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STABILITY OF PENCILS OF CUBIC SURFACES IN P^3

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Abstract

In this paper we study the classification of pencils of cubic surfaces in P^3, up to projective equivalence. We obtain explicit vanishing criteria on the Plücker coordinates of a pencil for both stability and semi-stability; moreover, we give the equations defining pairs of generators for unstable and not properly stable pencils. Thus we extend the work of Miranda and Ballico [5, 1]. We give some geometric criteria for when a pencil is properly stable, and in particular, we give a characterization of smooth not properly stable pencils.

1. Introduction

A pencil of cubic surfaces is a line in the parameter space of cubic surfaces. Choose coordinates $[x, y, z, w]$ on P^3. Let F_A and F_B be two points spanning the line, and let F_A and F_B represent the cubic forms

$$\sum a_{ijk}x^i y^j z^k w^{3-i-j-k} \text{ and } \sum b_{ijk}x^i y^j z^k w^{3-i-j-k}$$

Form the 2×20 matrix

$$\begin{pmatrix} a_{000} & a_{001} & \cdots & a_{ijk} & \cdots & a_{021} & a_{300} \\ b_{000} & b_{001} & \cdots & b_{ijk} & \cdots & b_{021} & b_{300} \end{pmatrix}.$$

The 2×2 determinants

$$p_{ijklmn} = \begin{vmatrix} a_{ijk} & a_{lmn} \\ b_{ijk} & b_{lmn} \end{vmatrix}$$

of the matrix is the *Plücker coordinates* to the line spanned by F_A and F_B.

We wish to use these coordinates to study the stability of pencils of cubic surfaces. Part of this work is to be found in [4]. This work was done while the author was a student of K. Ranestad.

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2. The Criterium for the Stability of Cubic Pencils

The following proposition is a numerical criterion to determine if a pencil is unstable (not properly stable) in terms of the Plücker coordinates of the pencil. The proposition is a straightforward generalization of the corresponding result in [5].

PROPOSITION 2.1. *A pencil P is unstable* (*resp. not properly stable*) *if and only if there exists rational numbers r and s satisfying* $1 \geq r \geq s \geq -\frac{1}{2}-\frac{1}{2}r$ *and coordinates of* P^3 *such that if P is represented by the point* (p_{ijklmn}) *in these coordinates, then*

$$p_{ijklmn} = 0 \quad \text{whenever} \quad e_{ijklmn}(r,s) \leq 0 \ (\textit{resp.} < 0),$$

where

$$e_{ijklmn}(r,s) := (2i+2l+j+k+m+n-6) + r(2j+2m+i+k+l+n-6) + s(2k+2n+i+j+l+m-6).$$

The inequality $1 \geq r \geq s \geq -\frac{1}{2}-\frac{1}{2}r$ define a triangle in R^2 with corners $(-\frac{1}{3},-\frac{1}{3})$, $(1,-1)$ and $(1,1)$. The conditions $e_{ijklmn}(r,s) \leq 0$ or $e_{ijklmn}(r,s) < 0$ subdivides the triangle in a finite number of convex polygons, and on each of these polygons the truth or falsity of the inequalities are constant. An inspection of the conditions on the (p_{ijklmn}) in each of the polygons shows that these conditions are not independent.

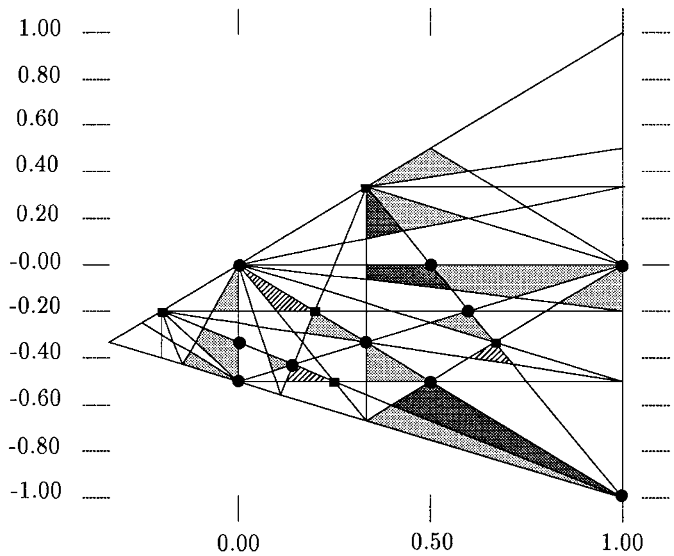

Figure 1. Minimal conditions for instability and non-proper stability.

The shaded and striped polygons in Figure 1, represent 'minimal' conditions for instability, while the black dots and squares represents minimal conditions for non-proper stability. In the next section we will translate these conditions into defining polynomials of generators of the pencils. Computations shows that the polynomials generated from the striped polygons are special cases of the polynomials generated from the shaded. Similarly, polynomials generated by the black squares are also special cases of polynomials generated by the black dots or by the shaded polygons.

3. The Stability Condition in Terms of Generators of a Pencil

Having computed the criteria for instability and non-proper stability in terms of Plücker coordinates, we will translate this into equations for generators of the pencils. Let A and B be to cubics generating a pencil P, and assume that P is unstable (not properly stable). Choose coordinates $[x, y, z, w]$ of P^3 as in Proposition 2.1, and let A and B have defining polynomials $F_A = \sum a_{ijk}x^iy^jz^kw^{3-i-j-k}$ and $F_B = \sum b_{ijk}x^iy^jz^kw^{3-i-j-k}$. The vanishing of some of the Plücker coordinates p_{ijklmn} give equations involving the coefficients a_{ijk} and b_{lmn}. After some algebraic manipulation, these equations are easily seen to be equivalent to the vanishing of the coefficients of *some pair* of cubics A' and B' in the pencil (not necessarily the original pair A and B). This part of the analysis is very tiresome to do by hand, and is therefore done by a computer, but we will indicate an algorithm.

To simplify the presentation of the algorithm, let $\mathbf{e}_1, \ldots, \mathbf{e}_{20}$ be a basis of P^{19}. Let $A = \sum_{i=1}^{20} a_i\mathbf{e}_i$ and $B = \sum_{i=1}^{20} b_i\mathbf{e}_i$ be two points in P^{19}. The Plücker coordinates p_{ij} to the line spanned by A and B is given by $p_{ij} = a_ib_j - a_jb_i$. It is easily seen that $p_{ij} = -p_{ji}$ and $p_{ii} = 0$, so we may choose a total ordering of the Plücker coordinates such that $i < j$.

We will use the ordering to solve the quadratic equations $p_{ij} = 0$ in an ascending order, starting with the least one.

Assume that, say, $p_{12} = 0$. Then (shown below) $a_1 = b_1 = 0$ or $b_1 = b_2 = 0$. When solving the next equation, say $p_{15} = 0$, we must solve for two conditions: $a_1 = b_1 = 0$ and $b_1 = b_2 = 0$. However, the solution of $p_{15} = 0$ gives two additional conditions on the coefficients which may influence the solutions of the next equation. So, when solving an equation $p_{ij} = 0$ we must take into account different conditions coming from previous equations.

It is natural to program the algorithm as a recursive algorithm as follows. Let the algorithm have the first equation as input. In the algorithm, solve the equation, and recursively solve the next equation for each of the two additional conditions on the coefficients. Continue until there are no more equations to solve. Now, some of the coefficients may be zero, and the rest of the coefficients may be arbitrary.

When solving $p_{ij} = 0$ we have three cases to consider:

(1) No previous conditions on a_i, b_i, a_j, b_j:

Then $p_{ij} = a_i b_j - a_j b_i = 0$ implies that $a_i = b_i = 0$ or $a_i \neq 0$ or $b_i \neq 0$.

Suppose that $a_i \neq 0$. Let $B' = B - \frac{b_i}{a_i} A$, where $b'_j = b_j - \frac{b_i}{a_i} a_j$. Now we have that $p'_{ij} = p_{ij}$ and $b'_i = 0$, so we may assume that $b_i = 0$. But then the equation $p_{ij} = a_i b_j = 0$ imply that $b_j = 0$.

By symmetry, we need not consider the case $b_i \neq 0$.

(2) Assume $b_i = 0$ is a previous condition:

Then $p_{ij} = a_i b_j = 0$ implies that $a_i = 0$, or $a_i \neq 0$ and $b_j = 0$.

(3) Assume $a_i = 0$ is a previous condition:

Then $p_{ij} = -a_j b_i = 0$ implies that $b_i = 0$, or $b_i \neq 0$ and $a_j = 0$.

Notation. *Let* $< M_1, \ldots, M_k >$ *denote the subspace of* $H^0(\mathsf{P}^3, \mathcal{O}_{\mathsf{P}^3}(3))$ *spanned by the monomials* M_i.

Proposition 3.1. *A pencil* P *is unstable if and only if there exists coordinates* $[x, y, z, w]$ *of* P^3 *and two generators* A, B *with equations* $\{F_A = 0\}$, $\{F_B = 0\}$, *respectively, satisfying one of the following conditions:*

(U1) $F_A, F_B \in \langle xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$

(U2) $F_A \in \langle x^2w, x^2z, x^2y, x^3 \rangle$
No restriction on F_B

(U3) $F_A \in \langle y^3, xy^2, x^2y, x^3 \rangle$
$F_B \in \langle z^2w, z^3, yw^2, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$

(U4) $F_A \in \langle xy^2, x^2z, x^2y, x^3 \rangle$
$F_B \in \langle z^2w, z^3, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$

(U5) $F_A \in \langle xy^2, x^2z, x^2y, x^3 \rangle$
$F_B \in \langle z^3, yw^2, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$

(U6) $F_A \in \langle xy^2, x^2w, x^2z, x^2y, x^3 \rangle$
$F_B \in \langle z^3, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$

(U7) $F_A \in \langle xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$
$F_B \in \langle z^3, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$

(U8) $F_A \in \langle xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$
$F_B \in \langle y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$

(U9) $F_A \in \langle xz^2, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$
$F_B \in \langle z^3, yz^2, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$

(U10) $F_A \in \langle xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U11) $F_A \in \langle y^2z, y^3, xz^2, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yz^2, y^2w, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U12) $F_A \in \langle yz^2, y^2z, y^3, xz^2, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yz^2, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U13) $F_A \in \langle yz^2, y^2z, y^3, xz^2, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yz^2, y^2w, y^2z, y^3, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U14) $F_A \in \langle y^3, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U15) $F_A \in \langle yz^2, y^2z, y^3, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yz^2, y^2z, y^3, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U16) $F_A \in \langle y^2w, y^2z, y^3, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle yz^2, y^2w, y^2z, y^3, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U17) $F_A \in \langle z^3, yz^2, y^2z, y^3, xz^2, xyz, xy^2, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^2w, z^3, yzw, yz^2, y^2w, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U18) $F_A \in \langle y^3, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle yw^2, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U19) $F_A \in \langle y^3, xy^2, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U20) $F_A \in \langle y^3, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(U21) $F_A \in \langle y^3, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

Proposition 3.2. *A pencil P is not properly stable if and only if P is unstable or there exists coordinates $[x, y, z, w]$ of P^3 and two generators A, B with equations $\{F_A = 0\}$, $\{F_B = 0\}$, respectively, satisfying one of the following conditions:*

(N1) $F_A \in \langle y^2w, y^2z, y^3, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle yw^2, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N2) $F_A \in \langle y^3, xy^2, x^2y, x^3\rangle$
No restriction on F_B.

(N3) $F_A \in \langle y^3, xy^2, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yw^2, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N4) $F_A \in \langle xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yw^2, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N5) $F_A \in \langle xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N6) $F_A \in \langle y^2z, y^3, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle yz^2, y^2w, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N7) $F_A \in \langle y^3, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N8) $F_A \in \langle xy^2, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^2w, z^3, yw^2, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N9) $F_A \in \langle yz^2, y^2z, y^3, xz^2, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yz^2, y^2w, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N10) $F_A \in \langle y^3, xy^2, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^2w, z^3, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N11) $F_A \in \langle yz^2, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yz^2, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N12) $F_A \in \langle xz^2, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^2w, z^3, yzw, yz^2, y^2w, y^2z, y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N13) $F_A \in \langle yz^2, y^2w, y^2z, y^3, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yz^2, y^2w, y^2z, y^3, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N14) $F_A \in \langle y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$
$F_B \in \langle y^3, xw^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

(N15) $F_A \in \langle x^2w, x^2z, x^2y, x^3\rangle$
No restriction on F_B.

(N16) $F_A \in \langle z^3, yz^2, y^2z, y^3, xz^2, xyz, xy^2, x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^2w, z^3, yzw, yz^2, y^2w, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle$

REMARK 3.3. Note that case (N15) and (N16) is equal to case (U2) and (U17), respectively.

NOTATION. *In the following, p will denote the point* $(0,0,0,1)$ *in* P^3*, and* T_p *denotes the tangent plane to a surface in p.* $f_i(x,y,z)$ *denotes a homogeneous form of degree i in the variables x,y and z.*

We will now give a description of the general A and B in each of the cases (N1) through (N16). The defining polynomial F to a cubic surface S in P^3

with a singularity in p, can be written as $F = f_2(x, y, x)w + f_3(x, y, z)$. The cone $\{f_2(x, y, x) = 0\}$ in P^3 is the tangent cone to S in p. The rank of the tangent cone, combined with how $\{f_2 = 0\}$ intersect $\{f_3 = 0\}$ as curves in the plane $\{w = 0\}$, determines the number of singularities on S and their ADE-classification. The results we need concerning cubic surfaces are to be found in [3, 2].

(N1) A is singular along $\{x = y = 0\}$. B is smooth and contains the line $\{x = y = 0\}$.

(N2) A is three planes intersecting along $\{x = y = 0\}$. A is quadric cone and a tangentplan. B is the general cubic surface in P^3.

(N3) A is a cuspidal cone with triple point p, double line $\{x = y = 0\}$ and cuspidal tangent $\{x^2 = 0\}$. B is smooth with $T_p = \{ax + by = 0\}$. The line $\{x = y = 0\}$ intersect B with multiplicity 3 in p.

(N4) A is the plane $\{x = 0\}$ and a quadric cone with double point on the line $\{x = y = 0\}$. The plane $\{x = 0\}$ is tangent to the quadric cone along the line $\{x = y = 0\}$. B is identical to $\{F_B = 0\}$ in case (N3).

(N5) A is the plane $\{x = 0\}$, and a smooth quadric with $T_p = \{ax + by = 0\}$. B is irreducible and singular, contains the line $\{x = y = 0\}$, has two A_1 singularities on $\{x = y = 0\}$, and has the plane $\{x = 0\}$ as tangent plane at a general point on $\{x = y = 0\}$. We have that $\{x \cap F_B = 0\}$ is the double line $\left\{x = y^2 = 0\right\}$ and a line through the point $(0, 0, 1, 0)$.

(N6) A is irreducible with an A_4 singularity in p, and A contains the line $\{x = y = 0\}$. We have that $\{x \cap F_A = 0\}$ is the double line $\left\{x = y^2 = 0\right\}$ and a line through the point $(0, 0, 1, 0)$. B is irreducible with an A_1 singularity in p. The tangent cone to B in p has the plane $\{x = 0\}$ as tangent plane along the line $\{x = y = 0\}$. B contains the line $\{x = y = 0\}$.

(N7) A is irreducible and singular along the line $\{x = y = 0\}$. The tangent cone to A along $\{x = y = 0\}$ has the plane $\{x = 0\}$ as a component. The tangent cone to A in p is the double plane $\left\{x^2 = 0\right\}$. B is smooth with $T_p = \{x = 0\}$, and the line $\{x = y = 0\}$ intersect B with multiplicity 3 in p.

(N8) A is the plane $\{x = 0\}$ and a quadric cone with double point p. The plane $\{x = 0\}$ is tangent to the quadric cone along $\{x = y = 0\}$. B is smooth with $T_p = \{ax + by = 0\}$.

(N9) A is irreducible with a D_4 singularity in p, and A contains the line $\{x = y = 0\}$. The tangent cone to A in p is the double plane $\left\{x^2 = 0\right\}$. B is irreducible with an A_1 singularity in p. The line $\{x = y = 0\}$ intersect B with multiplicity 3 in p. The tangent cone to B in p has the plane $\{x = 0\}$ as tangent plane along $\{x = y = 0\}$.

(N10) A is identical to $\{F_A = 0\}$ in case (N3). B is smooth with $T_p = \{x = 0\}$.

(N11) A is irreducible with an A_2 singularity in p, and A contains the line $\{x = y = 0\}$. The tangent cone to A in p has the plane $\{x = 0\}$ as a component. B is irreducible with an A_2 singularity in p. The tangent cone to B in p has the plane $\{x = 0\}$ as a component. The line $\{x = y = 0\}$ intersect B with multiplicity 3 in p.

(N12) A is the plane $\{x = 0\}$ and a smooth quadric. The plane $\{x = 0\}$ is the tangent plane to the smooth quadric in p. B is identical to $\{F_B = 0\}$ in case (N10).

(N13) A is irreducible with an A_3 singularity in p. The tangent cone to A in p is two planes intersecting along $\{x = y = 0\}$. A contains $\{x = y = 0\}$. B is irreducible with and A_2 singularity in p. The tangent cone to B in p is two planes intersecting along $\{x = y = 0\}$. The line $\{x = y = 0\}$ intersect B with multiplicity 3 in p.

(N14) A is irreducible with two A_2 singularities on the line $\{x = y = 0\}$, one of them is in p. A contains the line $\{x = y = 0\}$, and $\{x = 0\}$ is the tangent plane to A in a smooth point on the line $\{x = y = 0\}$. B is of the same type as A.

(N15) A is a double plane and a plane. B is the general cubic surface in P^3.

(N16) A is irreducible with a triple point in p. B is irreducible with an A_1 singularity in p.

4. Properly Stable Pencils

Definition 4.1. A pencil of cubic surfaces is smooth if it has a smooth member. A pencil that is not smooth, is singular. A pencil is irreducible if every member of the pencil is irreducible.

Proposition 4.2. *Let P be a pencil of cubic surfaces. P is properly stable if*

(a) *The base locus is smooth.*

(b) *P is smooth and the base locus is irreducible.*

(c) *P is smooth and irreducible. Every surface in P has at most isolated singularities.*

(d) *P is irreducible, and every surface has an A_1 singularity in the same point. The general surface in P has only one singularity, and the other surfaces have at most two A_1 singularities.*

Proof. The proof of (a), (c) and (d) is a straight forward inspection of the description of (N1) through (N16).

(b) Assume that the base locus D has a singular point q of multiplicity r.

By blowing up B in q and considering the strict transform of D, it is easily seen that D is reducible if $r \geq 6$.

Assume that P is a smooth not properly stable pencil. We will show that the base locus D is reducible. In cases (N1), (N2), (N4), (N8), (N12) and (N15) one of the generators is reducible or D has a multiple line as a component.

Assume that P is a general pencil satisfying case (N3). D will have a multiple line as a component if the coefficient of z^3 in F_B is zero, so we may assume that the coefficient is non-zero. B is a smooth surface, so not both the coefficients of xw^2 and yw^2 in F_B is zero simultaneous.

If the coefficients of x^2z or y^3 in F_A is zero, then A is reducible. If the coefficients of x^2z and y^3 in F_A is non-zero, then D has a singularity of multiplicity of 7 in p, hence D is reducible.

Similary, the base locus of a general pencil satisfying case (N7) or (N10) is reducible.

Remark 4.3. Any cubic surface, except one with a triple line, can occur in a stable pencil.

Remark 4.4. If the base locus is irreducible and singular, the pencil may be not properly stable.

Proof. Let P be a general pencil satisfying case (N16). Fix a general B, and let A vary. As A vary, we get a linear system of intersection curves D on B. From the description of case (N16) in the previous section, we see that B has an A_1 singularity in the point p. By using Bertini on the minimal model of B the result follows.

Definition 4.5. We call a singularity an unode if its tangent cone is a double plane.

Notation. *Let C and D be two curves on a smooth surface, then we denote the intersection multiplicity of C and D in a point q by $I(C \cap D, q)$.*

Theorem 4.6. *Let P be a smooth pencil of cubic surfaces. Then P is not properly stable if and only if at least one of the following conditions is true:*

(a) *The base locus contains a multiple line.*
(b) *The base locus contains a singular plane cubic and a singular curve C_6 of degree 6 with a common singular point.*
(c) *P contains a surface that has a triple line.*
(d) *P contains a surface A singular along a line L such that the tangent cone in a general point on L contains the same plane. There exists a smooth surface B in P and a point q in the base locus, such that one of the following is true:*

(d1) *A is a cone over cuspidal cubic, or a specialization, with triple point q. The line L lies on the tangent plane to B in q.*
If $\deg TC_q(A \cap B) = 3$ *and the multiplicity of L in* $TC_q(A \cap B)$ *is 2, then L intersect a general surface in P with multiplicity at least 3 in q*
If $\deg TC_q(A \cap B) > 3$ *or the multiplicity of L in* $TC_q(A \cap B)$ *is greater than 2, then L intersect a general surface in P with multiplicity at least 2 in q.*

(d2) *A is a quadric cone union a tangent plane, or a specialization. The base locus has a plane cubic* C_3 *and a curve* C_6 *of degree 6 as components.* $I(L \cap C_3, q) \geq 3$ *and* $I(L \cap C_6, q) \geq 2$*; or* $I(L \cap C_3, q) \geq 2$ *and the triple point of A is the point q.*

(d3) *The tangent cone to A in a general point on L has* $T_q(B)$ *as a component, and has an unode or a triple point in q. L intersect a general surface in P with multiplicity at least 3 in q.*

REMARK 4.7. Case (d3) is not 'closed' under specialization in the sense that A may be specialized to a triple plane which intersect $T_q(B)$ in a triple line. But the tangent cone in a general point on L of all the other specializations of A has $T_q(B)$ as a component.

PROOF. We will now prove the theorem. One way of the proof is straight forward. If P is smooth and not properly stable, then P has generators A and B satisfying one of the cases (N1), (N2), (N3), (N4), (N7), (N8), (N10), (N12) or (N15).

Assume now that the pencil is general in each case. We see that (N1) satisfy case (a), (N12) satisfy case (b), while (N2) and (N15) satisfy case (c). Case (N10) and (N3) satisfy case (d1), (N8) and (N4) satisfy case (d2), while (N7) satisfy case (d3).

We shall now prove the converse of the theorem. From now on we will assume that A and B are two generators of P, and that B is smooth.

LEMMA 4.8. *Let P be a smooth pencil of cubic surfaces. If the base locus contains a double line, then P contains a surface with a double line.*

PROOF. We may assume that the double line is given by $\{x = y = 0\}$. The tangent plane to members of the pencil along the line $\{x = y = 0\}$ is given by $a(0, 0, z, w)x + b(0, 0, z, w)y = 0$. By assumption the map between projective lines is given by $(z, w) \longmapsto (a, b)$ is constant, hence a and b are linearly dependent as linear forms.

If P is a smooth pencil with a double line in the base locus, it follows from Lemma 4.8 and case (N1) that P is not properly stable. This proves case (a).

Lemma 4.9. *Let P be a smooth pencil of cubic surfaces. The base locus has a plane cubic as a component if and only if P contains a reducible surface.*

Proof. We may assume that the plane cubic is contained in the plane $\{x = 0\}$. The result follows by an inspection of the defining equations of two smooth members of P.

By Lemma 4.9 we may assume that A is reducible. From the description of the base locus and case (N12), case (b) follows. If P contains a member with a triple line, then P is not properly stable by case (N2) or (N15). This proves case (c), and we shall now prove case (d1).

By the description of A and the base locus, we see that A is the cone over a plane cuspidal cubic or its specialization. We may assume that L does not lie on B, $q = p$ and $L = \{x = y = 0\}$.

If the tangent cone in p to the base locus is the triple line L^3, then P is not properly stable by case (N10). If the tangent cone in p is L^2 and another line, then L intersect B with multiplicity at least 2 in p. If the multiplicity is 3, then p is not properly stable by case (N3).

We shall now prove case (d2). We may assume that L does not lie upon B. If we assume that $q = p$, $L = \{x = y = 0\}$ and that the tangent plane of the quadric cone is $\{x = 0\}$, then it is easily seen that any pencil satisfying case (d2), has generators with defining polynomials as in case (N4) or (N8). This completes the proof of case (d2), and now remains case (d3).

Assume that A is as in case (d3). We may assume that the point q is equal to p, L does not lie on B and $L = \{x = y = 0\}$. If the fix plane is $\{x = 0\}$, then the unode has tangent cone $\left\{x^2 = 0\right\}$.

The line L lies on the tangent plane to B in p. If L intersect B with multiplicity 3, the P is not properly stable by case (N7). This completes the proof of case (d) and the theorem.

Corollary 4.10. *There exists smooth and properly stable pencils P such that:* (a) *the base locus is irreducible and singular,* (b) *the base locus contains a line,* (c) *the base locus contains a plane cubic or* (d) *the base locus has three plane cubics as components.*

Proof. (a) In the linear system of the cubic surfaces which have a common tangent plane in a point q, there exists two smooth surfaces such that their intersection is an irreducible curve with a node in q. Use Proposition 4.2.

(b) Let A be a general cubic surface with an A_1 singularity in the point p, and assume that A contains the line $\{x = y = 0\}$. Let B be a general smooth surface containing the line $\{x = y = 0\}$. By Bertini, the base locus has the line $\{x = y = 0\}$ and a smooth curve of degree 8 as components, and the only singularity is a node in p. By Theorem 4.6 the pencil is properly stable.

(c) Let A be a general plane and a general smooth quadric, and let B be a general smooth cubic surface. By Bertini, the base locus has a smooth plane cubic and a smooth, irreducible sextic curve intersecting transversally, as components. Moreover, the singularities of the base locus are nodes, and each of the lines in the tangent cones of the nodes intersect B in two distinct points. Use Theorem 4.6.

(d) Let A be three general planes intersecting transversally, and let B be a general smooth cubic surface. By Bertini, the base locus of the pencil P generated by A by B has three smooth plane cubics as components, and each of the components intersect each other transversally. The singularities of the base locus is nodes, and no line on A intersect each of the plane cubics with multiplicity at least 3. From Theorem 4.6 we see that P is properly stable.

Lemma 4.11. *Let $F = f_2(x, y, z)w + f_3(x, y, z)$. Suppose that $\{f_2 \cap w = 0\}$ is a smooth conic, and suppose that $\{f_2 \cap f_3 \cap w = 0\}$ is six points, counted with multiplicity. Then the surface $\{F = 0\}$ is irreducible and has only isolated double points, one of them in p.*

Proof. If a cubic surface S has a singularity in the point p, then S has a defining polynomial $F = f_2(x, y, z)w + f_3(x, y, z)$. By using the classification of cubic surfaces in [2] and comparing rank and degree of tangent cones, the result follows.

Proposition 4.12. *There exists singular and irreducible properly stable pencils such that:* (a) *the base locus contains five lines or* (b) *the base locus contains four lines, with one of them double.*

Proof. See [2, 3] for details and proofs. (a) Let C be a smooth conic in the plane $\{w = 0\}$ with defining polynomial $f_2(x, y, z)$. Let $p_1, \dots, p_5$ be five general points on C. There exists a plane pencil with generators $f_3^A(x, y, z)$ and $f_3^B(x, y, z)$ such that no member of the plane pencil has C as a component and such that the base locus of the pencil is the points $p_1, \dots, p_5$. Let P be the pencil of cubic surfaces corresponding to the line $f_2(x, y, x)w + \lambda_1 f_3^A(x, y, z) + \lambda_2 f_3^B(x, y, z)$ with $[\lambda_1, \lambda_2] \in \mathsf{P}^1$. By Lemma 4.11 it follows that every member of P is irreducible with isolated double points. Every surface will have an A_1 singularity in the same point p, and one surface will have an additional A_1 singularity. Also, every surface in the pencil contains the lines going through p and p_i. By inspection of the cases (N1) through (N16) it follows that P is properly stable.

(b) Let C be as above. Let $p_1, \dots, p_4$ be general points on C, but assume that p_1 is counted with multiplicity 2. We can construct a pencil P of irreducible cubic surfaces with a singularity in p containing the lines going through p and p_i. The line going through p and p_1 is double. Every member of P will have an A_1 singularity in p. One member has an additional A_3 singularity, while the rest has one or two additional A_1 singularities. By inspection it follows that P is properly stable.

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