MATH. SCAND. 84 (1999), 11–22

ANNIHILATING COMPLEXES OF MODULES

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Abstract

For a complex X of modules over a commutative ring R the *weak annihilator* is defined by $\mathrm{Ann}_R X = \bigcap_{i\in \mathsf{Z}} \mathrm{Ann}_R H_i(X)$, the intersection of the annihilators of the homology modules, *homotopy annihilator* $\mathrm{hann}_R X$ as the kernel of the map $R \to \mathrm{H}_0(\mathbf{R}\mathrm{Hom}_R(X, X))$, and when X is homologically bounded, say $\mathrm{H}_i(X) = 0$ for $|i| > n$, the *small annihilator* is $\mathrm{ann}_R X = \mathrm{Ann}_R H_{-n}(X) \cdots \mathrm{Ann}_R H_n(X)$, the product of the annihilators of the homology modules. Various properties of annihilators are investigated; in particular it is proved that for suitably bounded complexes X and Y the homotopy annihilator $\mathrm{hann}_R X$ is contained in $\mathrm{hann}_R \mathbf{R}\mathrm{Hom}_R(X, Y)$ and $\mathrm{hann}_R(X \otimes^{\mathbf{L}}_R Y)$.

Introduction

Let R be a commutative ring with unity. For an R-module M its annihilator $\mathrm{Ann}_R M$ carries a substantial amount of information on the structure of M. As for a complex X of R-modules, some structural information is encoded in the annihilators of the homology modules $\mathrm{Ann}_R \mathrm{H}_i(X)$ for $i \in \mathsf{Z}$.

To reflect the structure of ideals $\mathrm{Ann}_R \mathrm{H}_i(X)$, various inclusion relations were investigated in literature (cf. for example [4], [5]). A classical example is a textbook result, given in [3].

TEXTBOOK THEOREM. *Assume $K = K(\mathbf{x}, R)$ is a Koszul complex on the variables $\mathbf{x} = (x_1, \dots, x_n)$ over the ring R. Then the ideal $(x_1, \dots, x_n)$ annihilates the homology modules $\mathrm{H}_i(K \otimes_R X)$ for any complex X and for all $i \in \mathsf{Z}$.*

More elaborate results concerning annihilators of homology modules of a dualizing complex were given in [4]:

THEOREM 1 of [4]. *Given a commutative Noetherian local ring $(R, \mathfrak{m})$ of dimension n, let $F = 0 \to F_0 \to \cdots \to F_{-r} \to 0$ be a complex of finite free modules over R with $H_i(F)$ of finite length for all i. Assume the ring R possesses a dualising complex $D = 0 \to D_n \to \cdots \to D_0 \to 0$. Set $\mathfrak{a}_i = \mathrm{Ann}_R \mathrm{H}_i(D)$. Then $\mathfrak{a}_j \cdots \mathfrak{a}_0 \subseteq \mathrm{Ann}_R H_{-j}(F)$ for $j = 0, 1, \dots, n$.*

Received October 29, 1996.

Finally, [5] gives a number of inclusion theorems on annihilators of local cohomology modules; the results there are more subtle than those concerning non-vanishing of local cohomology. We cite

SATZ 2.3.1 of [5]. *For a complex* $X = 0 \to X_0 \to \cdots \to X_{-s} \to 0$ *of finite modules over a Noetherian local ring* $(R, \mathfrak{m})$ *of dimension* n *set* $\mathfrak{a}_i = \mathrm{Ann}_R \mathrm{H}_{-j}(X)$. *Then one has* $\mathfrak{a}_0 \cdots \mathfrak{a}_{-j} \subseteq \mathrm{Ann}_R \mathrm{H}^j_{\mathfrak{m}}(X)$ *for all* $j = 0, 1, \ldots, s$.

This paper is an attempt to find a unified approach to results incorporating annihialtors of homology modules; to provide a language for intepretating such results and to give a correct framework for possible generalizations. In this paper we define the *small annihilator* ann C of a homologically bounded complex C (that is $\mathrm{H}_i(C) = 0$ for $|i| \gg 0$) and the *homotopy annihilator* hann$X$ of any complex X to be certain ideals in R invariant under homotopy equivalence. We also introduce the *weak* or *naive annihilator* Ann X of a complex X as the intersection of annihilators of all its homology modules. All three annihilators are really extensions of a usual module annihilator concept (for a module M, all three of them are equal to the module-theoretic annihilator of M); moreover, they are all invariant under quasi-isomorphisms when passing to the derived category setting.

In section 2 we present a number of elementary properties of all three annihilators. Furthermore, we extend the inclusion result for linear module functors[1] to functors $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes^{\mathbf{L}}_R -$. Some examples are also given, mainly to illustrate the relation between small, homotopy and weak annihilators.

The most general question to be asked is: given a functor T taking complexes to complexes, what possible inclusions can exist between annihilators of a complex X and $T(X)$? However, in the derived category setting the conditions to be posed on T in order to get the inclusion hann$X \subseteq$ hann$T(X)$ are not known to the author; therefore the Annihilator Theorem and its corollaries discussed in Section 3 incorporate only small and weak annihilators. There, various inclusion theorems are proved for such annihilators; one then has Theorem 1 of [4] and Sätze 2.1.3, 2.3.3 of [5], as corollaries. Futhermore, a couple of applications to the study of dualizing complexes are formulated and proved.

[1] $\mathrm{Ann}_R\, M$ is contained in $\mathrm{Ann}_R\, F(M)$ for any *linear* functor F: R-modules $\to$ R-modules (we say that the functor F is linear if $F(a_X) = a_{F(X)}$ for any $a \in R$; a_X stands for multiplication by a on X).

1. Homological algebra of complexes of modules

Complexes. A complex X of R-modules is a sequence of maps $\{\partial_i : X_i \to X_{i-1}\}_{i\in \mathsf{Z}}$ where $\partial_i\partial_{i+1} = 0$ for all i. We use the following notation:

$$\begin{aligned} \mathrm{Z}_n^X &= \mathrm{Ker}\ \partial_n^X, \text{ the } \textit{kernel of } \partial_n^X, \\ \mathrm{B}_n^X &= \mathrm{Im}\ \partial_{n+1}^X, \text{ the } \textit{image of } \partial_{n+1}^X, \\ \mathrm{C}_n^X &= \mathrm{Coker}\ \partial_{n+1}^X, \text{ the } \textit{cokernel of } \partial_{n+1}^X, \\ \mathrm{H}_n(X) &= \mathrm{Z}_n^X / B_n^X, \text{ the } \textit{n-th homology module.} \end{aligned}$$

Then *infimum*, *supremum* and *amplitude* of X are defined by

$$\begin{aligned} \inf X &= \inf\{n \in \mathsf{Z} \mid \mathrm{H}_n(X) \neq 0\}, \\ \sup X &= \sup\{n \in \mathsf{Z} \mid \mathrm{H}_n(X) \neq 0\} \text{ and} \\ \mathrm{amp}\ X &= \sup X - \inf X. \end{aligned}$$

The *truncated* complexes $\mathscr{T}_{m\subset} X$ and $\mathscr{T}_{\supset_n} X$ are given by

$$\mathscr{T}_{m\subset} X = 0 \longrightarrow \mathrm{C}_m^X \xrightarrow{\overline{\partial}_m^X} X_{m-1} \xrightarrow{\partial_{m-1}^X} X_{m-2} \xrightarrow{\partial_{m-2}^X} \cdots$$

$$\mathscr{T}_{\supset_n} X = \cdots \xrightarrow{\partial_{n+3}^X} X_{n+2} \xrightarrow{\partial_{n+2}^X} X_{n+1} \xrightarrow{\tilde{\partial}_{n+1}^X} \mathrm{Z}_n^X \longrightarrow 0,$$

where $\overline{\partial}_m^X$ and $\widetilde{\partial}_{n+1}^X$ are the induced maps.

For $n \in \mathsf{Z}$ we denote by $\mathscr{S}^n X$ the complex with $(\mathscr{S}^n X)_i = X_{i-n}$ and $\partial_i^{\mathscr{S}^n X} = (-1)^n \partial_{i-n}^X$. If N is an R-module then the complex $0 \to N \to 0$, concentrated in degree 0, will be also denoted by N.

If Y is another R-complex then a *morphism* $\alpha : X \to Y$ is a collection of R-linear homomorphisms $\{\alpha_n : X_n \to Y_n\}$, with $\partial_n^Y \alpha_n = \alpha_{n-1}\partial_n^X$ for all integers n. A *quasi-isomorphism* is a morphism α such that the induced map $\mathrm{H}_n(\alpha)$ is an isomorphism for all n. Quasi-isomorphisms are denoted by $\simeq$.

Derived functors. The derived category of the category of modules over R is denoted by $\mathscr{D}(R)$. Isomorphisms in $\mathscr{D}(R)$ are labeled with $\simeq$ (as a morphism of complexes is a quasi-isomorphism if and only if its image in $\mathscr{D}(R)$ is an isomorphism, no notational confusion arises).

By $\mathscr{D}_+(R)$, $\mathscr{D}_-(R)$, $\mathscr{D}_b(R)$, $\mathscr{D}_0(R)$ we will denote the full subcategories of $\mathscr{D}(R)$ defined by $\mathrm{H}_n(X) = 0$ for, respectively $n \ll 0, n \gg 0, |n| \gg 0, n \neq 0$. We also write $\mathscr{D}^f(R)$ for the full subcategory consisting of complexes with $\mathrm{H}_n(X)$ finite for each $n \in \mathsf{Z}$. By means of obvious equivalences $\mathscr{D}_0(R)$ is identified with the category of R-modules and $\mathscr{D}_0^f(R)$ with that of finite R-modules.

The left derived functor of the tensor product functor of R-complexes is denoted by $- \otimes_R^{\mathbf{L}} -$, the right derived functor of the homomorphism functor of R-complexes is denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$ and the right derived functor of the local section functor of R-complexes with the support in the ideal $\mathfrak{a} \subseteq R$ is denoted by $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. The existence of appropriate resolutions (cf. [6]) guarantees then that for arbitrary $X, Y \in \mathscr{D}(R)$ there are complexes $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$, $X \otimes^{\mathbf{L}}_R Y$ and $\mathbf{R}\mathrm{Hom}_R(X, Y)$ which are defined uniquely up to isomorphism in $\mathscr{D}(R)$ and possess the expected functorial properties.

Homological dimensions. For a complex $X \in \mathscr{D}(R)$ define the *projective, injective* and *flat* dimension of X by

$$\begin{aligned}\mathrm{pd}_R\, X &= \sup_N(\sup\{i \in \mathsf{Z} \mid \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(X, N)) \neq 0\}),\\ \mathrm{id}_R\, X &= \sup_N(\sup\{i \in \mathsf{Z} \mid \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(N, X)) \neq 0\}),\\ \mathrm{fd}_R\, X &= \sup_N(\sup\{i \in \mathsf{Z} \mid \mathrm{H}_{i}(X \otimes^{\mathbf{L}}_R N) \neq 0\}),\end{aligned}$$

where N ranges over all R-modules. As shown in [1], these numerical invariants of X can be defined by the existence of a suitably bounded projective, injective or flat resolution of X.

We also cite the following Characterization Theorems of [1] for homological dimensions.

Flat Dimension Theorem. *For a complex $Y \in \mathscr{D}_b(R)$ its flat dimension* $\mathrm{fd}\, Y \leq n$ *if and only if* $\sup(Y \otimes^{\mathbf{L}}_R Z) \leq n + \sup Z$ *for all* $Z \in \mathscr{D}_b(R)$.

Projective Dimension Theorem. *For a complex $Y \in \mathscr{D}_b(R)$ its projective dimension* $\mathrm{id}\, Y \leq n$ *if and only if* $\inf \mathbf{R}\mathrm{Hom}_R(Y, Z) \geq -n - \sup Z$ *for all* $Z \in \mathscr{D}_b(R)$.

Injective Dimension Theorem. *For a complex $Y \in \mathscr{D}_b(R)$ its injective dimension* $\mathrm{id}\, Y \leq n$ *if and only if* $\inf \mathbf{R}\mathrm{Hom}_R(Z, Y) \geq -n - \sup Z$ *for all* $Z \in \mathscr{D}_b(R)$.

2. Annihilators of a complex

Definition. *For a complex $X \in \mathscr{D}(R)$ define:*

- *Weak annihilator of X by* $\mathrm{Ann}_R\, X = \bigcap_{i \in \mathsf{Z}} \mathrm{Ann}_R\, \mathrm{H}_i(X)$,
- *Homotopy annihilator of X by* $a \in \mathrm{hann}_R\, X \overset{def}{\iff} a_P \sim 0$ *[respectively,* $a_I \sim 0$*] for some (hence all) projective [respectively, injective] resolution(s) of X (it is well-defined, see below!)*
- *If X is bounded, the small annihilator of X by* $\mathrm{ann}_R\, X = \prod_{i \in \mathsf{Z}} \mathrm{Ann}_R\, \mathrm{H}_i(X)$.

Remark. As no boundedness conditions are posed on X in the definition of the homotopy annihilator, K-projective (K-injective) resolutions of X are needed, as defined in [6]. By notation abuse, in what follows, we omit the prefix K-; no ambiguity is caused, as a K-projective resolution of a complex in $\mathscr{D}_+(R)$ is, indeed, a projective one (verbatim for K-injectives and complexes in $\mathscr{D}_-(R)$).

The properties of these annihilators are summarized in the following

THEOREM.
(1) $\operatorname{hann}_R X$ *and* $\operatorname{ann}_R X$ *are well-defined.*
(2) $a \in \operatorname{Ann} X \Leftrightarrow \mathrm{H}(a_X) = 0$.
(3) $\operatorname{hann} X \subseteq \operatorname{Ann} \dot{X}$.
(4) $\operatorname{ann} X \subseteq \operatorname{hann} X$ *if* $X \in \mathscr{D}_b(R)$.
(5) $\operatorname{hann} X \subseteq \operatorname{hann} T(X)$ *where* T *is any of the functors* $\mathbf{R}\mathrm{Hom}_R(-, Y)$, $- \otimes^{\mathbf{L}}_R B$ *or* $\mathbf{R}\mathrm{Hom}_R(Z, -)$ *for* $Y, Z, B \in \mathscr{D}(R)$.

PROOF. (1) The small annihilator is well-defined since for a complex $X \in \mathscr{D}_b(R)$ we have $\operatorname{Ann}_R \mathrm{H}_i(X) = R$ for all but finitely many $i \in \mathsf{Z}$.

If $P \xrightarrow{\simeq} X \xrightarrow{\simeq} I$ are projective and injective, respectively, resolutions of X, then $a_P \sim 0 \Leftrightarrow a_I \sim 0$. Namely, we have the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & \mathrm{H}_0(\mathrm{Hom}_R(P,P)) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\cong} \\ \mathrm{H}_0(\mathrm{Hom}_R(I,I)) & \xrightarrow{\cong} & \mathrm{H}_0(\mathrm{Hom}_R(P,I)) \end{array}$$

Since a_P is homotopic to zero if and only if $a_P \in \mathrm{B}_0^{\mathrm{Hom}_R(P,P)}$ (the same is true for a_I) – and thus a is in the kernel of both φ and π, we get that $\operatorname{hann}_R X$ for a complex $X \in \mathscr{D}(R)$ is also well-defined.

(2) Multiplication by a annihilates all homology modules of X – that is, acts like zero map on the complex $\mathrm{H}(X)$ – if and only if $\mathrm{H}(a_X) = 0$.

(3) If P (or I) is a projective (injective) resolution of X such that $a_P \sim 0$ ($a_I \sim 0$) then $\mathrm{H}(a_P) = 0$ and $a \in \operatorname{Ann}_R X$.

(4) Take a projective resolution $P \xrightarrow{\simeq} X$. Then $\operatorname{ann}_R X = \operatorname{ann}_R P$; we also can safely assume that $P_i = 0$ for $i < 0$ (otherwise, set $\widehat{P} = \mathscr{S}^{-\inf X} P$; then $\mathrm{Hom}(P, P)$ is equal to $\mathrm{Hom}(\widehat{P}, \widehat{P})$); let also s denote $\sup P$. Pick an element $b \in \operatorname{ann}_R X$; we can assume that $b = a_0 a_1 \cdots a_s$, where $a_i \in \operatorname{Ann} \mathrm{H}_i(P)$ for $i = 0, \ldots, s$ (as an arbitrary element in $\operatorname{ann}_R X$ is a sum of such). We will prove that $b_P \sim 0$ by explicitly constructing the needed homotopy.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_s & \longrightarrow & P_{s-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & & & \downarrow{\scriptstyle b} & & \\ \cdots & \longrightarrow & P_s & \longrightarrow & P_{s-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & 0 \end{array}$$

We have, that $a_i \mathrm{Z}_i^P \subseteq \mathrm{B}_i^P$ for all $s \geq i \geq 0$, and $\mathrm{Z}_j^P = \mathrm{B}_j^P$ for $j \geq s$. For $i \geq s$ we set $a_i = 1$ and define inductively

$\alpha_i = 1_{P_i}, \ \widetilde{\alpha}_0 = \alpha_0$

σ_i extending $a_i\widetilde{\alpha}_i$ by means of the following diagram:

$$\begin{array}{ccccc} & & P_i & & \\ & \swarrow^{\sigma_i} & \downarrow{\scriptstyle a_i\tilde{\alpha}_i} & & \\ P_{i-1} & \xrightarrow{\partial^P_{i-1}} & \partial^P_{i+1} P_{i+1} & \longrightarrow & 0 \end{array}$$

and, finally, $\widetilde{\alpha}_{i+1} = a_i \cdots a_0 \alpha_{i+1} - \sigma_i \partial^P_{i+1}$.

For every i, $\partial_i \widetilde{\alpha}_i = 0$, therefore $a_i\widetilde{\alpha}_i$ maps P_i into $\partial_{i+1}P_{i+1}$ and thus the extension σ_i is well-defined.

As follows from the construction, the needed homotopy map will be

$$\sigma = (\ldots, \ 0, \ \sigma_s, \ \ldots, \ a_s \cdots a_{i+1}\sigma_i, \ \ldots, \ a_s \cdots a_1\sigma_0, \ 0, \ \ldots).$$

(5) Let $P \xrightarrow{\simeq} X \xrightarrow{\simeq} J$ be projective and injective resolutions of X; choose also a projective resolution $F \xrightarrow{\simeq} B$, and a projective, respectively injective resolutions $L \xrightarrow{\simeq} Z$ and $Y \xrightarrow{\simeq} I$. Then the corresponding representatives for $T(X)$ will be $\mathrm{Hom}(P, I)$, $\mathrm{Hom}(L, J)$ and $P \otimes F$. By definition of Hom-functor and tensor product for complexes, we get ($a \in \mathrm{hann}_R X$):

$$(*) \qquad a_P \sim 0 \ (a_J \sim 0) \Longrightarrow a_{\mathrm{Hom}(P,I)}, \ a_{P\otimes F}, \ a_{\mathrm{Hom}(L,J)} \sim 0.$$

Let us prove now that all three representatives are, indeed, resolutions (injective, projective and injective, respectively) of the corresponding $T(X)$. Pick an arbitrary acyclic complex E. By adjointness, we have the following isomorphisms:

$$\mathrm{Hom}(E, \mathrm{Hom}(P, I)) \cong \mathrm{Hom}(E \otimes P, I),$$
$$\mathrm{Hom}(P \otimes F, E) \cong \mathrm{Hom}(P, \mathrm{Hom}(F, E)).$$

F is K-projective; therefore $\mathrm{Hom}(F, E)$ is acyclic. As K-projectives are K-flat ([6], Prop. 5.8), $E \otimes P$ is also acyclic. By injectivity of I and projectivity of P we get that right-hand sides above are acyclic, which implies that $P \otimes F$ is K-projective and $\mathrm{Hom}(P, I)$ is K-injective. The same argument works for $\mathrm{Hom}(L, J)$ and thus $P \otimes F$, $\mathrm{Hom}(P, I)$ and $\mathrm{Hom}(L, J)$ can be used as resolutions for $T(X)$ in each case.[2] Then by $(*)$, a lies in $\mathrm{hann}_R\, \mathbf{R}\mathrm{Hom}_R(X, Y) \cap \mathrm{hann}_R(X \otimes^{\mathbf{L}}_R B) \cap \mathrm{hann}_R\, \mathbf{R}\mathrm{Hom}_R(Z, X)$.

[2] When complexes involved are in $\mathscr{D}_+(R)$, respectively $\mathscr{D}_-(R)$, the K-resolutions become the usual ones (bounded properly) and $\mathrm{Hom}(P, I)$, $P \otimes F$ and $\mathrm{Hom}(L, J)$ are then (the usual) injective, projective and injective resolutions of the corresponding $T(X)$.

For an R-module M all three annihilators of M are equal to the usual, module-theoretic annihilator of M.

Remark. The following argument shows that (5) is also true for the functor $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$, if the complex involved is in $\mathscr{D}_{-}(R)$.

Take the (usual) injective resolution $X \xrightarrow{\simeq} I$ and $a \in \operatorname{hann} X$. Then $\Gamma_{\mathfrak{a}}(I)$ represents $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$ and consists of injective modules (and, therefore, is a resolution of $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$). As $\Gamma_{\mathfrak{a}}(I)$ is a subcomplex of I, the maps a_I and $a_{\Gamma_{\mathfrak{a}}(I)}$ are homotopic to zero simultaneously. Therefore, $\operatorname{hann} X \subseteq \operatorname{hann} \mathbf{R}\Gamma_{\mathfrak{a}}(X)$.

Examples. We illustrate the given definition and properties.

Example 1. If $R = \mathsf{Z}/(8)$, $X = 0 \longrightarrow R \xrightarrow{4} R \xrightarrow{4} R \longrightarrow 0$, $a = 4$, then $a \in \operatorname{Ann} X$ but $a \notin \operatorname{hann} X$ (note that X is bounded and consists of modules that are both projective and injective).

Example 2. Consider the (short) Koszul complex $K_a = 0 \longrightarrow R \xrightarrow{a} R \longrightarrow 0$ for $a \in R$. Then $\mathrm{H}_0(K_a) = R/(a)$, $\mathrm{H}_1(K_a) = \{x | ax = 0\}$. As for anihilators, $\operatorname{Ann}_R H_1(K_a) \supseteq (a) = \operatorname{Ann}_R H_0(K_a)$ and thus $\operatorname{Ann}_R K_a = (a) = \operatorname{hann}_R K_a$ but $\operatorname{ann}_R K_a$ might be smaller.

If, e.g. $R = \mathsf{Z}/(8), a = 2$, then $\mathrm{H}_0(K_a) = \mathrm{H}_1(K_a) \cong \mathsf{Z}/(2)$ and $\operatorname{ann}_R K_a = (4) \neq \operatorname{Ann}_R K_a = (2)$.

In general, as the Koszul complex $K(\mathbf{x}, R)$ on the variables $\mathbf{x} = (x_1, \dots, x_n)$ is a bounded complex of free modules and thus $X \otimes_R K(\mathbf{x}, R)$ represents $X \otimes_R^{\mathbf{L}} K(\mathbf{x}, R)$, the inclusion $(x_1, \dots, x_n) = \operatorname{hann} K(\mathbf{x}, R) \subseteq \operatorname{hann}(X \otimes_R K(\mathbf{x}, R))$ is a consequence of (5). We also get the result from [3], Theorem 16.4 as an easy corollary:

Corollary. $(x_1, \dots, x_n) \subseteq \operatorname{Ann}_R \mathrm{H}_i(X \otimes_R K(\mathbf{x}, R))$ *for any complex* X.

Example 3. Take the ring $R = k[[x, y]]/x(x, y)$; define $\widetilde{x}$ and $\widetilde{y}$ as images of x and y under the residue map $k[[x, y]] = Q \to R$. Let now D_R denote the dualizing complex of R (see [2], Prop.V.2.1 for a definition and basic properties of D_R). Then D_R is quasi-isomorphic to a complex $\mathbf{R}\mathrm{Hom}_Q(R, Q)$ (since Q is regular, thus Gorenstein), considered as a complex of R-modules. The complex

$$L = 0 \longrightarrow Q \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} Q \oplus Q \xrightarrow{[x^2 \quad xy]} Q \longrightarrow 0$$

is a Q-projective resolution of R; thus $\mathrm{Hom}_Q(L, Q)$ represents D_R:

$$\mathrm{Hom}_Q(L, Q) = 0 \longrightarrow Q \xrightarrow{\left[\begin{smallmatrix} x^2 \\ xy \end{smallmatrix}\right]} Q \oplus Q \xrightarrow{[-y \quad x]} Q \longrightarrow 0,$$

thus $1 = \dim R = \operatorname{amp} D_R$; $\mathrm{H}_0(D_R) = k$, $\mathrm{H}_1(D_R) = R/(\widetilde{x})$. We see that $\operatorname{ann} D_R = 0$, and $\operatorname{Ann} D_R = (\widetilde{x})$.

If T stands for a functor $\mathbf{R}\mathrm{Hom}_R(D_R, -)$ then $T(D_R) \simeq R$, $\operatorname{Ann} T(D_R) = 0$ and $\operatorname{Ann} T(D_R) \not\supseteq \operatorname{Ann} D_R$.

3. Annihilator theorems

We would like now to extend the Annihilator Theorem for modules (for any linear functor F: R-modules $\to$ R-modules there is an inclusion $\operatorname{Ann}_R M \subseteq \operatorname{Ann}_R F(M)$) to complexes and functors $\mathscr{D}(R) \to \mathscr{D}(R)$. The ideal thing to prove would of course be that $\operatorname{hann}_R X \subseteq \operatorname{hann}_R T(X)$ for an appropriate class of functors. However, nothing is known to the author about conditions to be imposed on T; thus, in what follows we will deal with small and large annihilators only.

First, we formulate and prove the Annihilator Theorem in the most general setting, namely for a (possibly contravariant) *linear TP*[3] functor L: $\mathscr{D}(R) \to \mathscr{D}(R)$.

The Annihilator Theorem. *Given a linear TP functor L: $\mathscr{D}(R) \to \mathscr{D}(R)$ and a complex $X \in \mathscr{D}_b(R)$, we have the following inclusion:*

$$\operatorname{ann} X \subseteq \operatorname{Ann} L(X).$$

Remark. As we see from the example 3 in Section 2, this inclusion *cannot* be strengthened to $\operatorname{Ann} X \subseteq \operatorname{Ann} L(X)$.

Proof. The proof is carried out only for a covariant L as it can be used almost verbatim in the contravariant case.

We will use induction on $\operatorname{amp} X = \sup X - \inf X$. For induction base take X with zero amplitude. Then X is quasi-isomorphic (up to a shift) to the module $\mathrm{H}_0(X)$ and $\operatorname{ann} X = \operatorname{Ann}_R \mathrm{H}_0(X) \subseteq \operatorname{Ann}_R \mathrm{H}_i(L(\mathrm{H}_0(X))$ for all $i \in \mathsf{Z}$.

Let ℓ denote $\sup X$. Assume the theorem is true for all complexes with smaller amplitude. Consider the distinguished triangle $(\mathscr{S}^\ell \mathrm{H}_\ell(X), X, \mathscr{T}_{\ell-1\subset} X)$ in $\mathscr{D}(R)$ (See [2], Lemma I.7.2). By applying L to it we get another distinguished triangle $(L(\mathscr{S}^\ell \mathrm{H}_\ell(X)), L(X), L(\mathscr{T}_{\ell-1\subset} X))$ since L is TP and the long homology sequence

$$\cdots \longrightarrow \mathrm{H}_i(L[\mathscr{S}^\ell \mathrm{H}_\ell(X)]) \longrightarrow \mathrm{H}_i(L(X)) \longrightarrow \mathrm{H}_i(L(\mathscr{T}_{\ell-1\subset} X)) \longrightarrow \cdots$$

Then we know that

$$(*) \qquad \operatorname{Ann} \mathrm{H}_i(L(X)) \supseteq \operatorname{Ann} \mathrm{H}_i(L[\mathscr{S}^\ell \mathrm{H}_\ell(X)]) \cdot \operatorname{Ann} \mathrm{H}_i(L(\mathscr{T}_{\ell-1\subset} X))$$

[3] We say that L is *TP* (triangle-preserving) when it takes distinguished triangles into distinguished triangles; linearity means that $L(a_X) = a_{L(X)}$ for all $X \in D(R), a \in R$ (here a_X denotes the multiplication by a on X).

By the induction hypothesis $\operatorname{Ann} \mathrm{H}_i(L(\mathscr{T}_{\ell-1\subset}X)) \supseteq \operatorname{ann} \mathscr{T}_{\ell-1\subset}X$. Now $\mathrm{H}_i \circ L \circ \mathscr{S}^\ell(-)$ is linear and $\mathrm{H}_\ell(X)$ is a module, so $\operatorname{Ann} \mathrm{H}_i(L(\mathscr{S}^\ell \mathrm{H}_\ell(X))) \supseteq \operatorname{Ann}_R \mathrm{H}_\ell(X)$. Substituting this into $(*)$ we get

$$\operatorname{Ann} \mathrm{H}_i(L(X)) \supseteq \operatorname{ann} X \text{ for all } i \in \mathsf{Z},$$

and we are done.

For "standard" commutative algebra functors $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes_R^{\mathbf{L}} -$ the Annihilator Theorem can be strengthened considerably, provided certain restrictive conditions are posed on one of the arguments. Theorems 1, 2 and 3 below are typical examples of this approach.

REMARK. We use the notation X/Y for $\mathbf{R}\mathrm{Hom}_R(Y, X)$.

THEOREM 1. *For* $X \in \mathscr{D}_b(R)$, $Y \in \mathscr{D}_b(R)$ *with* $\operatorname{pd} Y < \infty$ *there is an inclusion* $\operatorname{ann} \mathscr{T}_{j\subset}X \subseteq \operatorname{Ann} \mathscr{T}_{j-\operatorname{pd} Y\subset}(X/Y)$.

PROOF. We apply $-/Y$ to the distinguished triangle $(\mathscr{T}_{\supset j+1}X,\ X,\ \mathscr{T}_{j\subset}X)$ and take the long exact homology sequence:

$$\cdots \longrightarrow \mathrm{H}_i([\mathscr{T}_{\supset j+1}X]/Y) \longrightarrow \mathrm{H}_i(X/Y) \longrightarrow \mathrm{H}_i([\mathscr{T}_{j\subset}X]/Y) \longrightarrow \cdots$$

By Projective Dimension Theorem $\inf([\mathscr{T}_{\supset j+1}X]/Y) \geq j+1-\operatorname{pd} Y$. Thus, for $i \leq j - \operatorname{pd} Y$ the first term in this exact sequence is zero, i.e.

$$0 \to \mathrm{H}_i(X/Y) \to \mathrm{H}_i([\mathscr{T}_{j\subset}X]/Y) \quad \text{is exact;}$$

thus $\operatorname{Ann} \mathrm{H}_i(X/Y) \supseteq \operatorname{Ann} \mathrm{H}_i([\mathscr{T}_{j\subset}X]/Y)$.

By the previous theorem, $\operatorname{Ann} \mathrm{H}_i([\mathscr{T}_{j\subset}X]/Y) \supseteq \operatorname{ann} \mathscr{T}_{j\subset}X$, and letting i range over all integers $\leq j - \operatorname{pd} Y$ we are done.

It is natural to formulate a dual statement.

THEOREM 2. *For* $X \in \mathscr{D}_b(R)$ *with* $\operatorname{id} X < \infty$, $Y \in \mathscr{D}_b(R)$ *there is an inclusion* $\operatorname{ann} \mathscr{T}_{\supset j}Y \subseteq \operatorname{Ann} \mathscr{T}_{-j-\operatorname{id} X\subset}(X/Y)$.

PROOF. Apply $X/-$ to the distinguished triangle $(\mathscr{T}_{\supset j}Y,\ Y,\ \mathscr{T}_{j-1\subset}Y)$ and take the long exact homology sequence:

$$\cdots \longrightarrow \mathrm{H}_i(X/[\mathscr{T}_{j-1\subset}Y]) \longrightarrow \mathrm{H}_i(X/Y) \longrightarrow \mathrm{H}_i(X/[\mathscr{T}_{\supset j}Y]) \longrightarrow \cdots$$

By Injective Dimension Theorem $\inf(X/[\mathscr{T}_{j-1\subset}Y]) \geq -\operatorname{id} X - j + 1$ and therefore the module $\mathrm{H}_i(X/[\mathscr{T}_{j-1\subset}Y])$ is zero for all $i \leq -j - \operatorname{id} X$, thus

$$0 \to \mathrm{H}_i(X/Y) \to \mathrm{H}_i(X/[\mathscr{T}_{\supset j}Y]) \text{ is exact;}$$

therefore $\operatorname{Ann} \mathrm{H}_i(X/Y) \supseteq \operatorname{Ann} \mathrm{H}_i(X/[\mathscr{T}_{\supset j}Y])$. The latter contains $\operatorname{ann} \mathscr{T}_{\supset j}Y$. Let now i range over all integers $\leq -j - \operatorname{id} X$.

Finally, there is a similar result for the $\otimes^{\mathbf{L}}_R$-functor.

THEOREM 3. *For* $X \in \mathscr{D}_b(R)$, $Y \in \mathscr{D}_b(R)$ *with* $\operatorname{fd} Y < \infty$ *there is an inclusion* $\operatorname{ann} \mathscr{T}_{\supset_j} X \subseteq \operatorname{Ann} \mathscr{T}_{\supset_{j+\operatorname{fd} Y}}(X \otimes^{\mathbf{L}}_R Y)$.

PROOF. The Flat Dimension Theorem implies that $\mathrm{H}_i([\mathscr{T}_{_{j-1}\subset} X] \otimes^{\mathbf{L}}_R Y) = 0$ for $i \geq \operatorname{fd} Y + j$. Therefore, taking a distinguished triangle $(\mathscr{T}_{\supset_j} X,\ X,\ \mathscr{T}_{_{j-1}\subset} X)$, applying $- \otimes^{\mathbf{L}}_R Y$ and taking the long exact homology sequence we get that

$$\mathrm{H}_i([\mathscr{T}_{\supset_j} X] \otimes^{\mathbf{L}}_R Y) \to \mathrm{H}_i(X \otimes^{\mathbf{L}}_R Y) \to 0 \text{ is exact and}$$
$$\operatorname{Ann}_R \mathrm{H}_i(X \otimes^{\mathbf{L}}_R Y) \supseteq \operatorname{Ann}_R \mathrm{H}_i([\mathscr{T}_{\supset_j} X] \otimes^{\mathbf{L}}_R Y).$$

The latter ideal contains $\operatorname{ann} \mathscr{T}_{\supset_j} X$ for all $i \geq \operatorname{fd} Y + j$.

One also has a number of corollaries; none of them is new but nevertheless it is an illustration to the approach.

COROLLARY 1. *For a dualizing complex D over R one has the* $\operatorname{ann} D = 0$.

PROOF. By definition of D we have $R \simeq \mathbf{R}\mathrm{Hom}_R(D, D)$. Therefore,

$$\operatorname{ann} D \subseteq \operatorname{Ann} \mathbf{R}\mathrm{Hom}_R(D, D) = \operatorname{Ann} R = 0.$$

Note, that as a consequence of (5) in the Characterization Theorem from Section 2, we get a stronger result: $\operatorname{hann} D \subseteq \operatorname{hann} R = 0$.

We also have Paul Roberts' result as a

COROLLARY 2 (THEOREM 1 OF [4]). *Given a commutative Noetherian local ring* $(R, \mathfrak{m})$ *of dimension n, let* $F = 0 \to F_0 \to \cdots \to F_{-r} \to 0$ *be a complex of finite free modules over R with* $H_i(F)$ *of finite length for all i. Assume the ring R possesses a dualising complex* $D = 0 \to D_n \to \cdots \to D_0 \to 0$. *Then* $\operatorname{ann} \mathscr{T}_{_j\subset} D \subseteq \operatorname{Ann}_R H_{-j}(F)$ *for* $j = 0,\ 1,\ \ldots,\ n$.

PROOF. Since F has homology of finite length and thus $\operatorname{Supp} F = \bigcup_\ell \operatorname{Supp} \mathrm{H}_\ell(F) \subseteq V(\mathfrak{m}) = \{\mathfrak{m}\}$, $F \simeq \mathbf{R}\Gamma_{\mathfrak{m}}(F)$. The latter complex is just $(\mathbf{R}\mathrm{Hom}_R(F, D))^\vee$ by the Local Duality Theorem as stated in [2], Thm.V.6.2 ($^\vee$ denotes Matlis dual: $X^\vee = \mathrm{Hom}(X, \mathrm{E}(k))$, where $\mathrm{E}(k)$ is the injective envelope of $k = R/\mathfrak{m}$). We have

$$\begin{aligned}\mathrm{H}_{-j}(F) = \mathrm{H}_{-j}(\mathbf{R}\Gamma_{\mathfrak{m}}(F)) &= \mathrm{H}_{-j}([\mathbf{R}\mathrm{Hom}_R(F, D)]^\vee) = \\ &= [\mathrm{H}_j(\mathbf{R}\mathrm{Hom}_R(F, D))]^\vee;\end{aligned}$$

and thus $\operatorname{Ann}_R \mathrm{H}_{-j}(F) = \operatorname{Ann}_R \mathrm{H}_j(\mathbf{R}\mathrm{Hom}_R(F, D))$. Now, the complex F is of non-positive projective dimension, so Theorem 1 applies:

$\mathrm{H}_j(\mathbf{R}\mathrm{Hom}_R(F,D))$ (and, therefore, $\mathrm{H}_{-j}(F)$) is annihilated by $\operatorname{ann}\mathscr{T}_{j\subset}D$.

Corollary 3 (Satz 2.3.1 of [5]). *For a complex* $X = 0 \to X_0 \to \cdots \to X_{-s} \to 0$ *of finite modules over a Noetherian local ring* $(R, \mathfrak{m})$ *of dimension* n *one has* $\operatorname{ann}\mathscr{T}_{\supset_{-j}}X \subseteq \operatorname{Ann}_R \mathrm{H}^j_{\mathfrak{m}}(X)$ *for all* $j = 0, 1, \ldots, s$.

Proof. $\mathrm{H}^j_{\mathfrak{m}}(X) = \mathrm{H}_{-j}(\mathbf{R}\Gamma_{\mathfrak{m}}(X))$ by definition of local cohomology modules. Local Duality Theorem implies that $\mathrm{H}_{-j}(\mathbf{R}\Gamma_{\mathfrak{m}}(X)) = [\mathrm{H}_j(\mathbf{R}\mathrm{Hom}_R(X,D))]^{\vee}$, thus $\operatorname{Ann}_R\dot{\mathrm{H}}^j_{\mathfrak{m}}(X) = \operatorname{Ann}_R \mathrm{H}_j(\mathbf{R}\mathrm{Hom}_R(X,D))$.
Since $\operatorname{id} D = 0$, the result follows from Theorem 2.

Finally, there are two Theorems which were stated incorrectly in [5] (Satz 2.3.3 and Korollar 2.3.4) and which we obtain here in their correct form.

Remark. Following [5] (section 2.1), we construct a complex of flat modules K with $\sup K = -\operatorname{depth} R$, $\inf K = -\dim R = -d$ (in particular $\operatorname{fd} K = 0$) such that $\mathbf{R}\Gamma_{\mathfrak{m}}(X) \simeq \mathbf{R}\Gamma_{\mathfrak{m}}(X \otimes^{\mathbf{L}}_R R) \simeq X \otimes^{\mathbf{L}}_R K$ for all $X \in \mathscr{D}^f_b(R)$.

Thus we have the following correction to (2.3.3 of [5]):

Corollary 4. *For a complex* $X = 0 \to X_0 \to \cdots \to X_{-s} \to 0 \in \mathscr{D}^f_b(R)$ *and* $Y \in \mathscr{D}_b(R)$ *of finite flat dimension there are inclusions:*

$$\operatorname{ann}\mathscr{T}_{\supset_{-n}}X \subseteq \operatorname{Ann}\mathscr{T}_{\supset_{-n+\operatorname{fd}Y}}\mathbf{R}\Gamma_{\mathfrak{m}}(X \otimes^{\mathbf{L}}_R Y)$$
$$\operatorname{ann}\mathscr{T}_{\supset_{-n}}\mathbf{R}\Gamma_{\mathfrak{m}}(X) \subseteq \operatorname{Ann}\mathscr{T}_{\supset_{-n+\operatorname{fd}Y}}\mathbf{R}\Gamma_{\mathfrak{m}}(X \otimes^{\mathbf{L}}_R Y),$$

for all $n = 1, 2, \ldots, s$.

Proof. As

$$\mathbf{R}\Gamma_{\mathfrak{m}}(X \otimes^{\mathbf{L}}_R Y) \simeq \mathbf{R}\Gamma_{\mathfrak{m}}(X) \otimes^{\mathbf{L}}_R Y \simeq (X \otimes^{\mathbf{L}}_R K) \otimes^{\mathbf{L}}_R Y \simeq X \otimes^{\mathbf{L}}_R (Y \otimes^{\mathbf{L}}_R K),$$

the first formula follows from Theorem 3, since $\operatorname{fd}(Y \otimes^{\mathbf{L}}_R K) \leq \operatorname{fd} Y$. The same theorem applied to $X \otimes^{\mathbf{L}}_R K$ and Y gives the second one.

To correct the statement of Korollar 2.3.4 we do the following

Observation. For all $X \in \mathscr{D}^f_b(R)$, $Y \in \mathscr{D}^f_b(R)$ of finite projective dimesion we have the isomorphisms

$$\mathbf{R}\mathrm{Hom}_R(Y,X) \simeq \mathbf{R}\mathrm{Hom}_R(Y, X \otimes^{\mathbf{L}}_R R) \simeq X \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(Y,R).$$

Note that $\mathbf{R}\mathrm{Hom}_R(Y,R)$ is also of finite projective dimension: $\operatorname{pd}\mathbf{R}\mathrm{Hom}_R(Y,R) = -\inf Y = \operatorname{fd}\mathbf{R}\mathrm{Hom}_R(Y,R)$.

The correct statement of Korollar 2.3.4 reads:

Corollary 5. *For* $X \in \mathscr{D}_b^f(R)$, $Y \in \mathscr{D}_b^f(R)$ *of finite projective dimension one has*

$$\operatorname{ann} \mathscr{T}_{\supset_{-n}} X \subseteq \operatorname{Ann} \mathscr{T}_{\supset_{-n-\inf Y}} \mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_R(Y, X))$$
$$\operatorname{ann} \mathscr{T}_{\supset_{-n}} \mathbf{R}\Gamma_{\mathfrak{m}}(X) \subseteq \operatorname{Ann} \mathscr{T}_{\supset_{-n-\inf Y}} \mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_R(Y, X)),$$

for all $n = 1, 2, \ldots, s$

Proof. Follows by Observation above and Corollary 4.

Acknowledgements. I would like to thank prof. Hans-Bjørn Foxby for introducing me to Hyperhomological Methods and for his invaluable help and advice during the preparation of this article.

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