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EXACTNESS OF A RANK ONE QUANTUM INDUCTION FUNCTOR

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Abstract

We give a short and elementary proof of the exactness of the induction functor $H_A^0(U_A/U_A^0, -)$ for $U_q(\mathfrak{sl}_2)$.

1. Introduction

Let U be the quantized universal enveloping algebra (quantum group) associated to a simple finite dimensional Lie algebra $\mathfrak{g}$. Then U has a Poincaré-Birkhoff-Witt type decomposition $U = U^- U^0 U^+$. We may use a given module for the subalgebra U^0 to construct modules for U by "induction"; in this paper we study such a functor in the case $\mathfrak{g} = \mathfrak{sl}_2$. In [1] induction is studied for a quantum algebra over a certain localization of $A = \mathsf{Z}[q, q^{-1}]$, in particular, exactness is proved in [1, 2.11]. The proof involves (among other things) specialization to the case $q = 1$ and Kempf's vanishing theorem. It is also possible via other specializations to avoid this localization but the complete proof becomes quite long and non-trivial (an alternative proof may be given using Lusztig's canonical bases, see the related results on the quantum coordinate algebra in [3, 29.5].)

In this paper we give a short and elementary proof of the exactness of induction in the case $\mathfrak{g} = \mathfrak{sl}_2$ where U is an A-algebra (no localization). The result in this case is mentioned in [4, 2.3] but the proof sketched there is incorrect.

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2. Notation

Let $A = \mathsf{Z}[q, q^{-1}]$, q an indeterminate, and let U be the quantized universal enveloping algebra of type $\mathfrak{sl}_2$, i.e., U is the $\mathsf{Q}(q)$-algebra generated by $E,\ F,\ K,\ K^{-1}$ with relations

$$KK^{-1} = 1 = K^{-1}K, \tag{1}$$
$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Define for $c \in \mathsf{Z}$, $[c] = \frac{q^c - q^{-c}}{q - q^{-1}}$, and for $t \in \mathsf{N}$, $[t]! = \prod_{j=1}^{t}[j]$ and $\left[{c \atop t}\right] = \prod_{j=1}^{t} \frac{q^{c-j+1} - q^{-c+j-1}}{q^j - q^{-j}}$. In particular, $\left[{c \atop 0}\right] = 1$ and $[0]! = 1$, and $\left[{c \atop t}\right] = 0$ for $t > c \geq 0$. For all c, t as above, the $\left[{c \atop t}\right]$ belong to A. We define $E^{(r)} = \frac{1}{[r]!}E^r, F^{(r)} = \frac{1}{[r]!}F^r$; let U_A be the A-subalgebra of U generated by $E^{(r)},\ F^{(r)},\ K,\ K^{-1},\ (r = 0, 1, \ldots)$. We have a decomposition

$$U_A = U_A^- U_A^0 U_A^+ \tag{2}$$

[2, Thm. 6.7] where U_A^- is generated by the $F^{(r)}$, U_A^+ by the $E^{(r)}$, and U_A^0 by $K,\ K^{-1},\ \left[{K; c \atop t}\right]$.

Define for $c \in \mathsf{Z}, t \in \mathsf{N}$

$$\left[{K; c \atop t}\right] = \prod_{j=1}^{t} \frac{Kq^{c-j+1} - K^{-1}q^{-c+j-1}}{q^j - q^{-j}};$$

these elements belong to U_A^0.

For $m \in \mathsf{Z}$ we define a *character* $\chi_m : U_A^0 \to A$ (cf. [1], Lemma 1.1) by

$$\chi_m(K^{\pm}) = q^{\pm m},\ \chi_m\left(\left[{K; c \atop t}\right]\right) = \left[{m + c \atop t}\right],\ c \in \mathsf{Z},\ t \in \mathsf{N} \tag{3}$$

and for a U_A^0-module M the m'th *weight space* (of type 1, cf. [1, 1.2])

$$M_m = \{v \in M |\ \forall u \in U_A^0 :\ uv = \chi_m(u)v\}$$

We may consider A as a U_A^0-module by letting $u \in U_A^0$ act as multiplication by $\chi_m(u)$; this U_A^0-module is (by abuse of notation) written simply as χ_m.

Let M be a U_A-module, and define

$$\mathscr{F}(M) = \Big\{v \in \bigoplus_{\nu} M_\nu \ \Big|\ E^{(r)}v = 0 = F^{(r)}v \text{ for } r \gg 0\Big\};$$

$\mathscr{F}M$ is a submodule of M (cf. the proof of Lemma 3 below) and we say that M is *integrable* if $\mathscr{F}M = M$. Let $\mathscr{U}_A$ be the category of U_A-modules and let $\mathscr{C}_A$ be the full subcategory of $\mathscr{U}_A$ whose objects are the integrable U_A-modules; then $\mathscr{F}$ is a functor $\mathscr{U}_A \to \mathscr{C}_A$. We let $\mathscr{C}'_A$ denote the category of "integrable" U_A^0-modules (meaning that they are direct sums of their weight spaces.)

We define an *induction functor* as in [1, 1.9–10],

$$H^0(U_A/U_A^0, -) = \mathscr{F} \circ Hom_{U'_A}(U_A, -) : \mathscr{C}'_A \to \mathscr{C}_A; \tag{4}$$

where, if M is a U_A^0-module, U_A acts on $Hom_{U_A^0}(U_A, M)$ as follows:

$$(uf)(x) = f(xu), \; x, u \in U_A, \; f \in Hom_{U_A^0}(U_A, M) \tag{5}$$

3. Exactness of the induction functor

Proposition 1. *Let $m \in \mathsf{Z}$. If $m < 0$ then $H_A^0(\chi_m) = 0$. If $m \geq 0$ then $H_A^0(\chi_m)$ is a free A-module; it has a basis $e_0, e_1, \ldots, e_m$ such that for all $r \geq 0$ and all $i \in \{0, \ldots, m\}$ we have*

$$e_i \in H_A^0(\chi_m)_{m-2i}$$
$$E^{(r)} e_i = \begin{bmatrix} i \\ r \end{bmatrix} e_{i-r}, i = 0, \ldots, m$$
$$F^{(r)} e_i = \begin{bmatrix} m-i \\ r \end{bmatrix} e_{i+r}, i = 0, \ldots, m$$

where we set $e_s = 0$ for $s < 0$ or $s > m$.

Proof. Same as [1, Proposition 4.1].

Let $T : U_A \to U_A$ be an automorphism of A-algebras, and let M be a representation of U_A, i.e. an A-algebra homomorphism $\rho_M : U_A \to \mathrm{End}_A(M)$. We define a T-twisted representation ${}^T M$ by letting U_A act on M by the homomorphism ${}^T\rho_M = \rho_M \circ T$. If $T(U_A^0) \subseteq U_A^0$ we can twist U_A^0 representations in the same way.

Lemma 2. *Let $T : U_A \to U_A$ be an A-algebra endomorphism with $T(U_A^0) \subseteq U_A^0$ and let V be a U_A^0-module. Then T induces a homomorphism of U_A modules*

$$\phi : {}^T Hom_{U_A^0}(U_A, V) \longrightarrow Hom_{U_A^0}(U_A, {}^T V), \qquad f \longmapsto f \circ T$$

(*Recall that the untwisted U_A-module structure is given by* (5).) *Moreover, if T is an isomorphism* (*of A-algebras*) *then ϕ is a module isomorphism* (*with inverse $f \mapsto f \circ T^{-1}$*).

Proof. This is straightforward.

Lemma 3. *The functor* $\mathscr{F} : \mathscr{U}_A \to \mathscr{C}_A$ *is a left exact and commutes with direct sums.*

Proof. Let M be a U_A-module. First we show that $\mathscr{F}(M)$ is indeed a U_A-module: for example, if $x \in \mathscr{F}(M)$, say $E^{(s)}x = 0$ for $s > s_0$ and $F^{(t)}x = 0$ for $t > t_0$, then $E^{(r)}x$ and $F^{(r)}x$ are also in $\mathscr{F}(M)$ (for all $r \in \mathsf{N}$), for $F^{(t)}F^{(r)}x = 0$ and, using Kac's formula (compare [3], 3.1.9),

$$E^{(s+r)}F^{(r)}x = \sum_{i=0}^{r} F^{(r-i)} \begin{bmatrix} K; 2i - 2r - s \\ i \end{bmatrix} E^{(s+r-i)}x = 0$$

(and similarly for $E^{(r)}x$). It is easy to see that $\mathscr{F}$ is a functor.

To show that this functor is left exact, it suffices to prove that it preserves kernels. Let $\phi : M \to N$ be a morphism of $\mathscr{U}_A$:

$$\ker(\mathscr{F}\phi) = \ker(\phi|\mathscr{F}M) = \ker\phi \cap \mathscr{F}M = \mathscr{F}(\ker\phi)$$

It is easy to see that $\mathscr{F}(M \oplus N) = \mathscr{F}(M) \oplus \mathscr{F}(N)$ for all M, N in $\mathscr{U}_A$.

Corollary 4. *The functor* $H_A^0(U_A/U_A^0, -) : \mathscr{C}'_A \to \mathscr{C}_A$ *is left exact and commutes with direct sums.*

In the rest of this section we shall work only with one specific automorphism T, namely the one given by

$$(6) \qquad K \longmapsto K^{-1}, \quad E \longmapsto F, \quad F \longmapsto E$$

(using (1) one checks that this is an A-algebra automorphism with $T(U_A^0) \subset U_A^0$.)

Corollary 5. *With T as in* (6)*, there is a U_A-isomorphism*

$$^{T}H^0(U_A/U_A^0, V) \cong H^0(U_A/U_A^0, {}^{T}V)$$

$$f \longmapsto f \circ T$$

Proof. First, ϕ of Lemma 2 is an isomorphism. From the identity

$$(7) \qquad ({}^{T}M)_m = {}^{T}(M_{-m}), \quad (M \text{ any } U_A\text{-module})$$

and from $T(E^{(r)}) = F^{(r)}$, $T(F^{(r)}) = E^{(r)}$ we deduce that ${}^{T}\mathscr{F}(M) = \mathscr{F}({}^{T}M)$; with this identification $\mathscr{F}\phi$ is the required isomorphism.

One may check that T-twist (with T given by (6)) is an equivalence functor from $\mathscr{U}_A$ to itself (In particular, the functor is faithfully exact.) The restriction of this functor maps $\mathscr{C}_A$ to itself.

Lemma 6. *If $m \in \mathsf{Z}$, $V \in \mathscr{C}_A^0$ and $V_n = 0$ for $n < -m$, then there is an isomorphism*

(8) $$H^0_A(U_A/U^0_A, V)_m \cong \{(a_{rs})_{(r,s)\in \mathsf{N}\times\mathsf{N}} | a_{rs} \in V_{m+2(s-r)}, a_{rs} = 0 \text{ for } s \gg 0 \text{ and all } r\}$$

(9) $$f \longmapsto (f(F^{(r)}E^{(s)}))$$

PROOF. First we observe that any $f \in H^0_A(U_A/U^0_A, V)_m$ is given uniquely by its values on $F^{(r)}E^{(s)}$, $r, s \geq 0$ (since these constitute a basis for U_A over U^0_A, see [2, 6.7]). Put $a_{rs} = f(F^{(r)}E^{(s)})$; since f has weight m we get

$$\begin{aligned} q^m a_{rs} = q^m f(F^{(r)}E^{(s)}) &= (K.f)(F^{(r)}E^{(s)}) = f(F^{(r)}E^{(s)}K) \\ &= q^{2(r-s)}Kf(F^{(r)}E^{(s)}) = q^{2(r-s)}Ka_{rs} \end{aligned}$$

(and similarly for the other generators of U^0_A) so a_{rs} has weight $m + 2(s-r)$. Conversely, if $a_{rs} \in V_{m+2(s-r)}$ for all $r, s \geq 0$ then

(10) $$(uF^{(r)}E^{(s)} \mapsto ua_{rs}),\ u \in U^0_A$$

defines a function $U_A \to V$ that clearly belongs to $Hom_{U^0_A}(U_A, V)_m$.

Consider first any $f \in H^0_A(U_A/U^0_A, V)_m$:

(11) $$\begin{aligned} &\exists s_0 > 0\ \forall s_1 > s_0 &&: E^{(s_1)}.f = 0 \\ \iff &\exists s_0 > 0\ \forall s_1 > s_0\ \forall r, s \geq 0 &&: f(F^{(r)}E^{(s)}E^{(s_1)}) = 0 \\ \iff &\exists s_0 > 0\ \forall s_1 > s_0\ \forall r, s \geq 0 &&: \begin{bmatrix} s+s_1 \\ s \end{bmatrix} a_{r,s+s_1} = 0 \\ \iff &\exists s_0 > 0\ \forall s_1 > s_0\ \forall r \geq 0 &&: a_{r,s_1} = 0 \end{aligned}$$

This proves that f is indeed sent to the RHS of (8).

Conversely, let (a_{rs}) from the RHS of (8) be given, and consider the corresponding function, call it f, as given by (10). By (11) above we deduce that $E^{(s)}.f = 0$ for $s \gg 0$ and we need only show that a sufficiently high power of F kills f:

(12) $$\begin{aligned} &\exists j_0 > 0\ \forall j > j_0 &&: F^{(j)} \cdot f = 0 \\ \iff &\exists j_0 > 0\ \forall j > j_0\ \forall r, s \geq 0 &&: (F^{(j)} \cdot f)(F^{(r)}E^{(s)}) = 0 \\ \iff &\exists j_0 > 0\ \forall j > j_0\ \forall r, s \geq 0 &&: f(F^{(r)}E^{(s)}F^{(j)}) = 0 \\ \iff &\exists j_0 > 0\ \forall j > j_0\ \forall r, s \geq 0 &&: \\ &f\left(\sum_{t=0}^{\min\{j,s\}} \begin{bmatrix} r+j-t \\ r \end{bmatrix} F^{(r+j-t)} \begin{bmatrix} K; 2t-j-s \\ t \end{bmatrix} E^{(s-t)}\right) = 0 \end{aligned}$$

$$\iff \exists j_0 > 0\ \forall j > j_0\ \forall r, s \geq 0 :$$

$$\sum_{t=0}^{\min\{j,s\}} \begin{bmatrix} r+j-t \\ r \end{bmatrix} \begin{bmatrix} K; 2r+j-s \\ t \end{bmatrix} f\left(F^{(r+j-t)}E^{(s-t)}\right) = 0$$

$$\iff \exists j_0 > 0\ \forall j > j_0\ \forall r, s \geq 0 :$$

$$\sum_{t=0}^{\min\{j,s\}} \begin{bmatrix} r+j-t \\ r \end{bmatrix} \begin{bmatrix} m+s-j \\ t \end{bmatrix} a_{r+j-t,s-t} = 0$$

Note that for $r - s > m$ we get $m + 2(s - r) < -m$ and hence $a_{rs} = 0$ by the assumption that V has no weights below $-m$. We shall prove (12) by considering two cases:

$m + s - j < 0$: $(r + j - t) - (s - t) = r + j - s > r + m \geq m$, so $a_{r+j-s,s-t} = 0$ for all t.

$m + s - j \geq 0$: In this case $\left[\begin{smallmatrix} m+s-j \\ t \end{smallmatrix}\right] = 0$ for $t > m + s - j$; and if $0 \leq t \leq m + s - j$ we have $s - t \geq j - m$, whence it follows that $a_{r+j-t,s-t} = 0$ (according to (12)) if we choose j_0 greater than $m + s_0$ (and greater than 0), which we may do without loss of generality.

Lemma 7. *If $m \in \mathbb{Z}$ and*

$$0 \longrightarrow P \longrightarrow Q \xrightarrow{\pi} R \longrightarrow 0$$

is an exact sequence in $\mathscr{C}'_A$ and $P_n = Q_n = R_n = 0$ for $n < -m$ then there is an exact sequence of U_A^0-modules

$$0 \to H_A^0(U_A/U_A^0, P)_m \to H_A^0(U_A/U_A^0, Q)_m \xrightarrow{\tilde{\pi}} H_A^0(U_A/U_A^0, R)_m \to 0$$

Proof. According to Corollary 4 we only have to prove that $\tilde{\pi}$ is surjective. Choose an arbitrary $g \in H_A^0(U_A/U_A^0, R)_m$ and let $b_{rs} = g(F^{(r)}E^{(s)}) \in R_{m+2(s-r)}$, $r, s \geq 0$. For all $r, s \geq 0$ find $a_{rs} \in Q_{m+2(s-r)}$ such that $\pi(a_{rs}) = b_{rs}$ and $b_{rs} = 0 \Rightarrow a_{rs} = 0$ (π is surjective). As in (10) above, let $f \in Hom_{U_A^0}(U_A, V)_m$ be given by $uF^{(r)}E^{(s)} \mapsto a_{rs}$ ($u \in U_A^0$). By Lemma 6, $f \in H_A^0(U_A/U_A^0, Q)_m$ and clearly $\tilde{\pi}(f) = g$.

Theorem 8. *The functor $H_A^0(U_A/U_A^0, -) : \mathscr{C}'_A \to \mathscr{C}_A$ is exact.*

Proof. Since $H_A^0(U_A/U_A^0, V) = \bigoplus_m H_A^0(U_A/U_A^0, V)_m$, it will suffice to prove the exactness of each $H_A^0(U_A/U_A^0, -)_m$ (as a functor from $\mathscr{C}'_A$ to the category of A-modules.) So let an arbitrary fixed $m \in \mathbb{Z}$ be given. For any V in $\mathscr{C}'_A$ we define $V' = \bigoplus_{n \geq -m} V_n$ and $V'' = \bigoplus_{n < -m} V_n$; clearly $V = V' \oplus V''$. Given a short exact sequence in $\mathscr{C}'_A$

$$0 \longrightarrow P \longrightarrow Q \longrightarrow R \longrightarrow 0$$

we obtain two short exact sequences (of U_A^0-modules)

$$0 \longrightarrow P' \longrightarrow Q' \longrightarrow R' \longrightarrow 0$$

$$0 \longrightarrow P'' \longrightarrow Q'' \longrightarrow R'' \longrightarrow 0$$

Using Lemma 7 we find an exact sequence

$$0 \to H_A^0(U_A/U_A^0, P')_m \to H_A^0(U_A/U_A^0, Q')_m \to H_A^0(U_A/U_A^0, R')_m \to 0 \tag{13}$$

and by the exactness of the T-functor an exact sequence

$$0 \longrightarrow {}^T P'' \longrightarrow {}^T Q'' \longrightarrow {}^T R'' \longrightarrow 0$$

Since $({}^T P'')_n = {}^T(P''_{-n}) = 0$ for $-n \geq -m$, i.e. for $n \leq m$, we can apply Lemma 7 again to obtain an exact sequence

$$\begin{aligned} 0 &\to H_A^0(U_A/U_A^0, {}^T P'')_m \to H_A^0(U_A/U_A^0, {}^T Q'')_m \\ &\to H_A^0(U_A/U_A^0, {}^T R'')_m \to 0 \end{aligned}$$

Using Corollary 5 and (7) we deduce that the sequence

$$\begin{aligned} 0 &\to {}^T(H_A^0(U_A/U_A^0, P'')_m) \to {}^T(H_A^0(U_A/U_A^0, Q'')_m) \\ &\to {}^T(H_A^0(U_A/U_A^0, R'')_m) \to 0 \end{aligned}$$

is exact, and, since T is faithfully exact, that

$$\begin{aligned} 0 &\longrightarrow H_A^0(U_A/U_A^0, P'')_m \longrightarrow H_A^0(U_A/U_A^0, Q'')_m \\ &\longrightarrow H_A^0(U_A/U_A^0, R'')_m \longrightarrow 0 \end{aligned} \tag{14}$$

is exact. Finally, applying Corollary 4 to (13) and (14) yields an exact sequence

$$0 \longrightarrow H_A^0(U_A/U_A^0, P)_m \longrightarrow H_A^0(U_A/U_A^0, Q)_m \longrightarrow H_A^0(U_A/U_A^0, R)_m \longrightarrow 0$$

as desired.

4. Applications

We can define an induction functor

$$H^0(U_A/U_A^- U_A^0, -) = \mathscr{F} \circ \mathrm{Hom}_{U_A^- U_A'}(U_A, -)$$

from the category of integrable $U_A^- U_A^0$-modules to $\mathscr{C}_A$. This functor is left exact but not exact, so we let $H^i(U_A/U_A^- U_A^0, -)$ denote the ith derived functor (the category of integrable $U_A^- U_A^0$-modules has enough injectives). This functor is often written quite simply as $H_A^0(-)$ and the derived functors as

$H_A^i(-)$. As in [4, section 2] we may use Theorem 8 to prove vanishing theorems. We may extend the U_A^0-module χ_m to a $U_A^- U_A^0$-module by letting U_A^- act trivially. Then we have:

PROPOSITION 9 (Kempf vanishing). *Let* $m \geq 0$. *Then* $H_A^i(\chi_m) = 0$ *for* $i > 0$.

PROOF. [4, 2.4]

PROPOSITION 10. $H^i(-) = 0$ *for* $i > 1$

PROOF. [4, 2.5] or [1, 4.3]

Let k be a field where we choose a distinguished element $\xi \in k^\times$; we may then consider k as an A-algebra by $q \mapsto \xi$. We have then a quantum algebra $U_k = k \otimes_A U_A$ with a decomposition as (2), $U_k = U_k^- U_k^0 U_k^+$, where $U_k^0 = k \otimes_A U_A^0$ and similarly for U_k^- and U_k^+. We now consider $E^{(r)}$, $F^{(r)}$, K, K^{-1} and $\left[\begin{smallmatrix} K;c \\ t \end{smallmatrix}\right]$ as elements of U_k. We may then define for $m \in \mathsf{Z}$ characters $\chi_m : \ U_k \to k$ by composing the map in (3) with the algebra map $A \to k$. We also extend the concept of integrable modules to U_k-modules (resp. U_k^0-modules), and we have then an induction functor as in (4) which we denote $H_k^0(U_k/U_k^- U_k^0, -)$, or simply $H_k^0(-)$.

PROPOSITION 11. *Let* V *be an integrable* U_A^0-*module. Then* [4, 2.9]

$$H_k^0(k \otimes_A V) \cong k \otimes_A H_A^0(V)$$

PROOF. As in the proof of Theorem 8 we write $V = V' \oplus V''$ where $V' = \bigoplus_{n \geq -m} V_n$ and $V'' = \bigoplus_{n < -m} V_n$. In the same notation, $(k \otimes V)' = k \otimes (V')$ and $(k \otimes V)'' = k \otimes (V'')$ since $(k \otimes V)_n = k \otimes V_n$. Using Lemma 6 and a similar version for $H^0(U_k/U_k^0, -)$, we see that

$$\begin{aligned} (15) \quad k \otimes H^0(U_A/U_A^0, V')_m &= k \otimes \{(a_{rs}) | a_{rs} \in V_{m+2(s-r)},\ a_{rs} = 0, s \gg 0\} \\ &= \left\{(\overline{a}_{rs}) | \overline{a}_{rs} \in (k \otimes V')_{m+2(s-r)},\ \overline{a}_{rs} = 0, s \gg 0\right\} \\ &= H^0(U_k/U_k^0, k \otimes V')_m \end{aligned}$$

As in Corollary 5, we have for each U_k^0-module M an isomorphism of U_k-modules

$$(16) \qquad {}^T H^0(U_k/U_k^0, M) \cong H^0(U_k/U_k^0, {}^T M)$$

and then, using Corollary 5, (16), and (15) with ${}^T V''$ and $-m$ substituted for, respectively, V' and m,

$$
\begin{aligned}
{}^{T}((k \otimes H^0(U_A/U_A^0, V''))_m) &\cong k \otimes {}^{T}(H^0(U_A/U_A^0, V'')_m) \\
&= k \otimes H^0(U_A/U_A^0, {}^{T}V'')_{-m} \\
&\cong H^0(U_k/U_k^0, k \otimes {}^{T}V'')_{-m} \\
&= H^0(U_k/U_k^0, {}^{T}(k \otimes V''))_{-m} \\
&\cong {}^{T}(H^0(U_k/U_k^0, k \otimes V'')_m)
\end{aligned}
$$

whence we get

$$
H^0(U_k/U_k^0, k \otimes V'')_m \cong k \otimes H^0(U_A/U_A^0, V'')_m \tag{17}
$$

Finally, we take the direct sum of (15) and (17) and use that also $H^0(U_k/U_k^0, -)$ commutes with direct sums (cf. Corollary 4).

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