MATH. SCAND. 84 (1999), 119–136

# CONNECTIVITY AND COMPONENTS FOR $C^*$-ALGEBRAS

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## Abstract

As observed by Kaplansky, a $C^*$-algebra is indecomposable exactly when its primitive ideal spectrum is connected. We extend the list of properties relating indecomposability to connectivity and define a corresponding concept of component projections in the enveloping von Neumann algebra of the $C^*$-algebra in question. We prove that in two essentially different ways, the component structure thus defined is identical to the component structures of the spectra associated to the $C^*$-algebra. Finally, we also consider further notions of connectivity, *arcwise* and *local*, in this setting.


0.1. *Introduction*. Let $X$ be a locally compact Hausdorff space and consider how the topological notions "connectivity" and "component" may be phrased in terms of the algebra $\mathfrak{A} = C_0(X)$. It is easy to see that $X$ is connected if and only if $\mathfrak{A}$ has no non-trivial decompositions $\mathfrak{A} = \mathfrak{I}_0 \oplus \mathfrak{I}_1$, where $\mathfrak{I}_0, \mathfrak{I}_1$ are closed ideals of $\mathfrak{A}$. But in general – corresponding to the fact that components need not be open – we can not find the components as ideals of $\mathfrak{A}$.

We resort to Akemann and Pedersen's theory of open and closed projections in the enveloping von Neumann algebra to define a set of component projections. The complications inherent in this theory may be overcome, mainly due to the abelian nature of the notion of connectivity. With one notable exception, we show that one may work with component projections as in the commutative case, and we have found that the component structure thus defined is the same as the component structures of both the spectra $P(\mathfrak{A})$ and $\mathrm{Prim}(\mathfrak{A})$, and that they in turn coincide, even though the spectra are topologically very different.

To understand the technical relevance of our results, one must return to the foundations of non-commutative topology for $C^*$-algebras. As is indicated by the existence of non-commutative Urysohn lemmas ([3], [9]), a $C^*$-algebra is really a generalization of a *normal* topological space. To work

Received July 27, 1995.
Supported by Knud Høgaards Fond and the Danish Science Research Council.

with the topology, however, we must pay a price: $P(\mathfrak{A})$ is rarely locally compact, $\mathrm{Prim}(\mathfrak{A})$ is rarely Hausdorff, and the supremum of two closed projections is rarely closed. To work with a $C^*$-algebra as an object with strong separation properties, one must manouver carefully back and forth between these different pictures of the topology of $\mathfrak{A}$, and it is in this setting our results find their application.

In particular, they can be used to describe the component structure *at infinity* for a $C^*$-algebra $\mathfrak{A}$ and relate it to the component structure of the corona algebra $M(\mathfrak{A})/\mathfrak{A}$, as is done in our joint work with C.A. Akemann ([5]). The notion of local connectivity is also crucial there. Our paper [5] has appeared while the present paper was being considered for publication by Mathematica Scandinavica. To follow suggestions from the referee without rendering invalid the references from [5], we have had to resort to a few anomalies in our enumeration of theorems.

0.2. *Acknowledgments*. This project was initiated during the academic year 1991/92, in which the author was an exchange student at University of California at Santa Barbara, under the auspices of C. A. Akemann. It would never have existed without the numerous inspiring conversations with him that I have had the privilege of having, both as a student and as a co-author. Also, I wish to express my gratitude for the hospitality that has been extended to me during my stay in 1994/95 at The Fields Institute for Research in Mathematical Sciences, where the final stages of this work were carried out. Thanks are also due to the referee for suggesting many improvements to the results of the paper, and for his constructive criticism of the exposition chosen, several years ago, by an unexperienced author.

0.3. *Notation*. When $\mathfrak{H}$ is a Hilbert space, $B(\mathfrak{H})$ is the set of bounded operators here, $K(\mathfrak{H})$ the compact ones. We suppress reference to $\mathfrak{H}$ when $\dim\mathfrak{H} = \aleph_0$. Let $\mathfrak{A}$ be a $C^*$-algebra, unital or not. We denote the *universal representation* of $\mathfrak{A}$ by $(\pi_u, \mathfrak{H}_u)$, and as usual, since the von Neumann algebra $\pi_u(\mathfrak{A})''$ is isomorphic as a Banach space to the second dual of $\mathfrak{A}$, we write $\mathfrak{A}^{**}$ for the von Neumann algebra as well. We identify $\mathfrak{A}$ with its image in $\mathfrak{A}^{**}$ and use this as a default framework for constructions involving $\mathfrak{A}$, considering constructs such as the unitization $\mathfrak{A}$ and the multiplier algebra $M(\mathfrak{A})$ as subalgebras here. Denote by $\mathfrak{A}'$ the commutant of $\mathfrak{A}$ in $B(\mathfrak{H}_u)$. The sum of all irreducible representations of $\mathfrak{A}$, considered as a subrepresentation of $\pi_u$, is denoted by $\pi_a$; its cover in $\mathfrak{A}^{**}$ by $\mathsf{z}$. Central covers are taken in $\mathfrak{A}^{**}$ and denoted by $\mathsf{c}(\cdot)$.

Unless specified, closure in dual spaces is with respect to the *weak*$^*$ topology. In $\mathfrak{A}^*$ we denote the *quasi-states* by $Q(\mathfrak{A})$, the *states* by $S(\mathfrak{A})$ and the *pure states* by $P(\mathfrak{A})$. When $\varphi \in S(\mathfrak{A})$, we denote the GNS triple arising from $\varphi$ by $(\pi_\varphi, \mathfrak{H}_\varphi, \xi_\varphi)$. For a subset $E$ of $Q(\mathfrak{A})$, set

$$\mathrm{sat}(E) = \{\varphi(u^* \cdot u) | u \in \mathcal{U}(\mathfrak{A}^\sim)\}.$$

We say that $E$ is *saturated* when $E = \mathrm{sat}\, E$.

$\hat{\mathfrak{A}}$ is the set of (equivalence classes of) irreducible representations of $\mathfrak{A}$, $\mathrm{Prim}(\mathfrak{A})$ the set of primitive ideals, equipped with the Jacobson topology. Finally, we shall need the *Fell map* $\boldsymbol{\Phi} : P(\mathfrak{A}) \to \mathrm{Prim}(\mathfrak{A})$ defined by $\varphi \mapsto \ker \pi_\varphi$, and, for given $a \in \mathfrak{A}$, the map $\hat{a} : Q(\mathfrak{A}) \to \mathsf{C}$ given by $\hat{a}(\varphi) = \varphi(a)$.

0.4. *On projections in* $\mathfrak{A}^{**}$. The work by Tomita and Effros devised correspondences between hereditary subalgebras of a $C^*$-algebra $\mathfrak{A}$ and certain *weak*$^*$ closed subsets of $\mathfrak{A}^*$. This was carried further, and placed in a quasi-topological setting, with Akemann and Pedersen's characterization of those projections in the double dual $\mathfrak{A}^{**}$ that support *weak*$^*$ closed subsets of $\mathfrak{A}^*$. We will follow the approach in [20], using the projections of $\mathfrak{A}^{**}$ as our main tool for describing the structure of $\mathfrak{A}$. Reflecting this fact notationally, we will often need to consider the sets

$$F(p) = \{\varphi \in Q(\mathfrak{A}) | \varphi(1 - p) = 0\}$$

and $P(p) = F(p) \cap P(\mathfrak{A})$, as well as the hereditary subalgebra

$$\mathrm{her}(p) = p\mathfrak{A}^{**}p \cap \mathfrak{A}$$

*supported* by $p \in \mathfrak{A}^{**}$. As defined in [2], see also [20], a projection $p$ in $\mathfrak{A}^{**}$ is *closed* when $F(p)$ is closed. We also say that $p$ is *open* when $1 - p$ is closed, and that $p$ is *compact* if it is closed and dominated by an element of $\mathfrak{A}$ itself. Any $p$ is dominated by a smallest closed projection which we denote $\overline{p}$. In fact, $\mathrm{her}(\cdot)$ establishes a 1–1 correspondence between open projections and hereditary subalgebras.

We are going to depend on Akemann's and Effros' pioneering work in [2] and [10] for results on how to work with open and closed projections. One needs to note, however, that although the $C^*$-algebras considered in these papers are assumed to be unital, the results hold true in general. Details of this may be found in [11]. We record a few known observations (cf. [20, 2.6.3], [19, 5.4.10]):

LEMMA 0.1. *If $p$ is open, so is* $\mathsf{c}(p)$.

LEMMA 0.2. *If $\boldsymbol{\Phi}(\varphi) = \boldsymbol{\Phi}(\psi)$, then* $\varphi \in \overline{\mathrm{sat}\{\psi\}}$.

LEMMA 0.3. *If $C \subseteq P(\mathfrak{A})$ is saturated and closed, then there exists a central projection $x$ in $\mathfrak{A}^{**}$ with $P(x) = P(\overline{x}) = C$. We may choose $x \leq$* $\mathsf{z}$.

## 1. Connectivity

1.1. *On subsets of the pure state spectrum.* We start this section with some preliminary results, pertaining to the structure of certain subsets of $P(\mathfrak{A})$.

We shall see that every preimage of a connected set under the Fell map $\boldsymbol{\Phi}$ is also connected. In order to avoid repetitions when dealing with local connectivity we prove a little more, namely that preimages of arbitrary connected sets of $\mathrm{Prim}(\mathfrak{A})$ are connected, even after they are cut down by sets of the form

$$V_{a,\varepsilon} = \{\varphi \in P(\mathfrak{A}) | \varphi(a) < \varepsilon\} \tag{1}$$

for any $a \in \mathfrak{A}_+$ and $\varepsilon > 0$.

To work with the $V_{a,\varepsilon}$, we shall need the following elementary observation on Hilbert spaces of dimension 2.

Lemma 1.1. *Let $\varepsilon > 0$ and $a \in M_2(\mathsf{C}) = \mathsf{B}(\mathsf{C}^2)$ with $0 \leq a \leq 1$ be given. If $\xi, \eta \in \mathsf{C}^2$ are unit vectors satisfying*

$$(a\xi|\xi) < \varepsilon \qquad (a\eta|\eta) < \varepsilon,$$

*there is a continuous function $\zeta : [0,1] \to \mathsf{C}^2$ with $\zeta(0) = \xi, \zeta(1) = \eta$, taking on unit vectors, such that*

$$(a\zeta(t)|\zeta(t)) < \varepsilon, \qquad t \in [0,1].$$

The lemma says that sets of the form $V_{a,\varepsilon}$ are arcwise connected in $P(\mathsf{B}(\mathfrak{H}))$ when $\dim \mathfrak{H} = 2$. This is all we need for the following observation.

Proposition 1.2. *Let $a \in \mathfrak{A}_+$, $\varepsilon > 0$ and $\varphi \in P(\mathfrak{A})$ be given. The set*

$$\mathrm{sat}\{\varphi\} \cap V_{a,\varepsilon}$$

*is arcwise connected.*

Proof. Let $x$ denote the central cover of $\varphi$ in $\mathfrak{A}^{**}$. We need only construct a path of vector states on $x\mathfrak{H}_u$ lying in $V_{a,\varepsilon}$ to a given such state, $\psi$. If $\varphi \neq \psi$, the two corresponding unit vectors $\xi$ and $\eta$ span a two-dimensional subspace $E$ of $p\mathfrak{H}_u$. We apply Lemma 1.1 to $\varepsilon, E$, the compression of $a$ to $E$, $\xi$ and $\eta$.

Corollary 1.3. *A component of $P(\mathfrak{A})$ is saturated.*

Proposition 1.4. *Let $\mathfrak{A}$ be a $C^*$-algebra, and let $V_{a,\varepsilon}$ with $a \in \mathfrak{A}^\sim_+$ and $\varepsilon > 0$ be given. If $C \subseteq \boldsymbol{\Phi}(V_{a,\varepsilon})$ is connected, then so is $\boldsymbol{\Phi}^{-1}(C) \cap V_{a,\varepsilon}$.*

Proof. Let $D$ denote this set and note that $\boldsymbol{\Phi}(D) = C$ by our assumption. If $D$ is not connected, it is separated by non-empty open sets $G_0, G_1 \subseteq V_{a,\epsilon}$. We have

$$C = \boldsymbol{\Phi}(D) \subseteq \boldsymbol{\Phi}(G_0) \cup \boldsymbol{\Phi}(G_1),$$

and as the sets on the right are both open because $\boldsymbol{\Phi}$ is open, we get from connectivity of $C$ that

$$\boldsymbol{\Phi}(G_0) \cap \boldsymbol{\Phi}(G_1) \cap C \neq \emptyset.$$

This means that we can find $\psi_i \in G_i \cap \boldsymbol{\Phi}^{-1}(C) = G_i \cap D$ such that $\boldsymbol{\Phi}(\psi_0) = \boldsymbol{\Phi}(\psi_1)$. The set $\mathrm{sat}\{\psi_1\} \cap V_{a,\epsilon}$ is connected by Lemma 1.1 above, and by Lemma 0.2, $D' = (\mathrm{sat}\{\psi_1\} \cap V_{a,\epsilon}) \cup \{\psi_0\}$ is also connected. Obviously, $D' \subseteq D$, and it meets both $G_0$ and $G_1$. This contradicts connectedness of $D'$.

COROLLARY 1.5. *If $C$ is a connected subset of* $\mathrm{Prim}(\mathfrak{A})$, *$\boldsymbol{\Phi}^{-1}(C)$ is a connected subset of $P(\mathfrak{A})$.*

LEMMA 1.6. *When $a \in \mathfrak{A}_{sa}$, $\hat{a}(P(\mathfrak{A})) = \hat{a}(S(\mathfrak{A}))$ if and only if $\hat{a}(P(\mathfrak{A}))$ is convex.*

PROOF. The lemma follows from the equality $\hat{a}(S(\mathfrak{A})) = \mathrm{co}(\hat{a}(P(\mathfrak{A}))$, which is true in general and obvious in the unital case by the finite Krein-Milman theorem. For lack of a reference, we take in the non-unital case a shorter path to our more specialized claim. We have that $\mathrm{sp}(a)\backslash\{0\} \subseteq \hat{a}(P(\mathfrak{A}))$ and $\hat{a}(S(\mathfrak{A})) \subseteq \mathrm{co}(\mathrm{sp}(a))$. When $\hat{a}(P(\mathfrak{A}))$ is convex with $0 \in \hat{a}(P(\mathfrak{A}))$, we can argue with $Q(\mathfrak{A})$ as in the unital case. If $0 \notin \hat{a}(P(\mathfrak{A}))$, then by convexity we may assume that $\hat{a}(P(\mathfrak{A}))$ lies entirely within $\mathsf{R}_+$, so that $\hat{a}(P(\mathfrak{A})) = (0, \|a\|]$. We also note that $0 \notin \hat{a}(S(\mathfrak{A}))$, for if $\varphi \in S(\mathfrak{A})$ has $\varphi(a) = 0$, we can choose $\psi \in P(\mathfrak{A})$ with $L_\varphi \subseteq L_\psi$ from [20, 3.13.5]. As then $a \in L_\varphi \subseteq L_\psi$, $a \in L_\psi + L_\psi^* = \ker\psi$ by [20, 3.13.6]. We conclude that

$$\hat{a}(S(\mathfrak{A})) \subseteq \mathrm{co}(\mathrm{sp}(a))\backslash\{0\} \subseteq (0, \|a\|] = \hat{a}(P(\mathfrak{A})) \subseteq \hat{a}(S(\mathfrak{A})).$$

1.2. *Connected $C^*$-algebras.* A ring $R$ with the property that for any pair $I_0, I_1$ of ideals of $\mathfrak{A}$,

$$R = I_0 \oplus I_1 \Longrightarrow \{I_0, I_1\} = \{(0), R\}$$

is often called *indecomposable*. The idea of relating indecompososability to connectivity is as old as the theory of structure spaces itself, first noted by Jacobson in [14, Theorem 2] in the case of semisimple unital rings. The corresponding result for $C^*$-algebras, employing the primitive ideal spectrum, was found by Kaplansky in [16, 8.5]. The following theorem takes this a bit further, invoking the set of pure states in different ways.

THEOREM 1.7. *Let $\mathfrak{A}$ be a $C^*$-algebra. The following conditions are equivalent*

(i) $M(\mathfrak{A})$ *has no non-trivial central projections.*

(ii) *If* $1 = p_0 + p_1$ *in* $\mathfrak{A}^{**}$ *with* $p_0, p_1$ *central open projections, then* $\{p_0, p_1\} = \{0, 1\}$.

(iii) *If* $\mathfrak{A} = \mathfrak{I}_0 \oplus \mathfrak{I}_1$ *with* $\mathfrak{I}_0, \mathfrak{I}_1$ *ideals of* $\mathfrak{A}$, *then* $\{\mathfrak{I}_0, \mathfrak{I}_1\} = \{\{0\}, \mathfrak{A}\}$.

(iv) $\mathrm{Prim}(\mathfrak{A})$ *is connected.*

(v) $P(\mathfrak{A})$ *is connected.*

*and imply*

(vi) $\forall a \in \mathfrak{A} : \hat{a}(P(\mathfrak{A}))$ *is connected.*

(vii) $\forall a \in \mathfrak{A}_{sa} : \hat{a}(P(\mathfrak{A})) = \hat{a}(S(\mathfrak{A}))$.

*If* $\mathfrak{A}$ *is* $\sigma$*-unital, all the conditions are equivalent.*

Definition 1.8. A $C^*$-algebra $\mathfrak{A}$ is called *connected* if it satisfies (i)–(v) above.

Proof of Theorem 1.7. The first two conditions are equivalent by [20, 3.12.9]. That (ii), (iii) and (iv) are equivalent follows by the well-known correspondence between ideals, central open projections, and open sets of $\mathrm{Prim}(\mathfrak{A})$. That (v) $\Longrightarrow$ (iv) is clear by continuity of $\Phi$, (iv) $\Longrightarrow$ (v) follows by Corollary 1.5, and (v) $\Longrightarrow$ (vi) is a consequence of the continuity of $\hat{a} : P(\mathfrak{A}) \to \mathsf{C}$. To get (vi) $\Longrightarrow$ (vii), we apply Lemma 1.6. Finally, assume that $\mathfrak{A}$ has a strictly positive element $h$ and that (i) does not hold. Then there exist non-trivial central projections $p_0, p_1 \in M(\mathfrak{A})$ with $p_0 + p_1 = 1$ and $a = hp_0 - hp_1 \in \mathfrak{A}_{sa}$. We claim that $\hat{a}(P)$ is not convex. When $\varphi \in P(\mathfrak{A})$, we have $\{\varphi(p_0), \varphi(p_1)\} = \{0, 1\}$, so $0 \notin \hat{a}(P(\mathfrak{A}))$. But since $p_0$ and $p_1$ are non-zero, $\mathrm{sp}(a)$, and hence $\hat{a}(P(\mathfrak{A}))$, contains both positive and negative elements.

Note that although the two spectra $P(\mathfrak{A})$ and $\mathrm{Prim}(\mathfrak{A})$ may be very different as topological spaces, they are connected simultaneously. This point of view will be expanded as we progress.

Remark 1.9. $0^\circ$: To see why $\sigma$-unitality must be taken into account in (vi) and (vii) above, consider $X = \mathsf{R} \times \mathsf{R}$ endowed with discrete topology in the first coordinate and the usual one in the second. In this case $X$ is far from connected, but any continuous function defined on it has connected range if it vanishes at infinity.

$1^\circ$: Condition (vii) is a *Lyapunov theorem* in the the language of [4], so the theorem above determines when a such a theorem holds true for *all* mappings $\hat{a} : S(\mathfrak{A}) \to \mathsf{R}$. The set $S(\mathfrak{A})$ is *weak*$^*$ compact, and by definition, $\hat{a}$ is continuous in this topology. If $\mathfrak{A}$ is not connected, by Theorem 1.7, we get $\hat{a}(S(\mathfrak{A})) \neq \hat{a}(\mathrm{ext}(S(\mathfrak{A})))$ for some $a \in \mathfrak{A}_{sa}$, and by the abstract Lyapunov theorem [4, 1.7] we conclude that the *facial dimension* of $S(\mathfrak{A})$ (see [4, p. 10]) is one. Connectivity does not imply that the facial dimension is strictly larger than one.

2°: Clearly any simple, or even prime, $C^*$-algebra is connected.

3°: Suppose $\mathfrak{B}$ is a connected $C^*$-algebra sitting as an essential ideal in $\mathfrak{A}$. If $\mathfrak{A} = \mathfrak{I}_0 \oplus \mathfrak{I}_1$, we may assume that $\mathfrak{I}_0\mathfrak{B} = (0)$, whence $\mathfrak{I}_0 = (0)$ also. This proves that $\mathfrak{A}$ is connected. We even have, as a direct consequence of Theorem 1.7(i), that $\mathfrak{B}$ is connected precisely when $M(\mathfrak{B})$ is.

Corollary 1.11. *The $C^*$-algebra $\mathfrak{A}$ is prime if and only if every hereditary subalgebra of $\mathfrak{A}$ is connected.*

Proof. As every hereditary subalgebra of a prime $C^*$-algebra is prime, we get the forward implication by Remark 1.9 2°. On the other hand, if $\mathfrak{A}$ is not prime, ideals $\mathfrak{I}_0$ and $\mathfrak{I}_1$ exist with $\mathfrak{I}_0\mathfrak{I}_1 = 0$. Clearly $\mathfrak{I}_0 \oplus \mathfrak{I}_1$ is not connected.

## 2. Components of $\mathfrak{A}^{**}$

2.1. *Connected projections.*

Definition 2.1. Let $p \in \mathfrak{A}^{**}$ be a projection. We say that $p$ is *connected* if whenever $q_0, q_1$ are central open projections of $\mathfrak{A}^{**}$ such that

$$p = pq_0 + pq_1, \tag{2}$$

then $\{pq_0, pq_1\} = \{0, p\}$.

In fact, as the referee has pointed out to us, a projection $p$ is disconnected precisely when there are ideals $\mathfrak{I} \subseteq \mathfrak{J}$ such that $\mathfrak{J}/\mathfrak{I} \cong \mathfrak{B}_1 \oplus \mathfrak{B}_2$, and $p$ lives in $(\mathfrak{J}/\mathfrak{I})^{**}$ and meets both pieces.

Note that by Theorem 1.7, $\mathfrak{A}$ is connected if and only $1 \in \mathfrak{A}^{**}$ is. A pair $\{q_0, q_1\}$ of central open projections is said to *separate* $p$ when (2) holds. It is said to be *trivial* (with respect to $p$) if $\{pq_0, pq_1\} = \{0, p\}$. In these words, $p$ is connected when every separating pair is trivial. Note that a minimal projection is automatically connected, as is a minimal central projection. Also, $p = 0$ is connected.

Proposition 2.2. *If $p \in \mathfrak{A}^{**}$ is an open projection, then $p$ is connected if and only if* $\mathrm{her}(p)$ *is.*

Proof. By [20, 3.11.9], the strong closure $\mathrm{her}(p)^-$ relative to $\mathfrak{A}^{**}$ is $p\mathfrak{A}^{**}p$. Furthermore, $\mathrm{her}(p)^-$ and $\mathrm{her}(p)^{**}$ are isomorphic under a normal isometry that preserves $\mathrm{her}(p)$ (cf. [20, 3.7.9]), so, using [20, 3.11.9], we may identify the open projections of $\mathrm{her}(p)^{**}$ with those of $\mathfrak{A}^{**}$ that lie under $p$.

Assume that $\mathrm{her}(p)$ is connected and let $\{q_0, q_1\}$ be a separation of $p$. By

[2, II.7], $pq_0$ and $pq_1$ are open. They are also central as elements of $p\mathfrak{A}^{**}p$, so as explained above the separation is trivial. Conversely, a separation of the identity in $\mathrm{her}(p)^{**}$ gives a separation $p = p_0 + p_1$, where $p_0, p_1$ are open central projections of $p\mathfrak{A}^{**}p$. For each $i \in \{0,1\}$, $\mathsf{c}(p_i)$ is open by Lemma 0.4, so since $p\mathsf{c}(p_i) = p_i$, we get that $\{\mathsf{c}(p_0), \mathsf{c}(p_1)\}$ is a separation of $p$ and $\{p_0, p_1\}$ is trivial.

Proposition 2.3. *Let $p, q$ and $p_i, i \in I$ be projections of $\mathfrak{A}^{**}$.*

(i) *$p$ is connected if and only if $\mathsf{c}(p)$ is.*

(ii) *If $p$ is connected, so is any $q$ with $p \leq q \leq \overline{p}$.*

(iii) *Assume that every $p_i$ is connected and that $q$ is minimal central and for all $i \in I$, $p_i q \neq 0$. Then $\bigvee_I p_i$ is connected.*

(iv) *Assume that every $p_i$ is connected and that there exists $\varphi \in P(\mathfrak{A})$ such that for all $i \in I$, $\varphi(p_i) > 0$. Then $\bigvee_I p_i$ is connected.*

Proof. For (i), use that the mapping $a\mathsf{c}(p) \mapsto ap$ is an isomorphism between $\mathfrak{A}'p$ and $\mathfrak{A}'\mathsf{c}(p)$, cf. [20, 2.6.7]. To prove (ii), assume that $\{x_0, x_1\}$ separates $q$, say with $px_0 = 0, px_1 = p$. Since $x_0$ is open, $1 - x_0$ is closed, so as $p \leq 1 - x_0$, also $q \leq \overline{p} \leq 1 - x_0$, whence $x_0 q = 0$. The claim in (iii) is trivial when all $p_i$ are central, and we can reduce to this case by (i). (iv) follows by applying (iii) to the central cover of $\pi_\varphi$.

Remark 2.4. We have generalized all the basic results about connected sets except one: That $\bigcap_1^\infty G_n$ is connected when $\{G_n\}$ is a decreasing sequence of connected and closed sets in a compact Hausdorff space, cf. [22, 28.2]. As a first surprise, there is *no* corresponding result in a unital $C^*$-algebra; indeed we can find a decreasing sequence $(p_n)_1^\infty$ of closed, connected projections in a unital $C^*$-algebra with $\bigwedge_{n=1}^\infty p_n$ not connected.

For this, let $q \in \mathsf{B}$ be a projection with infinite rank and corank. Let $\mathfrak{A} = C^*(\mathsf{K}, q, 1)$ and denote the central covers of the two irreducible representations given by $\mathfrak{A}/\mathsf{K} = \mathsf{C} \oplus \mathsf{C}$ by $y_0$ and $y_1$. Write $x = \mathsf{z} - y_0 - y_1$. Let $p_n$ be an descending sequence of projections of finite corank, converging strongly to 0. With $p = \vee p_n$ in $\mathfrak{A}^{**}$ we get that $p\mathsf{z} = y_0 + y_1$. Applying [20, 3.11.9] one gets central open projections $q_0, q_1 \in \mathfrak{A}^{**}$ with $q_i \mathsf{z} = x + y_i$. We get from $pq_0q_1\mathsf{z} = 0$, using that the $pq_i$ are closed by [2, II.7] and applying [2, II.7] twice, that $pq_0$ and $pq_1$ are orthogonal and $p = pq_0 + pq_1$.

2.2. *Component projections.*

Definition 2.5. A *component projection* of $\mathfrak{A}$ is a maximal connected projection of $\mathfrak{A}^{**}$.

Combining Proposition 2.3 (iii) with Zorn's lemma, one gets:

PROPOSITION 2.6. *Any connected projection is dominated by a component projection.*

PROPOSITION 2.7. *Every component projection of $\mathfrak{A}$ is closed and central. If two component projections are different, they are orthogonal.*

PROOF. The first claim is clear from Proposition 2.3 (i)–(ii). If $p, q$ are component projections and $pq \neq 0$ then, since $pq = p \wedge q$ is closed, $pq\mathsf{z} \neq 0$ from [2, II.16] and we can find $\varphi \in P(\mathfrak{A})$ such that $\varphi(pq) > 0$. Consequently $\varphi(p), \varphi(q) > 0$ and by Proposition 2.3 (iv), $p \vee q$ is connected. By maximality, $p = p \vee q = q$.

REMARK 2.8. Let $x$ denote the sum of all component projections of $\mathfrak{A}^{**}$. From what we have already seen, $x$ must dominate $\mathsf{z}$, but in general, $x < \mathsf{1}$. For instance one may conclude from Proposition 3.6 below that the component projections of $C(X)$ is exactly the set of minimal projections of $C(X)^{**}$ when $X$ is totally disconnected. Hence in this case $x = \mathsf{z}$.

## 3. Component structures

3.1. *Preliminaries*. When attempting to describe the component structure of a general topological space $X$, one can choose at least two different strategies. One is to focus attention on the Boolean algebra $\mathrm{Lat}(X)$ consisting of the family of clopen sets endowed with the natural settheoretic operations. Applying a Wallman type compactification, one may derive for this the *Stone space* $\sigma X$ ([21, I.8]) which is the closest one gets to a space of components. The other strategy is to forget about the set of clopen sets and focus attention directly on the set of components. In this case, one can only describe the components structure in coarse terms like cardinality. These two foci are clearly not independent, but the Boolean algebra does not even determine the cardinal of the set of components, even though it appears to carry more information. In fact, the following is all that can be said.

PROPOSITION 3.1. *The map*

$$C \mapsto \{f \in \mathrm{Lat}(X) | f \geq 1_C\}$$

*sends the set of components of $X$ to the set of ultrafilters in* $\mathrm{Lat}(X)$. *When $X$ is compact, the map is onto. When $X$ is also Hausdorff, the map is a bijection.*

We shall not need the result; for a proof, see [12]. What is more relevant in this context is the restrictions on the result, all of which are necessary. For instance, the map is not onto for the locally compact Hausdorff space $X = \mathsf{N}$ and not injective for the locally compact Hausdorff space given by

$$\{0\} \times C \cup \left( \bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} \right\} \times [0,1] \right)$$

where $C$ is an open and non-connected subset of $[0,1]$. Mimicking this construction with a space that is not second countable, we even get that the cardinality of the sets of components may be strictly larger than the cardinality of the set of ultrafilters of $\mathrm{Lat}(X)$. We may also get, by identifying the points in the fibers over $\frac{1}{n}$, a compact, but non-Hausdorff example of the same phenomenon.

Even in a *unital* non-commutative setting, we are faced with a similar problem:

REMARK 3.2. It is possible to have

$$c < \bigwedge \{p \in \mathcal{Z}(M(\mathfrak{A})) | p \text{ is a projection}, p \geq c\},$$

even for a component $c$ in a unital $C^*$-algebra $\mathfrak{A}$. Consider

$$\mathfrak{A} = \left\{ f : \mathsf{N} \cup \{\infty\} \to M_2(\mathsf{C}) \,\middle|\, f(n) \to f(\infty), f(\infty) = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \right\}.$$

and let $x_n$ be the cover of the irreducible representation over $n$ for each $n \in \mathsf{N}$. Let $y_0, y_1$ be the covers of the irreducible representations over $\infty$. With $c = y_0$, any clopen projection which donimates $c$ also dominates $y_0 + y_1$, showing that the infimum is not $c$. And $c$ is a component because any dominating connected component $d$ must satisfy $y_0 = c\mathsf{z} \leq d\mathsf{z} \leq y_0 + y_1$.

In general, one may not infer much about connectivity of $p$ from connectivity of $P(p)$ or vice versa. When $\mathfrak{A}$ is simple, *any* projection is connected, but clearly not every subset of $P(\mathfrak{A})$ is connected. In the other direction, note that if $p \in \mathfrak{A}^{**}$ has $p\mathsf{z} = 0$, then $p \oplus p$ is not connected in $\mathfrak{A} \oplus \mathfrak{A}$, but $P(p \oplus p) = \emptyset$. Something can be said, however:

LEMMA 3.3. *Let $p$ be a projection in $\mathfrak{A}^{**}$.*
(i) *If $p$ is either open or closed, $P(p)$ connected $\Longrightarrow$ $p$ connected.*
(ii) *If $p$ is open, $P(p)$ connected $\Longleftrightarrow$ $p$ connected.*
(iii) *If $x$ is closed and central, $P(x)$ connected $\Longleftrightarrow$ $x$ connected.*

PROOF. For (i), assume that $p = px_0 + px_1$ is a non-trivial separation of $p$. We have that the pair of open sets $\{P(x_0), P(x_1)\}$ disconnects $P(p)$ by centrality. Furthermore, as both $px_0$ and $px_1$ are open or closed when $p$ is according to [2, II.5] or [2, II.7], the separation is non-trivial by [2, II.16]. The other implication of (ii) is a consequence of Theorem 1.7 and Proposition 2.2, since $P(p)$ and $P(\mathrm{her}(p))$ are homeomorphic (see [5, 1.1.3]). For the implication of (iii) not covered by (i), assume that $P(x) = U_0 \cup U_1$ disjointly with closed sets $U_i$. Both $U_i$ must be saturated, so we may find central projections $y_i$ with $P(y_i) = P(\overline{y_i}) = U_i$ by Lemma 0.3. By [2, II.5,II.17] and $P(\overline{y_0} \wedge \overline{y_1}) = \emptyset$, the closed central projections $\overline{y_0}$ and $\overline{y_1}$ are orthogonal, so $\overline{y_0} + \overline{y_1}$ is a projection. We have $P(\overline{y_0} + \overline{y_1}) \supseteq P(x)$, so $\overline{y_0} + \overline{y_1} \geq x$ according to [2, II.17]. Now $\{1 - \overline{y_0}, 1 - \overline{y_1}\}$ separates $x$, so by assumption we may assume that $(1 - \overline{y_0})x = 0, (1 - \overline{y_1})x = x$. We conclude that $\emptyset = P(\overline{y_1}x) = U_1$.

3.2. *Lattices of clopen sets and projections.*

PROPOSITION 3.4. *The Boolean algebras given by*
(i) *central projections of* $M(\mathfrak{A})$
(ii) *clopen central projections of* $\mathfrak{A}^{**}$
(iii) *clopen subsets of* $\mathrm{Prim}(\mathfrak{A})$
(iv) *clopen subsets of* $P(\mathfrak{A})$
*are all isomorphic.*

DEFINITION 3.5. We denote this Boolean algebra by Lat $\mathfrak{A}$.

PROOF OF PROPOSITION 3.4. Isomorphism of the three first lattices follow from [20, 3.12.9] and [20, 4.4.8]. An isomorphism between the latter two Boolean algebras is induced by the map $\Phi$. When $G$ is a clopen subset of $P(\mathfrak{A})$, we claim that $\Phi(G)$ is also clopen. As $\Phi$ is onto and open by [20, 4.3.3], this will follow by the claim

$$\Phi(G) \cap \Phi(P(\mathfrak{A}) \backslash G) = \emptyset.$$

To see this, assume that $\Phi(\varphi) = \mathfrak{J} = \Phi(\psi)$ for $\varphi \in G, \psi \notin G$. Then $\Phi^{-1}(\{\mathfrak{J}\})$ is non-trivially separated by $\{G, P(\mathfrak{A}) \backslash G\}$, contradicting Corollary 1.5. The maps are both Boolean algebra isomorphisms, in the case of $\Phi$ because $\Phi(P(\mathfrak{A}) \backslash G) = \mathrm{Prim}(\mathfrak{A}) \backslash G$ by the above.

3.3. *Components and component projections.*

PROPOSITION 3.6. *There is a canonical bijective correspondence between the sets of*
(i) *component projections of* $\mathfrak{A}^{**}$
(ii) *components of* $P(\mathfrak{A})$
(iii) *components of* $\mathrm{Prim}(\mathfrak{A})$

DEFINITION 3.7. The cardinal of these sets is denoted by $c_{\mathsf{K}}(\mathfrak{A})$. The number of elements in these sets, with values in $\{1, 2, \ldots, \infty\}$, is denoted by $\#_{\mathsf{K}}\mathfrak{A}$.

PROOF OF PROPOSITION 3.6. The correspondence between the first two sets is given by the map $p \mapsto P(p)$. When $x$ is a component projection of $\mathfrak{A}^{**}$, Lemma 3.3 (iii) applies according to Proposition 2.7, and so $P(x)$ is connected. Assume $P(x) \subseteq C$ where $C$ is a component. By Corollary 1.3, $C$ is saturated and we can hence by Lemma 0.3 take a central projection $y$ in $\mathfrak{A}^{**}$ with $P(y) = P(\overline{y}) = C$. But as $\overline{y}$ is connected by Lemma 3.3 (iii) again, $\overline{y} = x$ and the two sets of pure states agree. The map is thus well-defined. It is onto by Proposition 2.7 and 1–1 by [2, II.16].

As in Proposition 3.4, the correspondence between the last two sets is given by $\boldsymbol{\Phi}$. To see this, let $D_1, D_2$ be components of $P(\mathfrak{A})$ and assume that $\boldsymbol{\Phi}(D_1), \boldsymbol{\Phi}(D_2)$ are both contained in the component $C$ of $\mathrm{Prim}(\mathfrak{A})$. By Corollary 1.3, $\boldsymbol{\Phi}^{-1}(C)$ is connected, and by maximality of the $D_i$, $D_1 = \boldsymbol{\Phi}^{-1}(C) = D_2$. Clearly, then, also $C = \boldsymbol{\Phi}(D_i)$, so we have proven that $\boldsymbol{\Phi}$ sends components to components and is injective. The map is onto since $\boldsymbol{\Phi}$ is.

Arguing as in Theorem 1.7 we get

Proposition 3.8. *We have*

$$\begin{aligned}\#_{\mathsf{K}}\mathfrak{A} &= \sup\{n \in \mathsf{N} | p_1, \ldots, p_n \text{ non-trivial orthogonal central} \\ &\qquad\qquad\qquad\qquad\text{projections of } M(\mathfrak{A})\} \\ &\geq \sup\{\#_{\mathsf{K}}\hat{a}(P(\mathfrak{A})) | a \in \mathfrak{A}\}\end{aligned}$$

*with equality when* $\mathfrak{A}$ *is* $\sigma$*-unital.*

## 4. Components of $C^*$-algebra constructions

This section contains results relating the component structure of a $C^*$-algebra constructed from other $C^*$-algebras to those of its constituents. As in Section 3, we work both with the lattice and the component approach.

4.1. *Sums.* By the *union* ($\sqcup$) of Boolean algebras, we understand the Boolean algebra achieved from a disjoint union (also denoted by $\sqcup$) of the Stone spaces. Using the obvious maps of complemented ideals, we get:

Lemma 4.1. *There are natural isomorphisms between* $\mathrm{Lat}(\sum_I \mathfrak{A}_i)$, $\mathrm{Lat}(\prod_I \mathfrak{A}_i)$ *and* $\bigsqcup_I \mathrm{Lat}\, \mathfrak{A}_i$.

Proposition 4.2. *Let* $\mathfrak{A}_i$, $i \in I$, *be* $C^*$*-algebras. We have*
(i) $c_{\mathsf{K}}(\sum_I \mathfrak{A}_i) = \sum_I c_{\mathsf{K}}(\mathfrak{A}_i)$.
(ii) $\#_{\mathsf{K}}(\prod_I \mathfrak{A}_i) = \sum_I \#_{\mathsf{K}}\mathfrak{A}_i$.

Proof. It follows, with a little work, from the definition of the Kaplansky sum that $\sqcup_I P(\mathfrak{A}_i)$ is homeomorphic to $P(\sum_I \mathfrak{A}_i)$, and clearly (i) is a consequence of this. The second claim follows directly from Lemma 4.1.

Remark 4.3. The product $\prod$ may have many more components than the sum $\Sigma$. Consider the case $I = \mathsf{N}$ and $\mathfrak{A}_i = \mathsf{C}$.

4.2. *Limits*. As there is in general no relation between the component structure of a $C^*$-algebra and a quotient of it, there is nothing nice to say about inductive limits using morphisms which are not injective. With appropriate identifications, we can reduce all inductive limits with injective morphisms to $C^*$-algebras of the form $(\bigcup_\Lambda \mathfrak{A}_\lambda)^=$, where $(\mathfrak{A}_\lambda)_\Lambda$ is an upward directed set of subalgebras of $\mathfrak{A}$. We consider this situation only.

PROPOSITION 4.4. *Let $\mathfrak{A}$ be a $C^*$-algebra of the form $\mathfrak{A} = (\bigcup_{\lambda\in\Lambda} \mathfrak{A}_\lambda)^=$, where $(\mathfrak{A}_\lambda)_{\lambda\in\Lambda}$ is an upward directed set of subalgebras. Provided that either $\mathfrak{A}$ is unital or every $\mathfrak{A}_\lambda$ is a hereditary subalgebra,*

$$\#_{\mathsf{K}}\mathfrak{A} \leq \liminf_{\lambda\in\Lambda} \#_{\mathsf{K}}\mathfrak{A}_\lambda.$$

PROOF. In the unital case, assume that

$$1 = \sum_{i=1}^{n} p_i, \tag{3}$$

where all $p_i$ are non-zero central projections of $\mathfrak{A}$. Standard lifting arguments ([18, 3.2], [8, 4.6.6]) show that $\lambda_0$ exists with $p_i \in \mathfrak{A}_\lambda$ for every $i \in \{1, \dots, n\}$ and $\lambda \geq \lambda_0$. We get the claim from Proposition 3.8.

Under the second assumption, we can find open projections $q_\lambda \in \mathfrak{A}^{**}$ such that $\mathfrak{A}_\lambda = \mathrm{her}(q_\lambda)$. When $\lambda \leq \mu$, $q_\lambda \leq q_\mu$ by [20, 3.11.9]. Suppose that $\bigvee_\Lambda q_\lambda \neq 1$ and take $\xi \in \mathfrak{H}_u$ orthogonal to all $q_\lambda$. We get that $a_\lambda\xi = q_\lambda a_\lambda q_\lambda \xi = 0$ for all $a_\lambda \in \mathfrak{A}_\lambda$, hence $a\xi = 0$ for all $a \in \mathfrak{A}$, contradicting the fact that $\pi_u$ is non-degenerate by definition. Now assume (3) as above. As $q_\lambda \nearrow 1$, we can find $\lambda_0$ such that $q_\lambda p_i \neq 0$ for all $i \in \{1, \dots, n\}$ and all $\lambda \geq \lambda_0$. Clearly $q_\lambda p_i$ constitutes a clopen projection in $\mathfrak{A}_\lambda$, and $n \leq \#_{\mathsf{K}}\mathfrak{A}_\lambda$, $\lambda \geq \lambda_0$ as above.

REMARK 4.5. 1°: Equality does not hold in the above propositions. Consider

$$\mathsf{K}^\sim = \overline{\overline{\bigcup_{n=1}^\infty M_n(\mathsf{C}) \oplus \mathsf{C}}} \qquad C_0([0,\infty)) = \overline{\overline{\bigcup_{n=1}^\infty \{f \in \mathfrak{A} | f(m) = 0, m \in \mathsf{N}, m \geq n\}}}$$

2°: For an example demonstrating the necessity of either of the conditions (i) or (ii) to hold for the proposition above, consider

$$C_0(\mathsf{R}\backslash(-1,1)) = \overline{\overline{\bigcup_{n=1}^\infty \{f \in C_0(X) | f(m) = f(-m), m \in \mathsf{N}, m \geq n\}}},$$

where equality follows by the Stone–Weierstrass theorem.

4.3. *Tensor products*. We denote the algebraic tensor product of two $C^*$-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by $\mathfrak{A}_1 \odot \mathfrak{A}_2$ and the completion of this under the $C^*$-norm $\|\cdot\|_\beta$ by $\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2$. We write $\beta = *$ for the minimal norm, and omit the index entirely when one of the algebras is nuclear.

The results below are limited by different conditions on the algebras. However, we have no examples showing the necessity of such restrictions.

Proposition 4.6. *If $\mathfrak{A}_1, \mathfrak{A}_2$ are both separable $C^*$-algebras, and $\mathfrak{A}_1$ is nuclear, we have*

(i) $\mathrm{Lat}(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \simeq \mathrm{Lat}(\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2))$.

(ii) $c_{\mathsf{K}}(\mathfrak{A}_1 \otimes \mathfrak{A}_2) = c_{\mathsf{K}}(\mathfrak{A}_1) c_{\mathsf{K}}(\mathfrak{A}_2)$.

Proof. By [7], $\mathrm{Prim}(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \simeq \mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2)$. We apply Propositions 3.4 and 3.6.

Proposition 4.7. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unital $C^*$-algebras,*

$$\mathrm{Lat}(\mathfrak{A}_1 \otimes_\gamma \mathfrak{A}_2) \simeq \mathrm{Lat}(\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2)).$$

*for any $C^*$-norm $\|\cdot\|_\gamma$.*

Proof. By [6, Theorem 3], we have that $\mathcal{Z}(\mathfrak{A}_1 \otimes_\gamma \mathfrak{A}_2) = \mathcal{Z}(\mathfrak{A}_1) \otimes \mathcal{Z}(\mathfrak{A}_2)$. Combine Proposition 3.4 with the Dauns-Hoffman theorem.

Proposition 4.8. *For $C^*$-algebras $\mathfrak{A}_1, \mathfrak{A}_2$, $\#_{\mathsf{K}}(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2) = \#_{\mathsf{K}}\mathfrak{A}_1 \#_{\mathsf{K}}\mathfrak{A}_2$.*

Proof. As every pair of central multiplier projections gives rise to a multiplier projection of the tensor product by the natural embedding $M(\mathfrak{A}_1) \otimes_* M(\mathfrak{A}_2) \hookrightarrow M(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)$, and since this projection must be central because the embedding is the identity on $\mathfrak{A}_1 \otimes_* \mathfrak{A}_2$, the rightmost number is not larger than the leftmost. And since $\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2)$ has a dense homeomorphic image in $\mathrm{Prim}(\mathfrak{A}_1 \otimes \mathfrak{A}_2)$ ([17, 11]), the numbers must agree.

In general, $\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2$ need not be prime when the $\mathfrak{A}_i$ are simple. However, as we are grateful to R. Archbold for pointing out to us, such a tensor product always contains a largest proper ideal and is hence connected.

4.4. *Unitizations*. We already noted in Remark 1.9 3° that $M(\mathfrak{A})$ is connected exactly when $\mathfrak{A}$ is. Note, however, that when $\mathfrak{I}$ is an essential ideal in $\mathfrak{A}$, $\mathfrak{I}$ may have more components than $\mathfrak{A}$, counted with values in $\mathsf{N} \cup \{\infty\}$ also. The strong relation between the component structures of $\mathfrak{A}$ and $M(\mathfrak{A})$ is thus another special feature of this particular unitization. It extends to a local phenomenon.

The following definition can be found in [9, p. 939].

DEFINITION 4.10. For a hereditary subalgebra $\mathfrak{B}$ of a $C^*$-algebra $\mathfrak{A}$, we may define a hereditary subalgebra of $M(\mathfrak{A})$ by

$$M(\mathfrak{A},\mathfrak{B}) = \overline{\mathfrak{B}}^{\beta} = \overline{\mathfrak{B}} \cap M(\mathfrak{A}) = \{x \in M(\mathfrak{A}) | \mathfrak{A}x \subseteq \mathfrak{A}\mathfrak{B}, x\mathfrak{A} \subseteq \mathfrak{B}\mathfrak{A}\}.$$

Here the first closure is relative to strict topology, the second is relative to strong topology.

LEMMA 4.11. *Let $\mathfrak{A}$ be a $\sigma$-unital $C^*$-algebra, $\mathfrak{B}$ a hereditary subalgebra. The map*

$$\mathfrak{I} \mapsto M(\mathfrak{A},\mathfrak{I})$$

*is a lattice isomorphism between* Lat $\mathfrak{B}$ *and* Lat $M(\mathfrak{A},\mathfrak{B})$.

PROOF. By [9, 3.46a], when $\mathfrak{B} = \mathfrak{I}_0 \oplus \mathfrak{I}_1$, also

$$M(\mathfrak{A},\mathfrak{B}) = M(\mathfrak{A},\mathfrak{I}_0 \oplus \mathfrak{I}_1) = M(\mathfrak{A},\mathfrak{I}_0) \oplus M(\mathfrak{A},\mathfrak{I}_1),$$

so the map described really sends Lat($\mathfrak{B}$) to Lat($M(\mathfrak{A},\mathfrak{B})$), and it is clear that this map preserves the lattice structure. It is 1–1 since $\mathfrak{I} = M(\mathfrak{A},\mathfrak{I}) \cap \mathfrak{A}$, and onto since if $M(\mathfrak{A},\mathfrak{B}) = \mathfrak{J}_0 \oplus \mathfrak{J}_1$, we may write

$$\mathfrak{B} = (\mathfrak{J}_0 \cap \mathfrak{A}) \oplus (\mathfrak{J}_1 \cap \mathfrak{A}) \tag{4}$$

and get $\mathfrak{J}_0$ as the image of $\mathfrak{J}_0 \cap \mathfrak{A}$ by essentiality of $\mathfrak{A}$ in $M(\mathfrak{A})$. In (4), inclusion from left to right follows by writing $a \in \mathfrak{B}$ as $a = bc$ with $b, c \in \mathfrak{B}$.

## 5. Further notions of connectivity

5.2. *Local connectivity*. We can mimic the idea of local connectivity, cf. [22, 27.7], in the setting of $C^*$-algebras. First an important lemma:

LEMMA 5.5. *Let $\mathfrak{A}$ be a $C^*$-algebra. The following conditions are equivalent.*
(i) *All components of $\mathfrak{A}$ are open projections.*
(ii) *All components of $P(\mathfrak{A})$ are open sets.*
(iii) *All components of* Prim($\mathfrak{A}$) *are open sets.*
(iv) $\mathfrak{A} \simeq \sum_I \mathfrak{A}_i$, *where every $\mathfrak{A}_i$ is connected.*

PROOF. We have already established a bijective correspondence between the components in (i)–(iii), and one checks directly that the maps involved preserve openness. To see that the first three conditions imply (iv), assume that when $(c_i)_I$ is the set of component projections of $\mathfrak{A}$, every $c_i$ is open. It is hence a multiplier, so for every $i \in I$, $\mathfrak{A}_i = \mathfrak{A}c_i$ is an ideal of $\mathfrak{A}$, and we may define $f : \mathfrak{A} \to \sum_I \mathfrak{A}_i$ by

$$f(a) = (ac_i)_{i \in I}.$$

To see that these tuples vanish at infinity, we employ the results and notation in [20, 4.4]. Given $\varepsilon > 0$ and $a \in \mathfrak{A}_+$, $\{t \in \mathrm{Prim}(\mathfrak{A}) | \check{a}(t) \geq \varepsilon\}$ is compact by [20, 4.4.4] and is thus covered by finitely many of the clopen subsets $C_i$ of $\mathrm{Prim}(\mathfrak{A})$ given by $c_i = 1_{C_i}$ via Dauns-Hofmann's theorem. We conclude that the set $\{i \in I | \|ac_i\| \geq \varepsilon\}$ is finite whenever $a \in \mathfrak{A}_+$ and may extend that conclusion to all of $\mathfrak{A}$ by decomposing. The map $f$ is clearly a bijection. Finally, (iv) $\Longrightarrow$ (ii) follows from Lemma 4.1.

Theorem 5.6. *Let $\mathfrak{A}$ be a $C^*$-algebra. The following conditions are equivalent:*

(i) *Whenever $\mathfrak{B}$ is a hereditary subalgebra of $\mathfrak{A}$, all components of $\mathfrak{B}$ are open projections.*

(ii) *Whenever $\mathfrak{I}$ is an ideal of $\mathfrak{A}$, all components of $\mathfrak{I}$ are open projections.*

(iii) $\mathrm{Prim}(\mathfrak{A})$ *is locally connected.*

(iv) $P(\mathfrak{A})$ *is locally connected.*

Definition 5.7. A $C^*$-algebra having these properties is called *locally connected.*

Proof of Theorem 5.6. Trivially, (i) implies (ii). Applying Lemma 5.5, we get (ii) $\Longrightarrow$ (iii), and (iii) $\Longrightarrow$ (i) follows by noting that $\mathrm{Prim}(\mathfrak{B})$ is homeomorphic to the open set $\mathrm{Prim}(\mathfrak{A}) \backslash \mathrm{Hull}(\mathfrak{B})$ according to [20, 4.1.10] and applying Lemma 5.5 again. That (iv) implies (iii) follows from the fact that $\boldsymbol{\Phi}$ is continuous, open and onto. Assume that (iii) holds, and fix $\varphi \in P(\mathfrak{A})$. The sets $V_{a,\epsilon}$ form a form a base for the *weak*$^*$ topology on $P(\mathfrak{A})$, as is seen by first using compactness of $Q(\mathfrak{A})$ to see that one may separate at $\varphi$ with elements of $\mathfrak{A}$ with $\varphi(a) = 0$, and then adding up using positivity. Hence it suffices to prove that every neighborhood of the form $V_{a,\varepsilon}$ contains a connected neighborhood $W$ of $\varphi$. The set $\boldsymbol{\Phi}(V_{a,\varepsilon})$ is open, so an open connected set $C$ satisfying $\boldsymbol{\Phi}(\varphi) \in C \subseteq \boldsymbol{\Phi}(V_{a,\epsilon})$ can be found. Put $V = \boldsymbol{\Phi}^{-1}(C) \cap V_{a,\varepsilon}$ and note that by Proposition 1.4, $V$ is connected. It is clearly an open neighborhood of $\varphi$ contained in $V_{a,\epsilon}$.

Corollary 5.8. *Let $p$ be an open projection in a locally connected $C^*$-algebra $\mathfrak{A}$. When $(c_i)_I$ is the set of components of $p$,*

$$p = \sum_{i \in I} c_i.$$

*Furthermore, $(\mathsf{c}(c_i))_I$ is the set of components of $\mathsf{c}(p)$.*

Proof. Since $\sum_I c_i$ is open, and $P(\sum_I c_i) = P(p)$ by applying Proposition 2.7 to $\mathrm{her}(p)$, we get the first equality by [2, II.17]. Since every $\mathsf{c}(c_i)$ is open by Lemma 0.1 and connected by Proposition 2.3(i), all we need to show is that they are orthogonal. This is clear since the $c_i$ are central relative to $p$ by Proposition 2.7.

5.3. *Arcwise connectivity.*

PROPOSITION 5.9. *When $\mathfrak{A}$ is a connected, locally connected and separable $C^*$-algebra, then $P(\mathfrak{A})$ is arcwise and locally arcwise connected.*

PROOF. We have seen that $P(\mathfrak{A})$ is connected and locally connected. Combine [13, 3-17] and [20, 4.3.2]

REMARK 5.10. The most obvious reason why a given connected $C^*$-algebra $\mathfrak{A}$ has a set of pure states that is not arcwise connected is that the underlying central structure of $\mathfrak{A}$, is not arcwise connected. Of course $\mathfrak{A}$ might be $C(X)$ where $X$ is some connected, but not arcwise connected, space. Another problem arises when $P(\mathfrak{A})$ is too big for us to expect that a mapping defined on the second countable space $[0,1]$ can "reach" from one end to another. The following example demonstrates this.

Let $\mathfrak{A}$ be a $II_1$ factor on a separable Hilbert space $\mathfrak{H}$. $\mathfrak{A}$ is simple and has exactly $2^{\aleph}$ irreducible representations (as in [15, 10.4.15]). By letting $x$ be the central cover in $\mathfrak{A}^{**}$ of any such representation, we thus have $0 < x < \mathsf{z}$, and by centrality, $\{P_0, P_1\}$ is a non-trivial separation of $P(\mathfrak{A})$ into disjoint norm closed sets, where $P_0 = P(x), P_1 = P(1-x)$. Since $P(\mathfrak{A})$ is *weak*$^*$ connected, the $P_i$ can not be closed in this topology. However, we only need to know that the sets are *sequentially* closed, as the following argument will show. Take $\varphi_0 \in P_0$, $\varphi_1 \in P_1$ and assume that $\varphi_t$ is a *weak*$^*$ continuous path from $\varphi_0$ to $\varphi_1$. Let

$$t_0 = \inf\{t \in [0,1] | \varphi_t \in P_1\}$$

and assume that $\varphi_{t_0} \in P_1$. Since $t_0 > 0$ we can find a sequence $t_n \nearrow t_0$, and by continuity, $\varphi_{t_n} \to \varphi_{t_0}$ in the *weak*$^*$ topology. By [1, 5], $\varphi_{t_n} \to \varphi_{t_0}$ in norm, contradicting the fact that $P_0$ is norm closed. Similarly, $\varphi_{t_0} \in P_0$ leads to a contradiction, and such a path can not exist.

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