^p A_{n+1} e_{jj}^p$ onto $e_{1j}^p A_{n+1} e_{1j}^p$.

Now for each $1 \le i \le n+1$,

9. $||(E_{A'_{n+1} \cap M} - E_{A_{n+1}})(x_i)||_2^2 = \Sigma ||e_{jj}^p (E_{A'_{n+1} \cap M} - E_{A_{n+1}})(x_i) e_{jj}^p||_2^2,$

where the summation is over $1 \le j \le k_n$ and all minimal projections p in $A_n \cap Z$, and equality holds because of the orthorgonality of $e_{jj}^p M \, e_{jj}^p$ for different j and p, and e_{jj}^p being in the commutants of the two algebras $A'_{n+1} \cap M$ and A_{n+1}. Further e_{jj}^p is in these two algebras so (9) equals

$$\Sigma ||(E_{A'_{n+1} \cap M} - E_{A_{n+1}})(e_{jj}^p x_i e_{jj}^p)||_2^2 = \Sigma ||(E_{A'_{n+1} \cap M} - E_{A_{n+1}})(e_{1j}^p p x_i \, e_{j1}^p)||_2^2$$

by the note above on the isometries. Now $e_{11}^p A'_{n+1} \cap M \, e_{11}^p = e_{11}^p B'_\ell \cap M \, e_{11}^p$ and $e_{11}^p A_{n+1} e_{11}^p = e_{11}^p B_\ell e_{11}^p$ by the definition of A_{n+1} in terms of B_ℓ and its minimal projections. Thus (9) equals

$$\Sigma ||(E_{B'_\ell \cap M} - E_{B_\ell})(e_{1j}^p x_i e_{j1}^p)||_2^2 \le \Sigma \, 4^{-(n+1)} \delta_n^{-1} k_n^{-1} = 4^{-(n+1)}$$

as the summation extends over δ_n minimal projections p in $Z \cap A_n$ and k_n subscripts since $||e_{1j}^p y e_{j1}^p||_2 = ||e_{jj}^p y e_{jj}^p||_2$ for all p and j. This finishes the inductive construction.

Let $N = (\cup N_n)^{-w}$ and $A = (\cup A_n)^{-w}$, where $^{-w}$ denotes weak closure. Clearly A is self-adjoint abelian, N is injective and $A \subseteq N$. By (7) and the $||\cdot||_2$-density of $\{x_i : i \ge 0\}$ in the closed unit ball M_1, it follows that $||(E_{A'_n \cap M} - E_{A_n})(x)||_2$ tends to zero for all $x \in M$. Hence A is a masa by [Po 1, Lemma 1.2]. Now $N' \cap M \subseteq A' \cap M = A$, since A is a masa in M, so $N' \cap M \subseteq N' \cap A \subseteq N' \cap N$, which equals Z by (2) and Lemma 7. This proves Theorem 8.

4. Injectives containing Cartan subalgebras

THEOREM 9. *Let M be a type II_1 von Neumann algebra with separable predual and let A be a Cartan subalgebra of M. Then there is an injective von Neumann subalgebra N of M with relative commutant the centre Z of M such that A is a Cartan subalgebra of N.*

The hypothesis that A is a Cartan subalgebra of a type II_1 von Neumann algebra M applies in the following discussion and definitions and in Lemma 10.

Let $\mathcal{N}(A)$ be the normalizer of A in the unitary group of M, so $u \in \mathcal{N}(A)$ if and only if $u A u^* = A$. Let

$$\mathcal{G} = \{v \in M : v = up, u \in \mathcal{N}(A), p \in P(A)\},$$

so elements of $\mathcal{G}$ are partial isometries with $v^*v = p \in A$ and $vv^* = upu^* \in A$. Observe that $\mathcal{G}$ is a semigroup in M as $v = up$ and $t = wq$ implies that $vt = upwq = uw \cdot w^*pw \cdot q$ with $uw \in \mathcal{N}(A)$ and $w^*pw \cdot q$ a projection in A since A is abelian and $w^*pw \in A$. Further $\mathcal{G}^* = \mathcal{G}$ as $v = up$ implies $v^* = u^* \cdot upu^*$.

LEMMA 10. *Let $f_1, ..., f_k$ be a set of orthogonal equivalent projections in A. Then there are matrix units f_{ij} in $\mathcal{G}$ such that $f_{ii} = f_i$ for $1 \le i \le k$.*

PROOF. The first stage is to show that if e_1 and e_2 are equivalent orthogonal projections in A then there is a $v \in \mathcal{G}$ with $v^*v = e_1$ and $vv^* = e_2$. The construction of v is done by a standard maximal trick; this part of the proof is exactly as in the first part of the proof of [Po 1, Prop 3.6], which is repeated here for the readers convenience.

Let $\mathcal{F} = \{(p, w) : p \in P(A), w \in \mathcal{G}, p \le e_1, w^*w = p \text{ and } ww^* \le e_2\}$. Note $\mathcal{F}$ is not empty since $(0, 0) \in \mathcal{F}$. Define a partial order $\le$ on $\mathcal{F}$ by $(p_1, w_1) \le (p_2, w_2)$ if and only if $p_1 \le p_2$ and $w_1 = w_2 p_1$. By Zorn's Lemma we obtain a maximal totally ordered subfamily $\mathcal{F}_o$ of $\mathcal{F}$ (which has a countable cofinal subfamily since M has separable predual), and let $(p, v) \in \mathcal{F}$ be the maximal element of $\mathcal{F}_o$. Suppose $p \ne e_1$. Let $p_1 = e_1 - p = e_1 - v^*v$ and $q_1 = e_2 - vv^*$. Note that p_1 and q_1 are equivalent projections - this can be seen easily here using the central trace T as $T(p_1) = T(q_1)$.

If $p_1 u q_1 u^* = 0$ for each unitary $u \in \mathcal{N}(A)$, then the projection $g = \vee\{u q_1 u^* : u \in \mathcal{N}(A)\}$ satisfies $p_1 g = 0$, $q_1 g = q_1$ and $g \in P(Z)$, because $\mathcal{N}(A)$ generates M as a von Neumann algebra. Now $T(p_1) = T(q_1) = T(q_1)g = T(p_1)g = T(p_1 g) = 0$ contrary to $p_1 \ne 0$, where the first and third equalities hold because $p_1 \sim q_1$. This contradiction implies that $p_1 u q_1 u^* = p_o \ne 0$ for some $u \in \mathcal{N}(A)$. Now $p_o \le p_1$ and $up_1 \in \mathcal{G}$ by construction of $\mathcal{G}$. If $v = wp$ with $w \in \mathcal{N}(A)$ (recall $(p, v) \in \mathcal{F}$), then $v + up_o \in \mathcal{G}$

because $p + p_o \in P(A)(pp_o = 0)$ and there is a unitary $u_1 \in \mathcal{N}(A)$ such that $u_1(p + p_o) = v + up_o$. Hence $(p + p_o, v + up_o) \in \mathcal{F}$ and is strictly greater than (p, v), contradicting the maximality of the element (p, v). Hence $v \in \mathcal{G}$ satisfies $v^*v = e_1(= p)$ and $vv^* = e_2$.

Using the above choose $f_{1j}(1 \leq j \leq k)$ in $\mathcal{G}$ such that $f_{1j}f_{1j}* = f_1$ and $f_{1j}^*f_{1j} = f_j$ for $1 \leq j \leq k$. Define $f_{ij} = f_{1i}^*f_{1j}$ for $1 \leq i, j \leq k$. These f_{ij} are the required matrix units in $\mathcal{G}$.

Proof of Theorem 9. Let $\{a_n : n \in \mathsf{N}\}$ be a sequence of projections in A whose linear span is dense in A in the strong topology. Observe that A has such a sequence, because M and hence A has separable predual. Further note that as τ is a faithful normal tracial state on M, density in the $\|\cdot\|_2$-topology implies strong density in the unit ball of A.

By induction we construct an increasing sequence $N_o = \mathsf{C}1 \subset \mathsf{C}N_1 \subset \mathsf{C}N_2 \ldots$ of finite dimensional *-subalgebras of M with matrix units e_{ij}^q in each of the matrix direct summands satisfying

1. the centre of N_n is equal to $N_n \cap Z$,

2. if $Q(n)$ denotes the set of minimal idempotents in $N_n \cap Z$, then the matrix units $e_{ij}^q(1 \leq i, j \leq k(q))$ constructed in $q N_n$ are in $\mathcal{G}$ for all $q \in Q(n)$,

3. if A_n is the finite dimensional subalgebra generated by the set $\{e_{ii}^q : 1 \leq i \leq k(q), q \in Q(n)\}$, then $A_n \subset A$, and

4. $\|a_j - E_{A_n}(a_j)\|_2 \leq 2^{-n}$ for $1 \leq j \leq n$.

Recall that E_{A_n} denotes the τ preserving conditional expectation from M onto A_n. Suppose that N_n has been constructed. If $v = ue$ is in $\mathcal{G}$ with $u \in \mathcal{N}(A)$ and e a projection in A, then $v\, x\, v^* = u\, e\, x\, e\, u^* = u\, e\, x\, u^*$ is in A for all $x \in A$. Thus the finite set

$$F = \{\Sigma_q e_{1i}^q a_t e_{i1}^q : 1 \leq t \leq n + 1, 1 \leq i \leq k(q)\}$$

is contained in $\Sigma_q e_{11}^q A\ e_{11}^q$, where all sums over q run over $Q(n)$. Now by Lemma 5 there is a finite dimensional subalgebra A_o of A and $k \in \mathsf{N}$ such that $A_n \subseteq A_o$,

5. $\|(I - E_{A_o})(e_{1i}^q a_t e_{i1}^q)\|_2 \leq \tau(q)^{\frac{1}{2}}k_n^{-1}2^{-(n+1)}$ for all $q \in Q(n)$, $1 \leq i \leq k_n$ and $1 \leq t \leq n + 1$, and

6. if $Z_{n+1} = A_o \cap Z$, then for each minimal projection p in Z_{n+1}, the minimal projections $f_1^p, \ldots, f_k^p$ in pA_o are all equivalent.

Let $Q(n + 1)$ be the set of minimal projections in Z_{n+1}. For each $p \in Q(n + 1)$ and each $q \in Q(n)$ with $pq = p$, the projection $p\, e_{11}^q$ is a sum of a finite number of f_i^p, which are the minimal projections in pA_o. By renumbering the projections f_i^p (if necessary) we will assume that $p\, e_{11}^q = \Sigma_1^{k_0(p)}f_i^p$ for some $k_o(p)$ and all $pq = p$. As the projections

$f_i^p (1 \leq i \leq k)$ in A are orthogonal and equivalent, there is a set of matrix units f_{ij}^p in $\mathscr{G}$ so that $f_{ii}^p = f_i^p$ for $1 \leq i \leq k$ by Lemma 10. The matrix units $e_{\ell m}^p$ are defined by a tensor operation using f_{ij}^p, e_{s1}^q, e_{1t}^q for $pq = p$ and all i, j, s, t:

7. Let $e_{i,j,s,t}^p = e_{s1}^q f_{ij}^p e_{1t}^q$ for $p = qp (p \in Q(n+1)$, $q \in Q(n))$, $1 \leq i, j \leq k_o(p)$ and $1 \leq s, t \leq k(q)$.

Let N_{n+1} be the linear span of all $e_{i,j,s,t}^p$ defined in (7). Note that N_{n+1} is a *-subalgebra of M and that for each $p \in Q(n+1)$, $e_{i,j,s,t}^p$ are the matrix units of $p N_{n+1}$ corresponding to the tensor product $\mathsf{M}_{k_o(p)}(\mathsf{C}) \otimes \mathsf{M}_{k(q)}(\mathsf{C})$ with $1 \leq i, j \leq k_o(p)$ and $1 \leq s, t \leq k(q)$. Clearly the centre of N_{n+1} is the linear span of $Q(n+1)$ so is Z_{n+1}. By construction $e_{i,j,s,t}^q$ is in $\mathscr{G}$, because $\mathscr{G}$ is a semigroup under multiplication. The matrix element $e_{i,i,s,s}^p$ is of the form $uefeu^*$ with $u \in \mathscr{N}(A)$ and $e, f \in P(A)$, because $e_{ii}^q \in P(A)$ and $e_{1s}^q = (e_{s,1}^q)^*$, so is in A. Let A_{n+1} be the linear span of $e_{i,i,s,s}^q$ over all $1 \leq i \leq k_n$, $1 \leq s \leq k$, $p = pq$, $p \in Q(n+1)$ and $q \in Q(n)$. Let $k(p) = k_o(p)k(q)$. Lastly we check inequality (4) as (1), (2) and (3) have been done for $n+1$. Observe that $e_{11}^q A_o e_{11}^q \subset A_{n+1}$ for all $q \in Q(n)$ as the minimal projections in $e_{11}^q A_o e_{11}^q$ are the projections $e_{i,i,1,1}^p$ for $1 \leq i \leq k_o(p)$ and $pq = p$ with $p \in Q(n+1)$. Now note that for each $q \in Q(n)$,

8. $||(I - E_{A_{n+1}})(e_{ii}^q x \, e_{ii}^q)||_2 = ||(I - E_{A_o})(e_{1i}^q x \, e_{i1}^q)||_2$;

because dropping the q's, the map $L_{e_{i1}} R_{e_{1i}} (L = \text{left multiplication}, R = \text{right multiplication})$ from $e_{1i} A \, e_{i1}$ onto $e_{ii} A \, e_{ii}$ is an isometry since

$$||L_{e_{i1}} R_{e_{1i}}(e_{1i} x e_{i1})||_2^2 = \tau(e_{i1} e_{1i} x e_{i1} e_{1i} e_{i1} e_{1i} x^* e_{i1} e_{1i}) = ||e_{ii} x e_{ii}||_2^2.$$

Now $e_{ii}^q e_{jj}^r = 0$ if $q \neq r \in Q(n)$, or $q = r$ and $i \neq j$. The module property of the conditional expectation map $E_{A_{n+1}}$ with respect to $e_{ii}^q \in A_{n+1}$ implies that the spaces $(I - E_{A_{n+1}})(e_{ii}^q A \, e_{ii}^q)$ are orthogonal in $|| \cdot ||_2$ norm for different pairs (q, i). Hence

$$||(I - E_{A_{n+1}})(a_t)||_2^2 = \Sigma ||(I - E_{A_{n+1}})(e_{ii}^q a_t e_{ii}^q)||_2^2,$$

with the sum running over all $q \in Q(n)$ and all $1 \leq i \leq k_n$. Equation (8) implies that

$$
\begin{aligned}
&||(I - E_{A_{n+1}})(a_t)||_2^2 \\
= \;&\Sigma ||(I - E_{A_o})(e_{1t}^q a_t e_{i1}^q)||_2^2 \\
\leq \;&\Sigma \, \tau(q) k_n^{-1} . 4^{-(n+1)} \\
&\qquad\qquad\qquad\qquad \text{for } 1 \leq t \leq n+1 \text{ by (5)} \\
= \;&4^{-(n+1)}
\end{aligned}
$$

on recalling that $\Sigma\{q : q \in Q(n)\} = 1$ and that $1 \leq i \leq k_n$.

The inductive construction of Z_{n+1}, A_{n+1}, N_{n+1} is done with properties (1)–(4). Let $N = (\cup N_n)^{-w}$ and $B = (\cup A_n)^{-w}$, where $^{-w}$ denotes the weak closure.

Then $B \subseteq N$, $B \subseteq A$, and by (4), $B = A$ since the $|| \cdot ||_2$ density of the unit ball of B in that of A implies the weak density.

Now $N' \cap M \subset A' \cap M = A$, since A is maximal abelian in M; thus $N' \cap M = N' \cap A = N' \cap N$. By (1) of the inductive construction, the sequence N_n satisfies the hypotheses of Lemma 7 so that $N' \cap N = Z$ and N has trivial relative commutant in M as required.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF EDINBURGH
EDINBURGH, EH9 3JZ.
U.K.

DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION, TX 77843
USA