COUNTING SUBGROUPS OF NON-EUCLIDEAN CRYSTALLOGRAPHIC GROUPS

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Abstract

A method is obtained for counting the normal subgroups N of a noneuclidean crystallographic group Γ without reflections, with a given finite quotient group Γ/N; this has applications to the enumeration of regular coverings of orbifolds. The method, which involves Möbius inversion and character theory, is also applied to count normal surface subgroups and non-normal subgroups of finite index in Γ.

1. Introduction

The aim of this note is to describe and to illustrate a technique for computing the number $n_\Gamma(G)$ of normal subgroups N of a non-euclidean crystallographic group (or NEC group) Γ, with quotient group Γ/N isomorphic to a given finite group G. The first part of the method (due to P. Hall) uses Möbius inversion to reduce the problem to that of counting homomorphisms from Γ to various subgroups of G; this can be applied to any finitely generated group Γ, and hence to any NEC group. However the second part, the use of character theory to count such homomorphisms, seems to be effective only for those NEC groups which contain no reflections. Several simple illustrative examples are considered, where G is cyclic, dihedral, or of prime exponent. These methods are extensions of those used in [9], where Γ is assumed to be a surface group, that is, it contains neither reflections nor elliptic elements (see also [10] for a general survey of applications of character theory to surfaces). Izquierdo has taken a complementary approach in [8], where reflections are allowed, Γ has genus 0, and G is a dihedral group D_p for some prime p; her methods are completely different, relying on the special structure of these dihedral groups.

One can interpret these enumerative results in terms of orbifold coverings (see [8] for a general account of this connection): if $\mathcal{O}$ is the orbifold $\mathcal{H}/\Gamma$ corresponding to Γ, where $\mathcal{H}$ is the hyperbolic plane, then $n_\Gamma(G)$ is the

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number of equivalence classes of regular coverings of $\mathcal{O}$ with covering group G; the restriction that Γ should have no reflections is equivalent to the condition that the underlying surface of $\mathcal{O}$ should be without boundary. By refining these techniques to count the normal surface subgroups of Γ with quotient-group G, one can also count regular coverings of $\mathcal{O}$ by Klein surfaces, rather than by orbifolds. Similarly, a straightforward extension to non-normal subgroups of finite index allows one to count the non-regular finite coverings of $\mathcal{O}$ with a given permutation group as their monodromy group.

2. Counting normal subgroups

In [5], P. Hall developed a general technique for computing the number $n_\Gamma(G)$ of normal subgroups N of a finitely generated group Γ with a given finite quotient group $\Gamma/N \cong G$. These subgroups N are the kernels of the epimorphisms $\theta : \Gamma \to G$; the set $\mathrm{Epi}\,(\Gamma, G)$ of such epimorphisms is finite (since there are only finitely many ways of mapping the generators of Γ into G), so $n_\Gamma(G)$ is finite. If $\theta_1, \theta_2 \in \mathrm{Epi}\,(\Gamma, G)$, then $\ker \theta_1 = \ker \theta_2$ if and only if $\theta_2 = \theta_1 \circ \alpha$ for some automorphism α of G, so $n_\Gamma(G) = |\mathrm{Epi}\,(\Gamma, G)/\mathrm{Aut}\,G|$, the number of orbits of $\mathrm{Aut}\,G$ acting by composition on $\mathrm{Epi}\,(\Gamma, G)$. Since this action is fixed-point-free, every orbit has length $|\mathrm{Aut}\,G|$ and so

$$n_\Gamma(G) = \frac{|\mathrm{Epi}\,(\Gamma, G)|}{|\mathrm{Aut}\,G|} \; .$$

To count epimorphisms $\Gamma \to G$, one first counts the homomorphisms, and then eliminates those which map Γ onto proper subgroups of G. One can invert the equation

$$|\mathrm{Hom}(\Gamma, G)| = \sum_{K \leq G} |\mathrm{Epi}(\Gamma, K)| \, ,$$

to count epimorphisms in terms of homomorphisms, by introducing the *Möbius function* for G. This assigns an integer $\mu(K)$ to each subgroup K of G by the recursive formula

$$\sum_{H \geq K} \mu(H) = \delta_{K,G} = \begin{cases} 1 & \text{if } K = G, \\ 0 & \text{if } K < G. \end{cases}$$

The equation

$$|\mathrm{Epi}(\Gamma, G)| = \sum_{H \leq G} \mu(H)|\mathrm{Hom}(\Gamma, H)|$$

is then easily deduced, and this immediately gives

(1) $$n_\Gamma(G) = \frac{1}{|\mathrm{Aut}\, G|} \sum_{H \leq G} \mu(H)|\mathrm{Hom}(\Gamma, H)|\,.$$

For many groups G, it is a routine task to find $|\mathrm{Aut}\, G|$ and $\mu(H)$ for all $H \leq G$, so there remains the problem of computing $|\mathrm{Hom}(\Gamma, H)|$ for all $H \leq G$ (or at least, for all those $H \leq G$ with $\mu(H) \neq 0$). We shall use the character theory of finite groups to do this for various NEC groups Γ; for background information on NEC groups and Klein surfaces, see [1] or [14], for finite groups see [7], for character theory see [7] or [12], and for number theory see [6].

3. Orientable NEC groups without reflections

An NEC *group* Γ is a discrete group of isometries of the hyperbolic plane $\mathscr{H}$, with compact quotient space $\mathscr{H}/\Gamma$. First let us take Γ to be an orientable NEC group without reflections (that is, a co-compact Fuchsian group), so that Γ has signature

(S+) $$(g; +; [m_1, \ldots, m_r]; \{-\})$$

where $g, r, m_1, \ldots, m_r$ are integers with $g, r \geq 0$ and $m_i > 1$ for all i. Here g is the genus of $\mathscr{H}/\Gamma$, the symbol $+$ denotes orientability, the integers m_i are the periods of Γ, and $\{-\}$ denotes the absence of reflections. This means that Γ has generators

$$X_i \quad (i = 1, \ldots, r)\,, \quad A_j,\ B_j \quad (j = 1, \ldots, g)$$

and defining relations

$$X_i^{m_i} = 1 \quad (i = 1, \ldots, r)\,, \qquad \prod_{i=1}^{r} X_i . \prod_{j=1}^{g} [A_j, B_j] = 1\,.$$

It follows that the number

$$\sigma_\Gamma(H) = |\mathrm{Hom}\,(\Gamma, H)|$$

of homomorphisms $\Gamma \to H$ is equal to the number of solutions x_i, a_j, b_j in H of the simultaneous equations

(2′) $x_i^{m_i} = 1 \quad (i = 1, \ldots, r)$,

(2″) $\displaystyle\prod_{i=1}^{r} x_i . \prod_{j=1}^{g} [a_j, b_j] = 1$.

This number can be computed by means of the following result, which is proved in [9]. (The case $r = 0$, where Γ is an orientable surface group, is due to Frobenius [3] for $g = 1$, and to Mednykh [11] for $g > 1$. See §7.2 of [13] for similar results.)

THEOREM 1. *The number of solutions of equation* (2″) *in a finite group H, where each x_i lies in some union L_i of conjugacy classes of H, is equal to*

$$|H|^{2g-1} \sum_\chi \left\{ \chi(1)^{2-2g-r} \sum_{x_1 \in L_1} \chi(x_1) \ldots \sum_{x_r \in L_r} \chi(x_r) \right\},$$

where χ ranges over the irreducible complex characters of H.

Note that $\chi(1)$ is the degree of χ, the dimension of the corresponding $\mathbf{C}H$-module. In order to deduce a formula for $\sigma_\Gamma(H)$, we will choose the sets L_i to be the sets of the solutions $x_i \in H$ of equations (2′). First we need some notation. If m is a positive integer, H is a finite group and χ is a complex character of H, then let

$$H[m] = \{ h \in H \mid h^m = 1 \}$$

(which is a union of conjugacy classes of H), and let

$$\chi[m] = \sum_{h \in H[m]} \chi(h) .$$

Then Theorem 1 immediately implies:

COROLLARY 1. *If Γ has signature* (S+) *and H is any finite group, then*

$$\sigma_\Gamma(H) = |H|^{2g-1} \sum_\chi \chi(1)^{2-2g-r} \chi[m_1] \cdots \chi[m_r] ,$$

where χ ranges over the irreducible complex characters of H.

Given the character table of H, this result makes it straightforward to evaluate $\sigma_\Gamma(H)$. Character tables are available for many finite groups: see [2], for instance.

Before considering some examples, it is useful to introduce some notation: we let l denote the least common multiple of the periods $m_1, \ldots, m_r$ of Γ, and for each integer m we let ν_m be the number of i such that m divides m_i.

EXAMPLES. (3.1) Let $H = C_d$, a cyclic group of order d. In this case, one can evaluate $\sigma_\Gamma(H)$ directly, without character theory, for instance by abelianising Γ and using the structure theory of finitely-generated abelian groups. However, it is useful to apply Corollary 1 here as a simple illustration of the method.

There are d irreducible characters χ of H, all of degree 1; these are the homomorphisms $H \to \mathsf{C}^*$, obtained by mapping a generator of H to a d-th root of 1 in C. If m is any positive integer, then $H[m]$ is the unique subgroup of order (m, d) in H, and so

$$\chi[m] = \begin{cases} (m, d), & \text{if } H[m] \leq \ker \chi, \\ \\ 0, & \text{otherwise.} \end{cases}$$

It follows that a character χ makes a non-zero contribution to the formula for $\sigma_\Gamma(H)$ if and only if $\ker \chi$ contains the subgroups $H[m_i]$ for all $i = 1, \ldots, r$, or equivalently $\ker \chi$ contains the subgroup $H[l]$ of order (l, d) which they generate, where $l = \operatorname{lcm}(m_1, \ldots, m_r)$. There are $d/(l, d)$ such characters, each with $\chi[m_i] = (m_i, d)$, so

$$\sigma_\Gamma(C_d) = \frac{d^{2g}}{(l, d)} \cdot \prod_{i=1}^{r} (m_i, d).$$

For instance, if d is a prime p then

$$\sigma_\Gamma(C_p) = \begin{cases} p^{2g-1+\nu_p}, & \text{if } \nu_p > 0, \\ \\ p^{2g}, & \text{if } \nu_p = 0. \end{cases}$$

We can now compute $n_\Gamma(G)$ where $G = C_n$, a cyclic group of order n. First, we have $|\operatorname{Aut}(G)| = \phi(n)$, where ϕ is Euler's function on N. Now G has a unique subgroup $H \cong C_d$ for each divisor d of n, and has no other subgroups; for each such H we have $\mu(H) = \mu(n/d)$, where μ on the right-hand side denotes the Möbius function on N. Equation (1) therefore gives

$$n_\Gamma(C_n) = \frac{1}{\phi(n)} \sum_{d|n} \mu\left(\frac{n}{d}\right) \sigma_\Gamma(C_d)$$

$$= \frac{1}{\phi(n)} \sum_{d|n} \left\{ \mu\left(\frac{n}{d}\right) \frac{d^{2g}}{(l, d)} \cdot \prod_{i=1}^{r} (m_i, d) \right\}.$$

For example, if n is a prime p then we have

$$n_\Gamma(C_p) = \begin{cases} (p^{2g-1+\nu_p} - 1)/(p-1) & \text{if } \nu_p > 0, \\ (p^{2g} - 1)/(p-1) & \text{if } \nu_p = 0. \end{cases}$$

(3.2) Let $H = D_p = \langle a, b \mid a^p = b^2 = (ab)^2 = 1 \rangle$, a dihedral group of order $2p$, where p is an odd prime. Apart from the identity element, this group has one conjugacy class consisting of the p involutions $a^i b$, and $(p-1)/2$ classes $\{a^{\pm i}\}$ of elements of order p. In addition to the principal character χ_1, the other irreducible characters χ of H are the alternating character $\chi_2(a^i b^j) = (-1)^j$, together with $(p-1)/2$ characters χ'_k $(k = 1, \ldots, (p-1)/2)$ given by $\chi'_k(a^i) = \zeta^{ik} + \zeta^{-ik}$ and $\chi'_k(a^i b) = 0$, where ζ is a primitive pth root of unity. It follows that

$$\text{if } (m, 2p) = 2p \quad \text{then} \quad \chi[m] = \begin{cases} 2p & \text{if } \chi = \chi_1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\text{if } (m, 2p) = p \quad \text{then} \quad \chi[m] = \begin{cases} p & \text{if } \chi = \chi_1 \text{ or } \chi_2, \\ 0 & \text{otherwise;} \end{cases}$$

$$\text{if } (m, 2p) = 2 \quad \text{then} \quad \chi[m] = \begin{cases} p + 1 & \text{if } \chi = \chi_1, \\ 1 - p & \text{if } \chi = \chi_2, \\ 2 & \text{otherwise;} \end{cases}$$

$$\text{if } (m, 2p) = 1 \quad \text{then} \quad \chi[m] = \begin{cases} 1 & \text{if } \chi = \chi_1 \text{ or } \chi_2, \\ 2 & \text{otherwise.} \end{cases}$$

If n_h denotes the number of i such that $(m_i, 2p) = h$ then Corollary 1 gives

$$\sigma_\Gamma(D_p) = (2p)^{2g-1} \Big\{ (2p)^{n_{2p}} . p^{n_p} . (p+1)^{n_2} + 0^{n_{2p}} . p^{n_p} . (1-p)^{n_2}$$

$$+ \frac{p-1}{2} . 2^{2-2g-r} . 0^{n_{2p}} . 0^{n_p} . 2^{n_2+n_1} \Big\}$$

$$= (2p)^{2g-1} \Big\{ 2^{\nu_{2p}} . p^{\nu_p} . (p+1)^{n_2} + 0^{n_{2p}} . p^{n_p} . (1-p)^{n_2} + \frac{p-1}{2} . 0^{\nu_p} . 2^{2-2g-\nu_p} \Big\},$$

where we interpret 0^0 as meaning 1.

If we take $G = D_p$ then the subgroups $H \leq G$ are $H = D_p$, a unique subgroup $H = \langle a \rangle$ isomorphic to C_p, p subgroups $\langle a^i b \rangle \cong C_2$, and the trivial subgroup C_1. The values of $\mu(H)$ for these subgroups are $1, -1, -1$ and p respectively, and $|\text{Aut}(D_p)| = p(p-1)$, so equation (1) gives

$$n_\Gamma(D_p) = \frac{1}{p(p-1)} \Big\{ \sigma_\Gamma(D_{2p}) - \sigma_\Gamma(C_p) - p\sigma_\Gamma(C_2) + p\sigma_\Gamma(C_1) \Big\}.$$

We have just computed $\sigma_\Gamma(H)$ for $H = D_p$, and Example 3.1 deals with the remaining subgroups H, so one can substitute these values and obtain $n_\Gamma(D_p)$. As in most cases where $|G|$ is divisible by more than one prime, the resulting general formula is rather unwieldy, so we will omit it and instead give a simple example. Suppose that Γ has signature $(S+)$ with $m_1 = \cdots = m_r = p$ and $r \geq 1$; then we find that $\sigma_\Gamma(H) = 2^{2g}p^{2g-1+r}$, p^{2g-1+r}, 2^{2g} and 1 for $H = D_p, C_p, C_2$ and 1 respectively, so

$$n_\Gamma(D_p) = \frac{(2^{2g} - 1)(p^{2g-2+r} - 1)}{p - 1}.$$

(This formula is also valid in the case $r = 0$, as shown in [9].)

(3.3) If H has prime exponent p then $H[m] = H$ or 1 as p does or does not divide m. In the first case the orthogonality relations for the characters of H give $\chi[m] = |H|$ or 0 as $\chi = \chi_1$ (the principal character) or $\chi \neq \chi_1$, and in the second case each character satisfies $\chi[m] = \chi(1)$, the degree of χ. It follows that if $\nu_p > 0$ (so that $p|m_i$ for some i) then only χ_1 contributes to $\sigma_\Gamma(H)$, and we have

$$\sigma_\Gamma(H) = |H|^{2g-1+\nu_p};$$

if $\nu_p = 0$, on the other hand, then $\chi[m_i] = \chi(1)$ for all χ and all i, so that

$$\sigma_\Gamma(H) = |H|^{2g-1} \sum_\chi \chi(1)^{2-2g}.$$

(In particular, if H is an elementary abelian p-group $C_p \times \cdots \times C_p$, so that there are $|H|$ characters χ of degree 1, then

$$\sigma_\Gamma(H) = \begin{cases} |H|^{2g-1+\nu_p}, & \text{if } \nu_p > 0, \\ |H|^{2g}, & \text{if } \nu_p = 0; \end{cases}$$

this generalises the result in Example 3.1 for $H = C_p$.)

If G has exponent p then so does every non-trivial subgroup $H \leq G$, and one can apply these formulæ, together with the values of $|\text{Aut}(G)|$ and $\mu(H)$, to determine $n_\Gamma(G)$. For example, let G be the unique non-abelian group of order p^3 and exponent p (where $p > 2$). Apart from G itself, the subgroups $H \leq G$ with $\mu(H) \neq 0$ are the $p + 1$ maximal subgroups, all isomorphic to $C_p \times C_p$ and satisfying $\mu(H) = -1$, together with their intersection (the Frattini subgroup $\Phi(G) \cong C_p$, also equal to the centre $Z(G)$), for which $\mu(H) = p$. It follows from the Burnside Basis Theorem ([7], III.3.15) that a pair of elements generate G if and only if their images generate the Frattini

factor group $G/\Phi(G) \cong C_p \times C_p$; any two such pairs are equivalent under a unique automorphism of G, so by counting such pairs we see that $\mathrm{Aut}(G)$ has order $(p^3 - p)(p^3 - p^2) = p^3(p^2 - 1)(p - 1)$. Thus equation (1) becomes

$$n_\Gamma(G) = \frac{1}{p^3(p^2 - 1)(p - 1)}\left\{\sigma_\Gamma(G) - (p + 1)\sigma_\Gamma(C_p \times C_p) + p\sigma_\Gamma(C_p)\right\}.$$

Since G has p^2 irreducible characters of degree 1, and $p - 1$ of degree p, Corollary 1 gives

$$\sigma_\Gamma(G) = \begin{cases} p^{3(2g-1+\nu_p)}, & \text{if } \nu_p > 0, \\ p^{3(2g-1)}(p^2 + (p - 1)p^{2-2g}), & \text{if } \nu_p = 0. \end{cases}$$

We have already evaluated $\sigma_\Gamma(H)$ for the elementary abelian groups $H = C_p \times C_p$ and C_p, so writing $2g - 1 + \nu_p = h$ we obtain

$$n_\Gamma(G) = \begin{cases} p^{h-2}(p^h - 1)(p^{h-1} - 1)/(p^2 - 1)(p - 1), & \text{if } \nu_p > 0, \\ p^{h-1}(p^{h+1} - 1)(p^{h-1} - 1)/(p^2 - 1)(p - 1), & \text{if } \nu_p = 0. \end{cases}$$

4. Non-orientable NEC groups without reflections

Now let Γ be an non-orientable NEC group without reflections, so that Γ has signature

(S−) $(g; -; [m_1, \ldots, m_r]; \{-\})$

where $g, r, m_1, \ldots, m_r$ are integers with $g \geq 1, r \geq 0$ and $m_i > 1$ for all i. Thus Γ has generators

$$X_i \quad (i = 1, \ldots, r), \quad A_j \quad (j = 1, \ldots, g)$$

and defining relations

$$X_i^{m_i} = 1 \quad (i = 1, \ldots, r), \qquad \prod_{i=1}^{r} X_i . \prod_{j=1}^{g} A_j^2 = 1.$$

In this case, the number

$$\sigma_\Gamma(H) = |\mathrm{Hom}\,(\Gamma, H)|$$

of homomorphisms $\Gamma \to H$ is equal to the number of solutions x_i, a_j in H of the simultaneous equations

(3′) $x_i^{m_i} = 1 \quad (i = 1, \ldots, r),$

(3″)
$$\prod_{i=1}^{r} x_i \cdot \prod_{j=1}^{g} a_j^2 = 1 \, .$$

The formula for $\sigma_\Gamma(H)$ is similar to that in the orientable case, but it contains one extra ingredient. Let χ be an irreducible character of H, afforded by a representation ρ. The *Frobenius-Schur indicator* of χ (or of ρ) is defined to be

$$c_\chi = \frac{1}{|H|} \sum_{h \in H} \chi(h^2) \, .$$

We have

$$c_\chi = \begin{cases} 1, & \text{if } \rho \text{ is real,} \\ -1, & \text{if } \chi \text{ is real but } \rho \text{ is not real,} \\ 0, & \text{if } \chi \text{ is not real.} \end{cases}$$

The following result is proved in [9]; the case $r = 0$ is due to Frobenius and Schur [4].

THEOREM 2. *The number of solutions of equation* (3″) *in a finite group H, where each x_i lies in some union L_i of conjugacy classes of H, is equal to*

$$|H|^{g-1} \sum_{\chi} \left\{ c_\chi^g \chi(1)^{2-g-r} \sum_{x_1 \in L_1} \chi(x_1) \cdots \sum_{x_r \in L_r} \chi(x_r) \right\},$$

where χ ranges over the irreducible complex characters of H.

COROLLARY 2. *If Γ has signature* (S−) *and H is any finite group, then*

$$\sigma_\Gamma(H) = |H|^{g-1} \sum_{\chi} c_\chi^g \chi(1)^{2-g-r} \chi[m_1] \cdots \chi[m_r],$$

where χ ranges over the irreducible complex characters of H.

EXAMPLES. (4.1) Let $H = C_d$, a cyclic group of order d. By our earlier description of the characters χ of H, we see that $c_\chi = 0$ unless either $\chi = \chi_1$, the principal character given by $\chi(h) = 1$ for all h, or else d is even and $\chi = \chi_2$, the alternating character which maps a generator of H to -1. Both of these characters have $c_\chi = 1$, so a similar argument to that in the orientable case gives

$$\sigma_\Gamma(C_d) = \delta d^{g-1} \cdot \prod_{i=1}^{g} (m_i, d)$$

where

$$\delta = \begin{cases} 1, & \text{if } d/(l,d) \text{ is odd,} \\ 2, & \text{if } d/(l,d) \text{ is even,} \end{cases}$$

and l is the least common multiple of the periods m_i. Thus if $(m)_2$ denotes the highest power of 2 dividing an integer m, then

$$\delta = \begin{cases} 1, & \text{if } (m_i)_2 \geq (d)_2 \text{ for some } i, \\ 2, & \text{if } (m_i)_2 < (d)_2 \text{ for all } i. \end{cases}$$

For instance, if we take d to be an odd prime p we obtain

$$\sigma_\Gamma(C_p) = p^{g-1+\nu_p},$$

whereas by taking $d = 2$ we have

$$\sigma_\Gamma(C_2) = \begin{cases} 2^{g-1+\nu_2}, & \text{if } \nu_2 > 0, \\ 2^g, & \text{if } \nu_2 = 0. \end{cases}$$

(4.2) The calculation of $\sigma_\Gamma(D_p)$ is similar to that in the orientable case (Example 3.2): the values of $\chi[m]$ are unchanged, and since every irreducible representation of a dihedral group is real, we have $c_\chi = 1$ for all χ. From this, one can deduce a formula for $n_\Gamma(D_p)$: equation (1) is the same as before, and Example 4.1 gives $\sigma_\Gamma(H)$ for the proper subgroups H of D_p. For a simple example, suppose that $m_1 = \cdots = m_r = p$ and $r \geq 1$. Then $\sigma_\Gamma(H) = 2^g p^{g-1-r}$, p^{g-1-r}, 2^g and 1 for $H = D_p, C_p, C_2$ and C_1 respectively, so

$$n_\Gamma(D_p) = \frac{(2^g - 1)(p^{g-2-r} - 1)}{p - 1}.$$

(This formula is the same as that obtained in the orientable case, except that there each exponent g is replaced with $2g$.)

(4.3) Let H have prime exponent p. We can use the values of $\chi[m]$ which we determined in Example 3.3. If $\nu_p > 0$ then since $c_\chi = 1$ for the principal character $\chi = \chi_1$, we have

$$\sigma_\Gamma(H) = |H|^{g-1+\nu_p};$$

if $\nu_p = 0$, however, then

$$\sigma_\Gamma(H) = |H|^{g-1} \sum_\chi c_\chi^g \chi(1)^{2-g}.$$

In this latter case, if p is odd then $c_\chi = 0$ for all $\chi \neq \chi_1$, so that

$$\sigma_\Gamma(H) = |H|^{g-1} = |H|^{g-1+\nu_p},$$

as before, whereas if $p = 2$ then $c_\chi = 1$ and $\chi(1) = 1$ for every χ (since H is abelian), so that

$$\sigma_\Gamma(H) = |H|^g.$$

As in the orientable case, these results extend those in Example 4.1 for $H = C_p$. One can now compute $n_\Gamma(G)$ when G has exponent p. For instance, if G is the non-abelian group of order p^3 and exponent p (where $p > 2$), then as before we have

$$n_\Gamma(G) = \frac{1}{p^3(p^2-1)(p-1)}\left\{\sigma_\Gamma(G) - (p+1)\sigma_\Gamma(C_p \times C_p) + p\sigma_\Gamma(C_p)\right\}.$$

Since p is odd, the above results give $\sigma_\Gamma(H) = |H|^h$ for each subgroup $H \leq G$ on the right-hand side, where $h = g - 1 + \nu_p$, so a little algebra yields

$$n_\Gamma(G) = \frac{p^{h-2}(p^h-1)(p^{h-1}-1)}{(p^2-1)(p-1)}.$$

5. Normal surface subgroups

Since Γ contains no reflections, its torsion elements are the conjugates of the powers of the elliptic generators X_i. Thus a normal subgroup N of Γ is torsion-free if and only if it contains no non-trivial powers of any X_i, that is, each X_i is mapped to an element x_i of order exactly m_i in the quotient-group $G = \Gamma/N$. This is equivalent to N being a surface group (orientable or non-orientable), or equivalently the orbifold $\mathscr{H}/N$ being a surface (without cone-points). In fact, in this situation $\mathscr{H}/N$ is a Klein surface without boundary, its dianalytic structure being induced by projection from $\mathscr{H}$; conversely, every compact Klein surface without boundary, other than the sphere, projective plane, torus and Klein bottle, arises in this way (these exceptional surfaces have spherical or euclidean uniformisations). One can apply Hall's theory as before to show that the number $n_\Gamma^s(G)$ of normal surface subgroups N in Γ, with $\Gamma/N \cong G$, is given by

$$(4) \qquad n_\Gamma^s(G) = \frac{1}{|\mathrm{Aut}\, G|} \sum_{H \leq G} \mu(H)\sigma_\Gamma^s(H),$$

where $\sigma_\Gamma^s(H)$ denotes the number of surface-kernel homomorphisms $\theta : \Gamma \to H$; these are the homomorphisms with torsion-free kernel, that is, such that $x_i = X_i\theta$ has order m_i for each $i = 1, \dots, r$.

For each finite group H and each integer $m \geq 1$, let $H\langle m \rangle$ denote the set of elements of order m in H, and for each character χ of H let

$$\chi\langle m \rangle = \sum_{h \in H\langle m \rangle} \chi(h) \,.$$

Theorems 1 and 2 immediately imply the two parts of the following result.

COROLLARY 3. *Let H be any finite group. If Γ has signature* (S+) *then*

$$\sigma_\Gamma^s(H) = |H|^{2g-1} \sum_\chi \chi(1)^{2-2g-r} \chi\langle m_1 \rangle \cdots \chi\langle m_r \rangle \,,$$

and if Γ has signature (S−) *then*

$$\sigma_\Gamma^s(H) = |H|^{g-1} \sum_\chi c_\chi^g \chi(1)^{2-g-r} \chi\langle m_1 \rangle \cdots \chi\langle m_r \rangle \,,$$

where in each case, χ ranges over the irreducible complex characters of H.

EXAMPLES. (5.1) Let us calculate $n_\Gamma^s(C_n)$. All subgroups H of C_n are cyclic, so first let $H = C_d$, let h be any generator of H, and let ζ be a primitive d-th root of 1 in C. Then the irreducible characters χ of H are the homomorphisms $\lambda_j : H \to S^1 \subset \mathsf{C}$, determined by mapping h to ζ^j $(j = 1, \ldots, d)$; the kernel of $\chi = \lambda_j$ has order $k = (j, d)$, and the image has order d/k. If m does not divide d then $\chi\langle m \rangle = 0$, so assume that $m | d$. Then $H\langle m \rangle$ consists of the elements of H which are generators of its unique subgroup C_m. Now $C_m \cap \ker \chi = C_{(m,k)}$, so χ maps C_m onto the group of $m/(m,k)$-th roots of unity, sending generators to primitive roots; each primitive root is the image of $\phi(m)/\phi(m/(m,k))$ elements of $H\langle m \rangle$, and since the sum of the primitive n-th roots of 1 in C is $\mu(n)$ for all n, we find that

$$\chi\langle m \rangle = \frac{\phi(m)\mu\bigl(m/(m,k)\bigr)}{\phi\bigl(m/(m,k)\bigr)} \,,$$

which we will abbreviate to

$$\phi(m)\Bigl(\frac{\mu}{\phi}\Bigr)\Bigl(\frac{m}{(m,k)}\Bigr) \,.$$

(This is the Ramanujan sum $c_m(j)$, the sum of the j-th powers of the primitive m-th roots of 1, given by

$$c_m(j) = \phi(m)\Bigl(\frac{\mu}{\phi}\Bigr)\Bigl(\frac{m}{(m,j)}\Bigr) \,,$$

where $(m,k) = (m,(j,d)) = (m,j,d) = (m,j)$ since $m|d$; see [6, §§ 5.6, 16.6] for details.)

Now let Γ have signature (S+). If $\sigma^s_\Gamma(C_d) > 0$ then each period m_i must divide d, and hence their least common multiple l must also divide d. If we assume that l divides d, then the above argument gives

$$\sigma^s_\Gamma(C_d) = d^{2g-1} \sum_\chi \left\{ \prod_{i=1}^r \phi(m_i) \left(\frac{\mu}{\phi}\right) \left(\frac{m_i}{(m_i,k)}\right) \right\}$$

$$= d^{2g-1} \prod_{i=1}^r \phi(m_i) . \sum_\chi \left\{ \prod_{i=1}^r \left(\frac{\mu}{\phi}\right) \left(\frac{m_i}{(m_i,k)}\right) \right\}$$

$$= d^{2g-1} \prod_{i=1}^r \phi(m_i) . \sum_{k|d} \left\{ \phi\left(\frac{d}{k}\right) \prod_{i=1}^r \left(\frac{\mu}{\phi}\right) \left(\frac{m_i}{(m_i,k)}\right) \right\},$$

where we have used the fact that there are $\phi(d/k)$ irreducible characters χ with $|\ker \chi| = k$ for each k dividing d.

If we take $G = C_n$ then all its subgroups H are cyclic groups C_d; we can therefore substitute the above formula for $\sigma^s_\Gamma(H)$ in equation (4), giving

$$n^s_\Gamma(C_n) = \frac{1}{\phi(n)} \sum_{d|n} \mu\left(\frac{n}{d}\right) \sigma^s_\Gamma(C_d)$$

$$= \frac{1}{\phi(n)} \sum_{l|d|n} \left\{ \mu\left(\frac{n}{d}\right) d^{2g-1} \prod_{i=1}^r \phi(m_i) . \sum_{k|d} \left\{ \phi\left(\frac{d}{k}\right) \prod_{i=1}^r \left(\frac{\mu}{\phi}\right) \left(\frac{m_i}{(m_i,k)}\right) \right\} \right\}$$

$$= \frac{1}{\phi(n)} \prod_{i=1}^r \phi(m_i) \sum_{l|d|n} \left\{ \mu\left(\frac{n}{d}\right) d^{2g-1} \sum_{k|d} \left\{ \phi\left(\frac{d}{k}\right) \prod_{i=1}^r \left(\frac{\mu}{\phi}\right) \left(\frac{m_i}{(m_i,k)}\right) \right\} \right\}.$$

For example, let n be a prime p. If $r \geq 1$ then the main sum is empty (and so $n^s_\Gamma(C_p) = 0$) unless $l = p$, that is, $m_i = p$ for each i. In this case, the only possible value for d in the summation is $d = p$, so $k = 1$ or p, and we find that

$$n^s_\Gamma(C_p) = p^{2g-1} \left((p-1)^{r-1} + (-1)^r \right).$$

If $r = 0$, on the other hand, then $l = 1$, so $d = 1$ or p, and we get

$$n^s_\Gamma(C_p) = \frac{p^{2g} - 1}{p - 1}.$$

(This agrees with our earlier value for $n_\Gamma(C_p)$, since Γ is torsion-free when $r = 0$.)

The calculation of $\sigma_\Gamma^s(C_d)$ is a little simpler in the non-orientable case, since the only irreducible characters χ of C_d with $c_\chi \neq 0$ are the principal character χ_1 and the alternating character χ_2 (when d is even); these two characters, which send a generator of C_d to ± 1 respectively, satisfy $c_\chi = 1$. As before, we have $\chi\langle m \rangle = 0$ unless m divides d, in which case $\chi_1\langle m \rangle = \phi(m)$ and (if d is even) $\chi_2\langle m \rangle = (-1)^{d/m}\phi(m)$. It follows that if ω_d denotes the number of i such that d/m_i is odd, then

$$\sigma_\Gamma^s(C_d) = \begin{cases} d^{g-1}\phi(m_1)\ldots\phi(m_r), & \text{if } l|d \text{ and } d \text{ is odd,} \\ 2d^{g-1}\phi(m_1)\ldots\phi(m_r), & \text{if } l|d, \ d \text{ is even and } \omega_d \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

so equation (4) gives

$$n_\Gamma^s(C_n) = \frac{1}{\phi(n)}\left\{ \sum_{\substack{l|d|n \\ d \text{ odd}}} \mu\left(\frac{n}{d}\right) d^{g-1} \prod_{i=1}^r \phi(m_i) + \sum_{\substack{l|d|n \\ d, \omega_d \text{ even}}} 2\mu\left(\frac{n}{d}\right) d^{g-1} \prod_{i=1}^r \phi(m_i) \right\}$$

$$= \frac{1}{\phi(n)} \prod_{i=1}^r \phi(m_i) \left\{ \sum_{\substack{l|d|n \\ d \text{ odd}}} \mu\left(\frac{n}{d}\right) d^{g-1} + \sum_{\substack{l|d|n \\ d, \omega_d \text{ even}}} 2\mu\left(\frac{n}{d}\right) d^{g-1} \right\}.$$

For example, let n be an odd prime p. If $r \geq 1$ then $n_\Gamma^s(C_p) = 0$ unless $l = p$, that is, $m_i = p$ for all i, in which case $n_\Gamma^s(C_p) = (p-1)^{r-1}p^{g-1}$; if $r = 0$ (so that $l = 1$) then $n_\Gamma^s(C_p) = (p^{g-1} - 1)/(p - 1)$. Similarly, if $n = p = 2$ and $r \geq 1$ we get $n_\Gamma^s(C_2) = 2^g$ or 0 as r is even or odd, while for $r = 0$ we get $n_\Gamma^s(C_2) = 2^g - 1$. As in the orientable case, one can confirm these results by considering epimorphisms from the finite abelian group $\Gamma/\Gamma'\Gamma^p$ onto C_p.

(5.2) Let $G = D_p$, where p is an odd prime, and suppose that Γ has periods $m_1 = \cdots = m_r = p$ with $r \geq 1$. A surface-kernel homomorphism $\Gamma \to D_p$ must have image $H \leq D_p$ containing elements x_i of order p, so $H = C_p$ or D_p and hence

$$n_\Gamma^s(D_p) = \frac{1}{p(p-1)}\left(\sigma_\Gamma^s(D_p) - \sigma_\Gamma^s(C_p)\right).$$

If Γ is orientable, then Example 5.1 shows that

$$\sigma_\Gamma^s(C_p) = p^{2g-1}(p-1)\left((p-1)^{r-1} + (-1)^r\right).$$

The two 1-dimensional characters χ of D_p satisfy $\chi\langle p \rangle = p - 1$, while the $(p-1)/2$ remaining characters (all 2-dimensional) satisfy $\chi\langle p \rangle = -2$, so Corollary 3 gives

$$\sigma_\Gamma^s(D_p) = (2p)^{2g-1}\sum_\chi \chi(1)^{2-2g-r}\chi\langle p\rangle^r = p^{2g-1}(p-1)\big(2^{2g}(p-1)^{r-1}+(-1)^r\big),$$

from which we obtain

$$n_\Gamma^s(D_p) = p^{2g-2}(p-1)^{r-1}(2^{2g}-1).$$

Similarly, in the non-orientable case we find that $\sigma_\Gamma^s(C_p) = (p-1)^r p^{g-1}$ and $\sigma_\Gamma^s(D_p) = 2p^{g-1}(p-1)\big(2^{g-1}(p-1)^{r-1}+(-1)^r\big)$, so

$$n_\Gamma^s(D_p) = p^{g-2}\big((2^g-1)(p-1)^{r-1}+2(-1)^r\big).$$

6. Non-normal subgroups and monodromy groups

It is possible to extend the method of this paper so that it applies to non-normal subgroups M of finite index in an NEC group Γ, with Γ inducing a given permutation group on their cosets. This is because the core of M (the intersection of its conjugates in Γ) is a normal subgroup N of finite index in Γ, and Γ/N is isomorphic to the finite transitive permutation group G induced by Γ acting on the cosets of M. The details are as follows.

Let G be a finite group with a faithful, transitive permutation representation on a set Ω. Our aim is to determine the number $n_\Gamma(G,\Omega)$ of subgroups M of Γ on whose cosets Γ induces a permutation group $(\Gamma/N,\Gamma/M)$ isomorphic to (G,Ω). (An isomorphism between permutation groups (G,Ω) and (G',Ω') consists of an isomorphism $G\to G'$, $g\mapsto g'$ and a bijection $\Omega\to\Omega'$, $\omega\mapsto\omega'$ such that $(\omega g)' = \omega'g'$ for all $\omega\in\Omega$ and $g\in G$.) This extends our earlier enumeration of normal subgroups, corresponding to the regular representation of G.

Firstly, the techniques of §§2—4 are used to find the number $n_\Gamma(G)$ of normal subgroups N of Γ with quotient-group Γ/N isomorphic to the abstract group G. For each such core N, the subgroups $M\geq N$ on whose cosets Γ induces a permutation group isomorphic to (G,Ω) are in one-to-one correspondence with the point-stabilisers $G_\omega = \{g\in G\mid \omega g=\omega\}$ ($\omega\in\Omega$) in such representations of G. Now these stabilisers form an orbit under $A = \operatorname{Aut} G$, so the number of them is equal to the index $|A:N_A(G_\omega)|$ in A of their normalisers $N_A(G_\omega) = \{\alpha\in A\mid (G_\omega)\alpha = G_\omega\}$, and hence

(5) $$n_\Gamma(G,\Omega) = |A:N_A(G_\omega)|.n_\Gamma(G).$$

The permutation group (G,Ω) is isomorphic to the monodromy group of the orbifold covering $\mathscr{H}/M\to\mathcal{O}=\mathscr{H}/\Gamma$, that is, the group of permutations of the sheets induced by lifting closed paths in $\mathcal{O}$. It follows therefore that

this method enumerates the coverings of $\mathcal{O}$ with a given permutation group as their monodromy group.

To determine how many of these coverings are by Klein surfaces, one restricts this enumeration to surface subgroups M, as in §5. A subgroup M is a surface group if and only if it contains no elliptic elements, that is, each of the generators x_i $(i = 1, \ldots, r)$ of G acts on Ω as a semi-regular permutation of order m_i, consisting entirely of cycles of length m_i. One can therefore find the number $n_\Gamma^s(G, \Omega)$ of surface subgroups M, on whose cosets Γ induces (G, Ω), by taking each L_i in Theorem 1 or 2 to be the set of elements of order m_i in G which act semi-regularly on Ω.

EXAMPLE (6.1). Let $G = D_p$ acting naturally on the set Ω of vertices of a regular p-gon, where p is an odd prime. A point-stabiliser G_ω ($\cong C_2$) lies in an orbit of length p under $A = \operatorname{Aut} D_p$, so equation (5) gives $n_\Gamma(D_p, \Omega) = p n_\Gamma(D_p)$; the calculation of $n_\Gamma(D_p)$ was discussed in Examples 3.2 and 4.2 in the orientable and non-orientable cases. To determine $n_\Gamma^s(D_p, \Omega)$ one needs to know the non-identity elements of D_p which are semi-regular on Ω; these are the rotations $a, a^2, \ldots, a^{p-1}$, which coincide with the elements of order p, so if some $m_i \neq p$ then $n_\Gamma^s(D_p, \Omega) = 0$. Let us therefore assume that $m_i = p$ for $i = 1, \ldots, r$. We may also assume that $r \geq 1$, for otherwise $n_\Gamma^s(D_p, \Omega) = n_\Gamma(D_p, \Omega)$ and there is no problem. In the orientable case, Example 5.2 shows that there are $n_\Gamma^s(D_p) = p^{2g-2}(p-1)^{r-1}(2^{2g} - 1)$ possible kernels N; each is the core of p surface subgroups M, so

$$n_\Gamma^s(D_p, \Omega) = p^{2g-1}(p-1)^{r-1}(2^{2g} - 1).$$

Similarly, in the non-orientable case there are

$$n_\Gamma^s(D_p) = p^{g-2}\big((2^g - 1)(p-1)^{r-1} + 2(-1)^r\big)$$

kernels N and hence there are

$$n_\Gamma^s(D_p, \Omega) = p^{g-1}\big((2^g - 1)(p-1)^{r-1} + 2(-1)^r\big)$$

surface subgroups M. In either case, each conjugacy class of these subgroups M consists of all those with a given core N, so the number of conjugacy classes, and hence the number of equivalence classes of surface coverings of $\mathcal{O}$ with monodromy group (D_p, Ω), is given by the above formulæ for $n_\Gamma^s(D_p)$.

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