MATH. SCAND. 86 (2000), 17–35

STRUCTURE SPACES AND DECOMPOSITION IN JB*-TRIPLES

L. J. BUNCE, C. H. CHU* and B. ZALAR

1. Introduction

Complex Banach spaces for which the group of biholomorphic automorphisms of the open unit ball acts transitively, alias JB*-triples, possess a ternary algebraic structure uniquely determined by the holomorphic properties of the open unit ball [21]. A large and important class of these spaces is comprised of the JC*-triples of [17] (known also as J^*-algebras) which are up to isometry the norm closed subspaces of $B(H, K)$, where H and K are complex Hilbert spaces, that are closed under the ternary product

$$\{xyz\} = \tfrac{1}{2}(xy^*z + zy^*x).$$

Hence, C^*-algebras are JC*-triples. On the other hand, the range of contractive projection on a C^*-algebra is a JC*-triple [13, 22, 27] but not necessarily a C^*-algebra. An "exceptional" class of JB*-triples involves certain subspaces of three by three matrices with complex Cayley numbers entries.

A detailed survey of JB*-triples recording recent developments including applications to quantum mechanics, complex holomorphy and operator algebras is to be found in [26].

Representation theory in terms of appropriate "irreducible" factors is a basic concept in algebra. In JB*-triples, for any integer $n \geq 1$, there are an infinite number of (appropriately "irreducible") Cartan factors at rank n. An additional complexity is the existence of six distinct generic types of Cartan factors.

The purpose of this paper is to investigate Cartan representation theory of JB*-triples. To this end we study the *structure space* of primitive M-ideals in some detail and we devise and apply techniques for decomposing JB*-triples into others with a simpler Cartan representation structure.

Received February 26, 1997; in revised form June 2, 1997.

1. Notation and preliminaries

A JB*-triple is a complex Banach space A with a ternary product $A^3 \to A$ given by $(a,b,c) \mapsto \{abc\}$ which, where $D(a,b)$ denotes the multiplication operator $x \mapsto \{abx\}$, satisfies

(i) $\{abc\}$ is symmetric and complex linear in a, c and conjugate linear in b,
(ii) $[D(a,b), D(c,d)] = D(\{abc\}, d) - D(c, \{dab\})$,
(iii) $D(a,a)$ is hermitian with positive spectrum,
(iv) $\|\{aaa\}\| = \|a\|^3$.

The conjugate linear operator $x \mapsto \{axb\}$ is denoted by $Q_{a,b}$. We write $Q_a = Q_{a,a}$. The elements a, b are said to be *orthogonal* if $D(a,b) = 0$ (equivalently $D(b,a) = 0$))

A subspace I of A is said to be an *ideal* of A if $\{AIA\} + \{AAI\} \subset I$ and to be an *inner ideal* of A if $\{IAI\} \subset I$. If I is a norm closed subspace, it is an ideal of A if $\{AII\} \subset I$ [6]. The annihilator, $I^{\perp} = \{x : \{xIA\} = 0\}$ of an ideal of A is also a norm closed ideal. By [2], the norm closed ideals of A are precisely the M-ideals.

A JBW*-triple is a JB*-triple with a (unique) predual [2, 18]. Frequent and tacit use shall be made of the facts [9, 2] that the second dual A^{**} of a JB*-triple A is a JBW*-triple containing A as a JB*-subtriple and that the triple product is separately weak* continuous in each variable in a JBW*-triple.

Associated with a tripotent e (i.e. $e = \{eee\}$) in A are the *Peirce projections*

$$P_2(e) = Q_e^2, \quad P_1(e) = 2(D(e,e) - Q_e^2), \quad P_0(e) = I - 2D(e,e) + Q_e^2$$

which are mutually orthogonal with sum I and ranges

$$P_j(e)(A) = A_j(e) = \left\{ x : \{eex\} = \frac{j}{2} x \right\}$$

giving $A = A_2(e) \oplus A_1(e) \oplus A_0(e)$.

JB*-algebras and their hermitian parts, *JB*-algebras, appear naturally as, for a tripotent e, the Peirce space $A_2(e)$ is a JB*-algebra with the identity e, product $x \circ y = \{xey\}$ and involution $x \mapsto \{exe\}$. If A is a JBW*-triple, then $A_2(e)$ is a JBW*-algebra. We refer to [15, 29] for the theory of JB-algebras and JB*-algebras.

The tripotent e of A is said to be *complete* if $A_0(e) = 0$ and to be *minimal* if $e \neq 0$ and $A_2(e) = \mathbb{C}e$. For $\rho \in \partial_e(A_1^*)$ (extreme points of the dual ball) there is a unique minimal tripotent e of A^{**} for which $\rho(e) = 1$, called the *support* $s(\rho)$ of ρ. The map $\rho \mapsto s(\rho)$ is a bijection from $\partial_e(A_1^*)$ onto the set of minimal tripotents of a JBW*-triple A^{**} [12].

A linear bijection between JB*-triples is an isometry if and only if it is a triple homomorphism (i.e. preserves the triple product). The JBW*-triples containing a minimal tripotent but without proper weak* closed ideals are called *Cartan factors* [8, 19] which, up to isometry, are as follows. For arbitrary Hilbert spaces and conjugation $j : H \to H$, the JB*-triples $B(H, K)$, $\{x \in B(H); x = -jx^*j\}$ and $\{x \in B(H); x = jx^*j\}$ characterize three families of Cartan factors. A fourth is given by the complex spin factors. The remaining two exceptional Cartan factors are the 1×2 matrices over the complex octonions $\mathbb{O}$ and the self-adjoint 3×3 matrices over $\mathbb{O}$.

A Cartan factor M is said to have infinite rank if it contains an infinite orthogonal family of tripotents. Otherwise, each maximal orthogonal family of minimal tripotents has the same finite cardinality, the rank of M. Apart from infinite dimensional spin factors and $B(H, \mathbb{C}^n)$, where $n < \infty$ and H is infinite dimensional, all other finite rank Cartan factors have finite dimension.

For unmentioned and further details of JB*-triples we refer to [26, 29].

2. Functional calculus and ideals

In this section we show that a JB*-triple is inundated with inner ideals that are naturally JB*-algebras and we describe certain other properties of inner ideals needed later. We begin with a description of triple functional calculus.

Given an element x of a JB*-triple A, we shall use A_x to denote the JB*-subtriple generated by x. If A is a C^*-algebra and $x \geq 0$, then A_x equals the C^*-algebra generated by x [17, Lemma 5.7.].

On the other hand it follows from [21] that for an arbitrary JB*-triple and $x \in A$ there exists a surjective linear isometry (hence a triple isomorphism) $\varphi : A_x \to C$ onto a commutative C^*-algebra generated by $\varphi(x) \geq 0$. Let $\tilde{\varphi} : A_x^{**} \to C^{**}$ be the bitransposed extension of φ. In these circumstances, we shall write

$$S(x) = \sigma(\varphi(x))\backslash\{0\}, \;\; f_t(x) = \varphi^{-1} f(\varphi(x)) \;\text{ if } f \in C_0(S(x)), \;\; e(x) = \tilde{\varphi}^{-1}(1),$$

and we note that this is unambiguous. For if $\psi : A_x \to D$ is another surjective linear isometry onto a commutative C^*-algebra D generated by $\psi(x) \geq 0$, then $\psi\varphi^{-1} : C \to D$ is a positive isometry and hence a *-automorphism sending $\varphi(x)$ to $\psi(x)$. So, $(\psi\varphi^{-1})(f(\varphi(x)) = f(\psi(x))$ if $f \in C_0(S(x))$. Similarly, $\tilde{\psi}\tilde{\varphi}^{-1}(1) = 1$. In particular,

$$A_x = \{f_t(x) : f \in C_0(S(x))\}.$$

Let $A(x)$ denote the norm closure of $\{xAx\}$. Then $A(x)$ is an inner ideal of A, as follows from the triple identity $Q_{\{aba\}} = Q_a Q_b Q_a$. With $y = \{xxx\}$ we have that $\varphi(y) = \varphi(x)^3$ also generates C and the functional calculus gives

$A_x = A_y \subset A(x)$. In particular, $A(x)$ is the smallest norm closed inner ideal of A containing x and $A(x)$ is weak* dense in $(A^{**})_2(e(x))$.

PROPOSITION 2.1. *Let A be a JB*-triple and let $x \in A$. Then $A(x)$ is a* JB*-*subalgebra of the* JBW*-*algebra $(A^{**})_2(e(x))$ and contains x as a positive element.*

PROOF. Let $\varphi : A_x \to C$ and its bitransition $\tilde{\varphi} : A_x^{**} \to C^{**}$ be as given above. Let $e = e(x)$ and put $y = \{xex\}$. Then $y \in A_x^{**}$ and $\tilde{\varphi}(y) = \varphi(x)^2$ lies in C and generates it both as a C*-algebra and as a JB*-triple. Hence, $y \in \tilde{\varphi}^{-1}(C) = A_x = A_y$. In particular, $x \in A_y \subset A(x)$. So, $A(y) = A(x)$.

Now let $a \in A$ and put $z = \{xax\}$. Then

$$\{zez\} = Q_x Q_a Q_x(e) = Q_x Q_a(y) \subset A(x)$$

and it follows that $A(x)$ is a norm closed Jordan subalgebra of $(A^{**})_2(e)$. To see that $A(x)$ is closed under involution $a \mapsto \{eae\}$, note first that $x = \{exe\}$ so that

$$Q_e(Q_y(A)) = Q_e Q_x Q_e(Q_x(A)) = Q_x^2(A) \subset A(x)$$

which gives $Q_e(A(x)) = Q_e(A(y)) \subset A(x)$ and proves that $A(x)$ is a JB*-subalgebra of $(A^{**})_2(e)$. With $f(\lambda) = \lambda^{\frac{1}{2}}, \lambda \geq 0$, we have $f_t(x) = \{ef_t(x)e\}$ and $x = \{f_t(x)ef_t(x)\}$. So, $x \in A(x)_+$.

REMARK 2.2. (a) Let $\pi : A \to B$ be a triple homomorphism between JB*-triples. Let $x \in A$ and put $y = \pi(x)$. Then it follows from the above proposition that the restriction $\pi : A(x) \to B(y)$ is a Jordan homomorphism of JB*-algebras. Further, $\pi(f_t(x)) = f_t(y)$ for all $f \in C_0(S(x))$.

(b) Let A be a weak* dense JB*-subtriple of a JBW*-triple M and let $x \in A$. Let $\pi : A^{**} \to M$ denote the weak* continuous projection onto M. Put $f = \pi(e(x))$. As π projects $(A^{**})_2(e(x))$ onto $M_2(f)$ and acts identically on $A(x)$, $A(x)$ is seen to be a weak* dense JB*-subalgebra of the JBW*-algebra $M_2(f)$ in the obvious way.

Next we describe some relevant ideal theory of inner ideals. If I is a norm closed inner ideal of a JB*-triple $A, T(I)$ shall denote the norm closed triple ideal of A generated by I.

LEMMA 2.3. *Let I be a norm closed inner ideal of a* JB*-*triple A and let J be a norm closed inner ideal of I. Then J is an inner ideal of A.*

PROOF. Let $x \in J$. By functional calculus, choose $y \in J$ such that $x = \{yyy\}$. Then

$$Q_x(A) = Q_y^2(Q_y(A)) \subset Q_y^2(I) \subset J.$$

LEMMA 2.4. *Let A be a* JB*-*triple. Let I be a norm closed inner ideal of A and let J be a norm closed triple ideal of A. Then*
(i) $I \cap J = \{IJI\}$,
(ii) $T(I \cap J) = T(I) \cap J$.

PROOF. (i) Given $x \in I \cap J$ take $y \in i \cap J$ with $x = \{yyy\}$. Then $x \in \{IJI\}$. This gives one inclusion and the other is clear.

(ii) Suppose first that $I \cap J = 0$. Given $x \in I$ and $y \in J$ we have $Q_x Q_y = 0$ so that $Q_{\{yxy\}} = Q_y Q_x Q_y = 0$ implying that $\{JIJ\} = 0$. By the fundamental identity

$$\{J\{JJI\}J\} \subset \{\{JJJ\}IJ\} - \{JJ\{JIJ\}\} = 0$$

which gives $\{JJI\} = 0$. In turn, we have

$$\{\{IJA\}JJ\} = \{IJ\{AJJ\} - \{AJ\{IJJ\}J\} = 0.$$

So, $\{IJA\} = 0$ giving $I \subset J^{\perp}$ and so $T(I) \subset J^{\perp}$. Hence, $T(I) \cap J = 0$.

Reverting to the general case, the canonical surjection $\pi : A \to A/T(I \cap J)$ gives $\pi(T(I)) = T(\pi(I))$ and $\{\pi(I)\pi(J)\pi(I)\} = \pi(\{IJI\}) = 0$. Therefore, by (i) together with the first part of the proof of (ii),

$$\pi(T(I) \cap J) = T(\pi(I)) \cap \pi(J) = 0$$

Hence, $T(I) \cap J \subset T(I \cap J)$, as required.

PROPOSITION 2.5. *Let I be a norm closed inner ideal of a* JB*-*triple A and let J be a norm closed triple ideal of I. Then $J = T(J) \cap I$.*

PROOF. Let f be a complete tripotent of I^{**} and, via [18, (4.2)], let e be a complete tripotent of A^{**} such that f is a projection of the JBW*-algebra $M = (A^{**})_2(e)$. Now, $J^{**} \cap N$ is a weak* closed Jordan ideal of the hereditary JBW*-subalgebra $N = \{fMf\}$ of M. Thus, by [11, Theorem], there is a central projection z of M such that $J^{**} \cap N = z \circ N$, where $\circ$ dentoes the Jordan product in M. In particular, $J^{**} \cap N$ is contained in the weak* closed triple ideal of A^{**}, $K = (A^{**})_2(z) + (A^{**})_1(z)$, and so lies in $(K \cap I^{**}) \cap N$. By [18, (4.2)] applied to I^{**} this gives $J^{**} \subset K \cap I^{**}$ from which it follows that $T(J)^{**} \subset K$. Hence,

$$T(J)^{**} \cap N \subset K \cap N = (z \circ M) \cap N \subset z \circ N = J^{**} \cap N.$$

But then $(T(J)^{**} \cap I^{**}) \cap N = J^{**} \cap N$ so that, as before, [17, (4.2)] gives $T(J)^{**} \cap I^{**} = J^{**}$. Intersecting both sides of which with A results in $T(J) \cap I = J$.

3. The structure space

The structure space of primitive M-ideals of a Banach space was introduced and investigated in [1]. A particularly important and very comprehensive reference on M-ideals is given by [16] to which we refer, together with [4], for any unmentioned details or M-structure in Banach spaces.

It was shown in [2] that the M-ideals of a JB*-triple A are the norm closed ideals. By a *primitive ideal* of A we shall mean primitive M-ideal. Thus P is primitive ideal of A if for some $\rho \in \partial_e(A_1^*)$ is the largest norm closed ideal of A contained in $\ker \rho$. Let Prim(A) denote the set of all primitive ideals of A and given $X \subset A$, $S \subset \text{Prim}(A)$ write

$$h(X) = \{P \in \text{Prim}(A) : X \subset P\},\ k(S) = \cap\{P \in \text{Prim}(A) : P \in S\}.$$

Primitive ideals are prime ideals (in the usual sense) and there is a unique topology on Prim(A), the *structure topology*, for which $hk(S)$ is the closure of S. Endowed with this structure topology, Prim(A) is referred to as the *structure space* of A. There is a bijective correspondance, $J \to h(J)$, between the norm closed ideals of A and the closed sets of Prim(A) and we have the homeomorphisms

$$h(J) \to \text{Prim}(A/J)$$

and

$$\text{Prim}(A)\backslash h(J) \to \text{Prim}(J) \quad (P \mapsto P \cap J)$$

for each norm closed ideal J of A.

A triple homomorphism, $\pi : A \to M$, into a JBW*-triple M has unique weak* continuous extension, $\tilde{\pi} : A^{**} \to M$, with $\tilde{\pi} : (A^{**}) = \overline{\pi(A)}$ [3], where here and later the bar refers to weak* closure. If M is a Cartan factor and $\overline{\pi(A)} = M$, then π is said to be a *Cartan factor* representation. The set of all Cartan factor representations of A is denoted by $C(A)$.

Given $\rho \in \partial_e(A_1^*)$, let A_ρ^{**} be the weak* closed ideal of A^{**} generated by the (minimal) support tripotent $s(\rho)$ [12]. Then A_ρ^{**} is a complemented Cartan factor in A^{**} [8, 18] and the restriction, $\pi_\rho : A \to A_\rho^{**}$, of the natural weak* continuous projection, $P_\rho : A^{**} \to A_\rho^{**}$, is a Cartan factor representation of A.

The following is contained in detail of [2, Theorem 3.6].

LEMMA 3.1. *Let A be a* JB*-*triple and let ρ be an extreme point of the dual ball. Then* $\ker \pi_\rho$ *is the largest norm closed ideal of A in* $\ker \rho$. *Hence,* $\text{Prim}(A) = \{\ker \pi_\rho : \rho \in \partial_e(A_1^*)\}$.

LEMMA 3.2. *Let $\pi : A \to M$ be a Cartan factor representation of a* JB*-*triple*

A. Then there exists $\rho \in \partial_e(A_1^*)$ and a surjective isometry $\varphi : A_\rho^{**} \to M$ such that $\varphi\pi_\rho = \pi$. Hence, $\mathrm{Prim}(A) = \{\ker \pi : \pi \in C(A)\}$.

PROOF. Let $J = \ker \tilde{\pi}$ where $\tilde{\pi} : A^{**} \to M$ is a weak* continuous extension of π onto M. Then the complement of J in A^{**}, $J^\perp \approx A^{**}/J \approx M$. Choose a minimal tripotent e of A^{**} contained in $J^\perp$ and let $\rho \in \partial_e(A_1^*)$ with $s(\rho) = e$, using [12, Proposition 4]. It follows that $A_\rho^{**} = J^\perp$ and that $\varphi\pi_\rho = \pi$.

PROPOSITION 3.3. *Let I be a norm closed inner ideal of a* JB*-*triple A. Then*

$$\beta : \mathrm{Prim}(A)\backslash h(I) \to \mathrm{Prim}(I) \quad (P \mapsto P \cap I)$$

is a homeomorphism.

PROOF. As a weak* closed inner ideal of a Cartan factor is a Cartan factor, it follows that a Cartan factor representation of A which fails to kill I restricts to a Cartan factor representation of I. It follows from Lemma 3.2 that β is well-defined.

On the other hand, given $\rho \in \partial_e(I_1^*)$ let $\bar{\rho} \in \partial_e(A_1^*)$ extend ρ. As $s(\rho)$ is minimal in the weak* closed inner ideal I^{**} it is also minimal in A^{**}. So $s(\rho) = s(\bar{\rho})$ and hence, $I_\rho^{**} \subset A_{\bar{\rho}}^{**}$. Let G be the complementary ideal of I_ρ^{**} in I^{**}. Let J be the norm closed ideal generated by in I_ρ^{**} in $A_{\bar{\rho}}^{**}$. Then $G \subset J^\perp$ by Lemma 2.4. Hence, $G \subset A_{\bar{\rho}}^{**}$ because $J^\perp = (\bar{J})^\perp = A_{\bar{\rho}}^{**}$. It follows that $P_{\bar{\rho}} : A^{**} \to A_{\bar{\rho}}^{**}$ restricts to $P_\rho : I^{**} \to I_\rho^{**}$ so that $\pi_{\bar{\rho}}$ restricts to π_ρ and hence $\ker \pi_\rho = I \cap \ker \pi_{\bar{\rho}}$. Therefore, β is surjective by Lemma 3.1.

By Lemma 2.4 together with primeness of primitive ideals, β is injective and for each norm closed ideal J of $A, \beta(h(J)\backslash h(I)) = h(I \cap J)$ (taken in I) so that β is a closed map. By Proposition 2.5, the right hand side of the equation runs through all closed sets of Prim(I) and so β is continuous.

REMARK 3.4. Let A be a JB*-algebra, $\pi : A \to M$ a Cartan factor representation and $\tilde{\pi} : A^{**} \to M$ its weak* continuous extension. Then with $e = \tilde{\pi}(1)$, π is reconstituted as a * *Jordan* Cartan factor representation, $\pi : A \to M_2(e)(= M)$ and induces by restriction (in the sense of 15, p.133) a Jordan type I factor representation of the JB-algebra A_{sa}. Thus, by restriction and by complexification in the opposite direction (cf. [29]) the structure space of A is naturally identified with the usual structure space [6] of the JC-algebra A_{sa}. We shall make frequent and often tacit use of this fact.

LEMMA 3.5. *Let A be a* JB*-*triple.*

(i) $\hat{x} : \mathrm{Prim}(A) \mapsto [0, \infty)$ $(P \mapsto \|x + P\|)$ *is lower semicontinuous for all x.*

(ii) *The sets $\{P \in \mathrm{Prim}(A) : \|x + P\| \geq \alpha\}$, where $\alpha > .0$ and $x \in A$, form a basis of quasi-compact sets for* Prim(A).

(iii) Prim(A) *is Hausdorff if $\hat{x}$ and only if is continuous for all $x \in A$.*

Proof. When A is a JB*-algebra and "$x \in A$" is replaced by "$x \in A_+$", (i), (ii) and (iii) follow as for C*-algebras (cf. [10, (3.3)], [25, (4.4)].

(i): Let $x \in A$. Via the triple and hence isometric embedding $A(x)/P \cap A(x) \to A/P$ we have $\|x + P\| = \|x + P \cap A(x)\|$ for each $P \in \mathrm{Prim}(A)$. Consider the open set $U = \mathrm{Prim}(A) \backslash h(A(x)) \approx \mathrm{Prim}(A(x))$. By Proposition 2.1., $A(x)$ can be realised as a JB*-algebra such that x is positive there. Therefore, the opening remark together with Proposition 3.3 imply that $\hat{x} : U \to [0, \infty)$ is lower semicontinuous. But for $\alpha \geq 0, \hat{x}^{-1}((\alpha, \infty))$ is contained in U by Proposition 3.3. Hence, $\hat{x}$ is lower semicontinuous on $\mathrm{Prim}(A)$.

(ii), (iii): Via the opening remark these follow by similar use of Proposition 2.1 and Proposition 3.3.

4. Rank and collinear systems

Let A be a JB*-triple. The *rank*, $\mathrm{rank}(\pi)$, of a Cartan factor representation, $\pi : A \to M$, is the rank of M. If for fixed n, where $1 \leq n < \infty$, $\mathrm{rank}(\pi) = n$ for all Cartan representations, then A is said to be of *constant rank n*. The JB*-triple is said to be of bounded rank if $\{\mathrm{rank}(\pi) : \pi \in C(A)\}$ is bounded.

In the Cartan factor M, the JB*-subtriple generated by all minimal tripotents is a simple norm closed ideal, $K(M)$, of M such that its second dual is isometric to M [7]. We have,

M has finite rank if and only if M is reflexive if and only if $K(M) = M$.

As seen in the proof of Proposition 3.3., given $\pi \in C(A)$ and $x \in A$ with $\pi(x) \neq 0$, π *restricts to a Cartan factor representation of* $A(x)$. In the following this induced representation is denoted by π_x.

Lemma 4.1. *Let A be a* JB*-*triple and $\pi : A \to M$ a Cartan representation of A .*

(i) *If* $\mathrm{rank}(\pi) < \infty$, *then there exists $x \in A$ with $\pi(x) \neq 0$ such that* $\mathrm{rank}(\pi) = \mathrm{rank}(\pi_x)$.

(ii) *If for all $x \in A$ with $\pi(x) \neq 0$ we have* $\mathrm{rank}(\pi_x) < \infty$, *then* $\mathrm{rank}(\pi) < \infty$.

Proof. (i): Suppose that the Cartan factor has a finite rank. Then M is reflexive so that $\pi(A) = M$. Choose $x \in A$ such that $\pi(x) = e$ is a complete tripotent of M. Then $\pi_x(A(x)) = M_2(e)$. Hence, $\mathrm{rank}(\pi_x) = \mathrm{rank}(\pi)$.

(ii): Let $x \in A$ be such that $\pi(x) \neq 0$ and $\mathrm{rank}(\pi_x) < \infty$. Then $\pi(A(x)) = \pi_x(A(x))$ is a reflexive, so weak* closed, inner ideal of M which implies that $\pi(A(x)) \subset K(M)$. Hence, given that the stated condition is satisfied, $\pi(A) \subset K(M)$. Now, the natural projection $Q : K(M)^{**} \to M$ is an isometry onto M and Q maps $\pi(A)^{**}$ onto $\overline{\pi(A)} = M$. Therefore, $\pi(A)^{**} = K(M)^{**}$ and so $\pi(A) = K(M)$. It follows that π has a finite rank. Otherwise, there is an infinite sequence (e_n) of orthogonal minimal tripotents in M. In this case $y = \sum \frac{e_n}{2^n} \in K(M)$ and, choosing $x \in A$ with $\pi(x) = y$ and putting $e = \sum e_n$, we obtain that $\pi_x : A_x \to M_2(e)$ is a Cartan factor representation of infinite rank. This contradiction concludes the proof.

For a JB*-triple A and natural number n we denote by $\mathrm{Prim}_n(A)$ the set of those primitive ideals $\ker \pi$ for which $\mathrm{rank}(\pi) \leq n$.

PROPOSITION 4.2. *Let A be a* JB*-*triple and n a natural number. Then* $\mathrm{Prim}_n(A)$ *is closed in* $\mathrm{Prim}(A)$.

PROOF. Take $\pi \in C(A)$ such that $\ker \pi \notin \mathrm{Prim}_n(A)$. By Lemma 4.1 there exists $x \in A$ such that $\pi(x) \neq 0$ and $\ker \pi \cap A(x) = \ker \pi_x \notin \mathrm{Prim}_n(A(x)) = F$, which is closed in $\mathrm{Prim}(A(x))$ as follows from Proposition 2.1 together with [5, Lemma 6]. Now $U = \beta^{-1}(\mathrm{Prim}(A(x))\backslash F)$, where β is the homeomorphism of Proposition 3.3, satisfies $U \cap \mathrm{Prim}_n(A) = \emptyset$, and U is an open neighbourhood of $\ker \pi$. This proves that $\mathrm{Prim}_n(A)$ is closed.

REMARK 4.3. Given a JB*-algebra A consider the functions $T_x : \mathrm{Prim}(A) \to [0, \infty]$, $x \in A^{**}_+$, given by $T_x(\ker \pi) = \mathrm{Tr}(\tilde{\pi}(x))$ where $\pi : A \to M$ and $\tilde{\pi} : A^{**} \to M$ is its weak* continuous extension and Tr is the Jordan trace on M. The functions T_x are lower semicontinuous for all $x \in A_+$ (cf. [5, Lemma 6]. Hence, T_x is lower semicontinuous whenever $x \in A^{**}$ is the strong limit of an increasing net in A_+. If A has constant rank n, then it follows as for C*-algebras (cf. [25, 4.4.10] that T_x is continuous for all $x \in A_+$ and that $\mathrm{Prim}(A)$ is Hausdorff.

LEMMA 4.4. *Let A be a* JB*-*triple of constant finite rank n. Then* $\mathrm{Prim}(A)$ *is Hausdorff.*

PROOF. Let $P_1, P_2 \in \mathrm{Prim}(A)$ with $P_1 \neq P_2$. By assumption, the canonical maps $\pi_i : A \to A?P_i = M_i$ belong to $C(A)$ $i = 1, 2$. For $i = 1, 2$, choose $x_i \in A$ such that $\pi_i(x_i) = e_i$ is a complete tripotent of M_i and let $a_i \in P_i$ such that $x_i - x_2 = a_1 + a_2$, which is possible because $P_1 + P_2 = A$. For $x = x_1 - a_1 = x_2 + a_2$ we have $\pi_1(x) = e_1$, $\pi_2(x) = e_2$. Hence, for $i = 1, 2$, the $Q_i = P_i \cap A(x)$ are, by Proposition 3.3, distinct elements of

$$\mathrm{Prim}_n(A(x))\backslash \mathrm{Prim}_{n-1}(A(x)) = \mathrm{Prim}(A(x))\backslash h(J) \approx \mathrm{Prim}(J),$$

where J is the closed ideal of $A(x)$ with hull equal to $\mathrm{Prim}_{n-1}(A(x))$ (where we let $J = 0$ if $n = 1$). But then J is a JB*-algebra of constant rank (using Proposition 2.1) so that $\mathrm{Prim}(J)$ is Hausdorff by Remark 4.3. Now Proposition 3.3 implies that P_1, P_2 are separated by open sets in $\mathrm{Prim}(A)\backslash h(A(x))$. Hence $\mathrm{Prim}(A)$ is Hausdorff.

In the following, which is inspired by [28, pp. 506–507], we let h be the continuous function on R satisfying

$$h((-\infty,\tfrac{1}{4})) = \{0\}, h([\tfrac{3}{4},\infty)) = \{1\} \text{ and } h \text{ is linear on } [\tfrac{1}{4},\tfrac{3}{4}].$$

Recall that $h_t(x)$ refers to the element of A_x given by the triple functional calculus (see §2). We shall also use the following: given tripotents e and f in a JBW*-triple such that e is minimal and $\|e-f\| < 1$, it follows that f is minimal too. For, indeed, $P_2(f)(e) = \alpha u$ where u is a minimal tripotent and $\alpha \in \mathsf{C}$ [12, Proposition 6] so that $\|f-\alpha u\| = \|P_2(f)(f-e)\| < 1$ which implies that u is invertible in the JBW*-algebra $M_2(f)$. Hence, $M_2(f) = M_2(u) \simeq \mathsf{C}$.

Lemma 4.5. *Let A be a* JB**-triple of constant finite rank n . Let $P_0 \in \mathrm{Prim}(A)$ and let $x \in A$ such that $x + P_0$ is a nonzero tripotent.*

(i) *$h_t(x) + P_0 = x + P_0$ and $h_t(x) + P$ is a nonzero tripotent for all P in some neighbourhood V of P_0.*

(ii) *If $x + P_0$ is minimal, then $h_t(x) + P$ is a minimal tripotent for all P in some neighbourhood W of P_0.*

Proof. (i) Regarding $A(x)$ as a JB*-algebra and $x \in A(x)_+$ by Proposition 2.1, we have that $x + Q_0$ is a non-zero projection in $A(x)/Q_0$ where $Q_0 = P_0 \cap A(x)$. As $\mathrm{Prim}(A)$ is Hausdorff by Lemma 4.4, so $\mathrm{Prim}(A(x))$ is Hausdorff by Proposition 3.3, and the argument on page 506 of [28] gives an open neighbourhood V of Q_0 such that $h(x) + Q_0 = x + Q_0$ and $h(x) + Q$ is a non-zero projection for all $Q \in V$. Now Proposition 3.3 together with triple functional calculus gives (i).

(ii) Let $\pi_0 : A \to A/P_0 = M$ be the quotient map and let $\pi_0(x) = e$ be a minimal tripotent of M. Choose a complete tripotent u of M such that the Type I_n JBW*-factor $M_2(u)$ contains e as a minimal projection (cf [18]). Choose $y \in A$ with $\pi(y) = u$ and let I be the norm closed ideal of the JB*-algebra $B = A(y)$ corresponding to $\mathrm{Prim}(B)\backslash\mathrm{Prim}_{n-1}(B)$. Then J is a JB*-algebra of constant rank n and $P_0 \cap J \in \mathrm{Prim}(J)$. Choose with $z \in J$ with $z \geq 0$ and $\pi_0(z) = e$. Transparent modification to the argument on page 507 of [28] now gives that $h(z) + Q$ is a minimal projection in J/Q for all Q in a neighbourhood of $P_0 \cap J$ in $\mathrm{Prim}(J)$. Via Proposition 3.3., this gives rise to a neighbourhood U of P_0 in $\mathrm{Prim}(A)$ such that $h_t(x) + P$ is a minimal tripotent for all $P \in U$. We may suppose that $U \subset V$, where V is given in (i). Now put $W = \{P \in U : \|h_t(x) - h_t(z) + P\| < 1\}$. Then $P_0 \in W$ and W is open by

Lemma 3.5 (iv) and Lemma 4.4. Finally, by the remark immediately preceding the statement, $h_t(x) + P$ is minimal for all $P \in W$.

Tripotents e and f in a JB*-triple A are said to be *collinear* if $e \in A_1(f)$ and $f \in A_1(e)$. If $e_1, \ldots, e_n$ are minimal and mutually collinear in A, we say that they form a *collinear system of length n*.

LEMMA 4.6. *Let A be a* JB*-*triple.*

(i) *If e and f are minimal tripotents in A and* $e \in A_1(f)$, *then* $f \in A_1(e)$.

(ii) *If* $e_1, \ldots, e_n$ *is a collinear system in A and*

$$T = \left(1 - \sum_{i=1}^{n} P_2(e_i)\right)(I - P_0(e_1)) \ldots (I - P_0(e_n)),$$

then $T(A) \subset \bigcap_{i=1}^{n} A_1(e_i)$.

PROOF. (i) This follows from [12, Lemma 2.1].

(ii) Let $e_1, \ldots, e_n$ be mutually collinear minimal tripotents. Put $a = y - \sum_{i=1}^{n} P_2(e_i)y$ where $y = (i - P_0(e_1)) \ldots (I - P_0(e_n))(x)$ and $x \in A$. The Peirce projections $P_k(e_i)$, $k = 0, 1, 2$, $i = 1, \ldots, n$, commute by [18, (1.10)], so $P_0(e_j)y = 0$, $j = 1, \ldots, n$ and

$$\begin{aligned} 2D(e_j, e_j)(a) &= 2D(e_j, e_j)y - \sum_{i \neq j} P_2(e_i)y - 2P_2(e_j)y \\ &= (2D(e_j, e_j) - P_2(e_j) - I)y + \left(y - \sum_{i=1}^{n} P_2(e_i)y\right) \\ &= -P_0(e_j)(y) + a \\ &= a. \end{aligned}$$

PROPOSITION 4.7. *Let A be a* JB*-*triple of constant rank and m a natural number. Then the set*

$$S = \{P \in \mathrm{Prim}(A) : A/P \text{ contains a collinear system of length } > m\}$$

is open in Prim(A).

PROOF. Let $P_0 \in S$. By assumption, A/P_0 contains minimal and mutually collinear tripotents $e_1, \ldots, e_n$, where $n > m$. Choose $x_1, \ldots, x_n \in A$ such that $x_i + P_0 = e_i$, $i = 1, \ldots, n$.

We shall show, by induction, that for all P in some neighbourhood of $P_0, A/P$ contains a collinear system of length n. To this end we make the following induction hypothesis.

Let $1 \le k < n$ and suppose that we have $y_1, \ldots, y_k \in A$ and a neighbourhood U of P_0 such that $y_i + P_0 = e_i$, $i = 1, \ldots, k$, and $\{y_1 + P, \ldots, y_k + P\}$ is a collinear system for all $P \in U$. By Lemma 4.5, we note that this hypothesis holds for $k = 1$. Put

$$y = \left(I = -\sum_{i=1}^{k} Q(y_1)^2\right)(2D(y_1, y_1) - Q(y_1)^2) \ldots (2D(y_k, y_k) - Q(y_k)^2)(x_{k+1})$$

and $y_{k+1} = h_t(y)$, where h is the function defined prior to Lemma 4.5. Then,

$$y + P_0 = (I - \sum P_2(e_i))(I - P_0(e_i)) \ldots (I - P_0(e_k))(e_{k+1}) = e_{k+1}.$$

Hence, by Lemma 4.5, $y_{k+1} + P_0 = e_{k+1}$ and there is a neighbourhood V of P_0 such that

(α) $\quad y_{k+1} + P$ is a minimal tripotent for all $P \in V$.

Also, by Lemma 4.6(ii) we have, for all $P \in U \cap V = W$,

$$y + P \in \bigcap_{i=1}^{k} (A/P)_1(y_i + P) \text{ so that } y_{k+1} + P \in \bigcap_{i=1}^{k} (A/P)_1(y_i + P),$$

(as latter is a JB*-subtriple of A/P) and hence $\{y_1 + P, \ldots, y_{k+1} + P\}$ is a collinear system by (α) together with Lemma 4.6(i). This completes the proof.

Elements a, b in a JC*-algebra are said to be *J-orthogonal* if $L_a(b) + \frac{1}{2}(ab + ba) = 0$. Let V_α be the spin factor that, when realised as a JC*-algebra, contains a maximal J-orthogonal family of symmetries $\{s_i\}_{i \in I}$ with card$(I) = \alpha$ (cf. [15, Chapter 6]).

Lemma 4.8. *Let V be a spin factor realised as a* JC**-algebra.*

(i) *If s, t are J-orthogonal symmetries, then $L_s^2 L_t^2 = L_t^2 L_s^2$.*

(ii) *If $s_1, \ldots, s_n$ are mutually J-orthogonal symmetries in V, then $(I - L_{s_1}^2) \ldots (I - L_{s_n}^2)(V)$ is elementwise J-orthogonal to s_i for all $i = 1, \ldots, n$.*

(iii) *Let $x^* = x \in V$ be nonzero and J-orthogonal to a symmetry in V. Then each nontrivial symmetry in the* JC*-algebra generated by x* (*there are two*) *is a scalar multiple of x.*

Proof. (i) is routine and (ii) then follows from the rule $L_{s_i}^3 = L_{s_i}$.

(iii) Let $t \ne \pm 1$ be a symmetry J-orthogonal to $x = \alpha \cdot 1 + \beta s$ where $\alpha, \beta \in \mathsf{R}$ and s is a nontrivial symmetry. We have $L_t(s) \in \mathsf{R}$ so that $\alpha = 0$. The JC-algebra generated by x is $\{\lambda \cdot 1 + \mu s : \lambda, \mu \in \mathsf{R}\}$, the only nontrivial symmetries in which are s and $-s$

For a JB*triple A and integer $n \ge 2$, let

$$S^n(A) = \{P \in \mathrm{Prim}(A) : A/P \approx V_m, m > n\}.$$

LEMMA 4.9. *Let A be a* JB*-*algebra for which all Cartan factor representations have rank 1 or 2 and let n be an integer with $n \geq 2$. Then $S^n(A)$ is open in* $\mathrm{Prim}(A)$.

PROOF. As $S^n(A) \subset \mathrm{Prim}(A)\backslash\mathrm{Prim}_1(A)$ we may suppose that A has constant rank 2. Let h be the real function given prior to Lemma 4.5 and let $f, g : \mathsf{R} \to \mathsf{R}$ be given by $f(\lambda) = \frac{\lambda(\lambda+1)}{2}, g(\lambda) = h(f(\lambda)) - h[(1 - h(f(\lambda))) \cdot (f(-\lambda))]$.

Let $P_0 \in \mathrm{Prim}(A)$. Let $x \in A_{\mathrm{sa}}$ such that $x + P_0 = s$ is a non-trivial symmetry in A/P_0. Then with $x_1 = f(x)$ and $x_2 = f(-x)$ we have $x_1 + P_0 = e_1, x_2 + P_0 = e_2$ are orthogonal minimal projections in A/P_0 with sum unity. So (cf. [27, pages 506–507] or Lemma 4.5) with $y_1 = h(x_1), y_2 = h((1 - y_1)x_2)$ we have that, $y_1 + P_0 = e_1, y_2 + P_0 = e_2$ and $y_1 + P, y_2 + P$ are orthogonal minimal projections in A/P for all P in a neighbourhood U of P_0. Note that $g(x) = y_1 - y_2$. Hence, $g(x) + P_0 = s$, and $g(x) + P$ is a non-trivial symmetry for all $P \in U$.

Now suppose that $P_0 \in S^n(A)$. Then, for some $m > n$, there exist $x_1, \ldots, x_m$ in A_{sa} such that $x_1 + P_0 = s_1, \ldots, x_m + P_0 = s_m$ are mutually J-orthogonal symmetries in A/P_0. We proceed by induction to show that there exist $y_1, \ldots, y_m \in A_{\mathrm{sa}}$ such that $y_1 + P, \ldots, y_m + P$ are mutually J-orthogonal symmetries in some neighbourhood of P_0.

Let $1 \leq k < m$. Suppose that $y_1, \ldots, y_k \in A_{\mathrm{sa}}$ have been chosen so that $y_i + P_0 = s_i, i = 1, \ldots, k$ and $y_1 + P, \ldots, y_K + P$, are mutually orthogonal symmetries for all P in a neighbourhood V of P_0.

Put $y = (I - L^2_{y_1}) \ldots (I - L^2_{y_k})(x_{k+1})$ and put $y_{k+1} = g(y)$. Then, by the first part of the proof, $y_{k+1} + P_0 = s_{k+1}$ and $y_{k+1} + P$ is a non-trivial symmetry in A/P for all P in a neighbourhood W of P_0. It follows from Lemma 4.8 (ii, iii), that $y_1 + P, \ldots, y_{k+1} + P$ are mutually J-orthogonal symmetries in A/P for all $P \in V \cap W$. Hence, $y_1, \ldots, y_{k+1}$ satisfy the inductive hypothesis and the result follows.

5. Decompositions of JB*-triples

We apply the structure space techniques developed earlier to study decomposition in JB*-triples. We are mostly interested in JB*-triples of bounded rank. Some results are more general. Relevant features and notation of finite rank Cartan factors are listed below for convenience. There are six generic types (cf. [17, 24]).

(1) Rectangular: $M_{n,\alpha} = B(H, K), 1 \leq n = \dim(K) \leq \alpha = \dim(H), n < \infty$ ($n \times \alpha$ matrices)

(2) Symplectic: $A_n, 4 \leq n < \infty$ (antisymmetric $n \times n$ matrices)

(3) Hermitian: $S_n, 2 \leq n < \infty$ (symmetric $n \times n$ matrices)
(4) Spin: $V_\lambda, 2 \leq \lambda$ ($\dim(V_\lambda) = \lambda + 1$)

In (1) and (4), the cardinals α and λ can be infinite. $M_{n,\alpha}$, for $n \geq 1$, and A_{2n}, A_{2n+1}, S_n for $n \geq 2$ are all of rank n. Spin factors have rank 2 and are, together with A_{2n} and S_n for $n \geq 2$, isometric to JC*-algebras. We have the isomorphisms $S_2 \approx V_2$, $M_{2,2} \approx V_3$, $A_4 \approx V_5$ and, for $n \geq 2$, $A_{2n} \approx M_n(\mathsf{H})_{sa} \otimes_{\mathsf{R}} \mathsf{C}$, $S_n \approx M_n(\mathsf{R})_{sa} \otimes_{\mathsf{R}} \mathsf{C}$. The factors $M_{1,\alpha}$ are the α-dimensional Hilbert spaces.

There are two exceptional factors.
(5) $B_{1,2}$: (1×2 matrices over the complex Cayley numbers)
(6) M_3^8: (self-adjoint 3×3 matrices over the complex Cayley numbers)

Let A be a JB*-triple. If for all $P \in \mathrm{Prim}(A)$, A/P is a finite rank rectangular Cartan factor, then A is said to be of *rectangular type.*

The appellations *symplectic*, *hermitian*, *spin* and *exceptional* are employed correspondingly.

If for a fixed finite rank Cartan factor M we have $A/P \approx M$ for all primitive ideals P, then A is said to be of *type M*. By convention, the zero triple is considered to be of every type.

We recall that by the Gelfand-Naimark theorem of [14] in a JB*-triple A there is a unique norm closed ideal J such that A/J is a JC*-triple and J is exceptional.

Theorem 5.1. *Let A be a* JB*-*triple and let $n \in \mathsf{N}$. There is a* (*unique*) *norm closed ideal J* of *A such that* $\mathrm{rank}(\pi) \leq n$ *for all* $\pi \in C(A/J)$ *and* $\mathrm{rank}(\pi) > n$ *for all* $\pi \in C(J)$.

Proof. This is the algebraic translation of Proposition 4.2.

Corollary 5.2. *Let A be a* JB*-*triple for which all non-exceptional Cartan factor representations have rank greater than* 3. *Then the exceptional ideal of A is a direct summand.*

Proof. Let J be the exceptional ideal of A. Then $h(J) = \{\ker(\pi) : \pi \in C(A), \pi \text{ is non-exceptional}\} = \{\ker(\pi) : \pi \in C(A), \mathrm{rank}(\pi) > 3\}$ is both open and closed in $\mathrm{Prim}(A)$. It follows that J is a direct summand.

Proposition 5.3. *Let A be a* JB*-*triple of bounded rank and let* $\{\mathrm{rank}(A/P) : P \in \mathrm{Prim}(A)\} = \{n_1\}_{i=1}^k$ *where* $1 \leq n_1 < n_2 < \ldots < n_k$. *Then there is a finite composition series of norm closed ideals,* $0 = J_0 \subset J_1 \subset \ldots \subset J_{k-1} \subset J_k = A$ *such that for* $r = 0, \ldots, k-1$, J_{r+1}/J_r *is non-trivial of constant rank* n_{k-r} *with Hausdorff structure space.*

Proof. This follows from Theorem 5.1 together with Lemma 4.4.

COROLLARY 5.4. *Let A be an exceptional* JB*-*triple. Then there is a norm closed ideal J of A* such that *J is of type M_3^8 and A/J is type $B_{1,2}$.*

REMARK 5.5. Let (e_{ij}) be the canonical matrix units of $M_{n,\alpha}$, where $n \leq \alpha < \infty$.

(a) The tripotents $e_{11}, \dots, e_{1\alpha}$ form a collinear system (see Section 4) in $M_{n,\alpha}$. Moreover, any collinear system S has cardinality $\leq \alpha$. Indeed, as two minimal tripotents are exchanged by some automorphism (see [24, §5]) we may suppose that $e_{11} \in S$. Then the collinearity and minimality of the elements of S implies by straightforward calculation that S is contained either in the linear span of $e_{11}, \dots, e_{1\alpha}$ or S is contained in the linear span of $e_{11}, \dots, e_{n1}$. By [8, Lemma on page 306] it follows that card$(S) \leq \alpha$.

(b) Let $n \geq 4$ and $f_{ij} = e_{ij} - e_{ji}$, $1 \leq i, j \leq n$. Then $\{f_{12}, \dots, f_{1n}\}$ is a collinear system in $A - n$. Let S be any collinear system in A_n. The claim now is that card$(S) \leq n - 1$. As before, by [24, §5], we may suppose that $f_{12} \in S$. In this case calculation shows that $S \backslash \{f_{12}\}$ is contained in the image of the injective triple homomorphism $\pi : M_{2,n-2} \to A_n$ given by $\pi(x) = \begin{bmatrix} 0 & x \\ -x^T & 0 \end{bmatrix}$. So, the claim follows from (a).

By the above results the study of JB*-triples of bounded rank reduces to that of constant rank. We shall now proceed to analyse JB*-triples of constant rank.

The main decomposition result is Theorem 5.8. The JB*-algebra version is known (cf. [6]), of which we shall make esssential use (in Lemma 5.7). In order to treat JB*-triples we need to come to grips with and synthesise phenomena that do not arise in JB*-algebras.

LEMMA 5.6. *Let A be a* JB*-*triple of constant rank n.*

(i) *If A is rectangular and $1 \leq n \leq \alpha < \infty$, then there is a norm closed ideal J of A such that all primitive quotients of J and A/J are respectively of the form $M_{n.\beta}$ where $\alpha < \beta \leq \infty$ and $M_{n,\beta}$ where $n \leq \beta \leq \alpha$.*

(ii) *If (up to isometry) $\{A/P : P \in \mathrm{Prim}(A)\} = \{M_{n,\alpha_i}\}_{i=1}^k$ where $1 \leq n \leq \alpha_1 < \ldots < \alpha_k < \infty$, then there are norm closed ideals in $A, 0 = J_0 \subset J_1 \subset \ldots \subset J_k \subset J_{k+1} = A$, such that J_{r+1}/J_r is non-trivial type $M_{n,\alpha_{k-r}}$ for $r = 0, \dots, k$.*

(iii) *If $n \geq 2$ and A is symplectic, then there is a norm closed ideal J of A such that J is type A_{2n+1} and A/J is type A_{2n}.*

PROOF. (i) If A is rectangular, then the set

$$C_\alpha(A) = \{P \in \mathrm{prim}(A) : A/P \approx M_{n,\beta}, \beta \leq \alpha\}$$

is closed in Prim(A) by Proposition 4.7 together with Remark 5.5 (a).

Thus $k(\mathsf{C}_\alpha(A))$ is the required ideal.

(ii) This follows from (i) by repeated application.
(iii) In this case, by Proposition 4.7 and Remark 5.5 (b),

$$S = \{P \in \mathrm{Prim}(A) : A/P \approx A_{2n}\}$$

is closed in $\mathrm{Prim}(A)$ and $J = k(S)$ is the required ideal.

Lemma 5.7. *Let A be a* JC*-*algebra of constant rank n where $3 \le n < \infty$. Then there are norm closed ideals of $A, J_1 \subset J_2$ such that J_1 is of type $A_{2n}, J_2/J_1$ is type $M_{n,n}$ and A/J_2 is type S_n.*

Proof. As all Type I factor representations (in the sense of 15, page 133) of the JC-algebra A_{sa} must be of Type I_n, this follows from [6, §5] because $A_{2n} \approx M_n(\mathsf{H})_{\mathrm{sa}} \otimes_{\mathsf{R}} \mathsf{C}$ and $S_{2n} \approx M_n(\mathsf{R})_{\mathrm{sa}} \otimes_{\mathsf{R}} \mathsf{C}$.

Theorem 5.8. *Let A be a* JC*-*triple of constant rank n, where $3 \le n < \infty$. Then there are norm closed ideals of $A, J_1 \subset J_2 \subset J_3$ such that*
(i) *J_1 is type A_{2n+1}*;
(ii) *J_2/J_1 is type A_{2n}*;
(iii) *J_3/J_2 is rectangular*;
(iv) *A/J_3 is type S_n.*

Proof. Let $P_0 \in \mathrm{Prim}(A)$. Let $x \in A$ and e be complete tripotent of $M = A/P_0$ such that $x + P_0 = e$. Then by Section 2, $A(x)/P_0 \cap A(x) \approx M_2(e)$ as JC*-algebras. Let I be the norm closed ideal of the JC*-algebra $A(x)$ such that

$$V = \mathrm{Prim}(A(x)) \backslash h(I) = \{Q : \mathrm{rank}(A(x)/Q) = n\}.$$

Then $P_0 \cap A(x) \in V$ and I is a JC*-algebra of constant rank n.

Now suppose that $P_0 \in S = \{P \in \mathrm{Prim}(A) : A/P$ is symplecic$\}$. Then $M \approx A_{2n}$ or A_{2n+1} so that $M_2(e) \approx A_{2n}$, and by Lemma 5.7 there is a non-zero norm closed ideal J of I all primitive quotients of which are isometric to A_{2n}. Then $P_0 \cap J \neq 0$. Let $K = T(J)$ be the norm closed ideal of A generated by J and let $P \in \mathrm{Prim}(A)$ such that $P \cap K \neq 0$. Then $P \cap J \neq 0$, by Lemma 2.4. Hence, $A_{2n} \approx J/P \cap J$ imbeds as a subtriple into A/P. As A_{2n} cannot be so embedded into $M_{n,\alpha}$ nor into S_n, we must have $A/P \approx A_{2n}$ or A_{2n+1}. Therefore, $P \in \mathrm{Prim}(A) \backslash h(K) \subset S$, which proves that S is open in $\mathrm{Prim}(A)$. Hence, there is a norm closed ideal J_2 of A such that J_2 is symplectic and A/J_2 has no symplectic primitive quotients. The required ideal $J_1 \subset J_2$ comes from Lemma 5.6 (iii).

Passing to A/J_2 we may assume that $J_2 = 0$ and emulate the above argument for $P_0 \in R = \{P \in \mathrm{Prim}(A) : A/P$ is rectangular$\}$. In this case, in the notations of the first paragraph of the proof, $A(x)/P_0 \cap A(x) \approx M_2(e) \approx$

$M_{n,n}$ as JC*-algebras. Applying Lemma 5.7 together with the fact that $M_{n,n}$ is not embeddable in S_n, we obtain by the same argument an ideal J_3 of A such that J_3 is rectangular and A/J_3 has no rectangular primitive quotients and so is type S_n.

It remains to deal with the general constant rank 2 case (Lemma 5.6(i) takes care of the general constant rank 1 case). Let $V \subset M$, where V is a spin factor and M is a JBW*-triple factor of rank 2. For convenience we tabulate the possible structure of M determined by $V = V_\alpha$, $\alpha \geq 3$.

V	V_3	V_4	V_5	$V_{\alpha>5}$
M	$M_{2,\alpha}$, A_5, $V_{\gamma\geq 3}$	A_5, $V_{\gamma\geq 4}$	A_5, $V_{\gamma\geq 5}$	$V_{\gamma>5}$

THEOREM 5.9. *Let A be a* JC*-*triple of constant rank* 2. *If A is a spin type and $2 \leq \gamma < \infty$, then $S^\gamma(A) = \{P \in \mathrm{Prim}(A) : A/P \approx V_\lambda, \lambda > \gamma$ is open in Prim*(A).

In general, there are ideals $J_1 \subset J_2 \subset J_3 \subset J_4 \subset J_5 \subset A$ such that

(i) J_1 *is spin type with* $\mathrm{Prim}(J_1) = S^5$;

(ii) J_2/J_1 *is type A_5*;

(iii) J_3/J_2 *is type V_5*;

(iv) J_3/J_2 *is type V_4*;

(v) J_5/J_4 *is rectangular*;

(vi) A/J_5 *is type V_2.*

PROOF. Let $P_0 \in (A)$. As in the proof of Theorem 5.8, for a complete tripotent $e \in M$ and $x \in A$ we have $A(x)/P_0 \cap A(x) \approx M_2(e)$ as JC*-algebras. As M is rank 2, $M_2(e)$ is a spin factor.

Assume that A is of spin type. Suppose that $P_0 \cap A(x) \in S^\gamma(A(x))$ which is open in Prim(A) by Lemma 4.9. Thus, by Proposition 3.3 and its notation $U = \beta^{-1}(S^\gamma(A(x)))$ is open neighbourhood of P_0 and $U \subset S^\gamma(A)$ by the table above. It follows that $S^\gamma(A)$ is open.

Reverting to the general case, the same argument shows that $S^5(A)$ is open. This gives the ideal J_1. Passing to A/J_1 we may assume that $S^5(A) = \phi$. In this case, suppose that $P_0 \in \{Q \in \mathrm{Prim}(A) : A/Q$ is symplectic$\} = S$. Then $M \approx A_5$ or $M \approx V_5 \approx A_4$ so that $M_2(e) \approx V_5$ and $P_0 \cap A(x) \in S^4(A(x))$. Hence, by Proposition 3.3 and Lemma 4.9 together with the above table, there is a neighbourhood of P_0 contained in S which is therefore open in Prim(A). By Lemma 5.6(iii), the corresponding ideal, J_3 contains the ideal J_2 as stated.

Proceeding, we now assume that S and $S^5(A)$ are empty to find, in the same way, that $S^3(A)$ is now open. This gives the ideal J_4.

Finally assume that $S^3(A)$ is empty and let

$$P_0 \in R = \{P \in \mathrm{Prim}(A) : A/P \text{ is rectangular}\}.$$

Then $M_2(e) = V_3$ and we find $P_0 \in \beta^{-1}(S^2(A(x))) \cap R$ from the first column of the table. It follows that R is open in $\mathrm{Prim}(A_)$, which gives the ideal J_5.

COROLLARY 5.10. *Let A be a* JB*-*triple of bounded rank such that all primitive quotients are finite dimensional and let, up to isometry, $\{A/P : P \in \mathrm{Prim}(A)\} = \{M_i\}_{i=1}^K$. Then there is a permutation π of $\{1, \ldots, k\}$ and norm closed ideals of $A, 0 = I_0 \subset I_1 \subset \ldots I_k \subset I_{k+1} = A$ such that I_{r+1}/I_r is non-trivial* type $M_{\pi(r)}$, for $r = 0, \ldots, k$.

ACKNOWLEDGEMENT. The authors are grateful for financial support from the British Council and the Slovene Ministry of Science and Technology.

REFERENCES

1. E.M. Alfsen and E.G. Effros, *Structure in real Banach spaces I, II,* Ann. of Math. 96 (1972), 98–128 and 129–173.
2. T. Barton and R. Timoney, *Weak* continuity of Jordan triple product and applications,* Math Scand. 59 (1986), 177–191.
3. T. Barton, T. Dang and G. Horn, *Normal representations of Banach triple systems,* Proc. Amer. Math. Soc. 102 (1987), 551–555.
4. E. Behrens, *M-structure and the Banach-Stone Theorem,* Lectue Notes in Math. 736, 1979.
5. H. Behncke and W. Bos, *JB-algebras with an exceptional ideal,* Math. Scand. 42 (1978), 306–312.
6. L.J. Bunce *Structure of representations and ideals of homogeneous type in Jordan algebras,* Quart. J. Math. Oxford 37 (1986), 1–10.
7. L.J. Bunce and C.H. Chu, *Compact operations, multipliers and the Radon- Nikodym property,* Pacific J. Math 153 (1992), 249–265.
8. T. Dang and Y. Friedman, *Classification of JBW*-triple factors and applications,* Math. Scand. 61 (1987), 292–330.
9. S. Dineen, *The second dual of a JB*-triple system,* Complex Analysis, Functional Analysis and Approximation theory, vol. 125, North Holland Math. Studies, 1986, pp. 67–69.
10. J. Dixmier, *C*-algebras,* North Holland, 1969.
11. C.M. Edwards, *On the centres of hereditary JBW subalgebras of a JBW algebra,* Math. Proc. Cambridge Philos. Soc. 85 (1979), 317–324.
12. Y. Friedman and B. Russo, *Structure of the predual of a JBW*-triple,* J. Reine Angew. Math. 356 (1985), 67–84.
13. Y. Friedman and B. Russo, *Solution of the contractive projection problem,* J. Funct. Anal. 60 (1985), 56–79.
14. Y. Friedman and B. Russo, *The Gelfand-Naimark theorem for JB*triples,* Duke Math. J. 53 (1986), 139–148.
15. H. Hanche-Olsen and E. Stormer, *Jordan Operator Algebras,* Pitman, London 1984.
16. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach Spaces and Banach algebras.* Lecture Notes in Math. 1547, 1993.
17. L.A. Harris, *A generalization of C*-algebras,* Proc. London Math. Soc. 42 (1982), 331–361.
18. G. Horn, *Characterization of the predual and ideal structure of a JBW*triple,* Math. Scand. 61 (1987), 117–133.
19. G. Horn, *Classification of JBW*-triples of type I,* Math. Z. 196 (1987), 271–291.

20. E. Horn and E Neher, *Classification of JBW*-triples,* Trans. Amer. Math. Soc. 306 (1988), 553–578.
21. W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces,* Math. Z. 138 (1983), 503–529.
22. W. Kaup, *Contractive projections on Jordan C*-algebras and generalizations,* Math. Scand. 54 (1985), 95–100.
23. W. Kaup, *On JB*-triples defined by fibre bundles,* Manuscripta Math. 87 (1987), 379–403.
24. O. Loos, *Bounded Symmetric Domains and Jordan Pairs* (1977), Irvine Lecture Notes.
25. G.K. Pedersen, *C*-algebras and their automorphism groups,* North Holland, 1979.
26. B. Russo, *Structure of JB*triples,* Jordan Algebras (W. Kaup, K. McCrimmon, H.P. Petterson, eds.), de Gruyter, 1994.
27. L.L. Stacho, *A projection principle concerning biholomorphic automorphisms of Banach spaces,* Acta Sci. Math. (Szeged) 46 (1982).
28. J. Tomiyama and M.Takesaki, *Application of fibre bundles to the certain class of C*-algebras,* Tohoku Math. J. 13 (1961), 498–522.
29. H. Upmeier, *Symmetric Banach Manifolds and Jordan C*-algebras,* North Holland, 1985.
30. J.D.M Wright, *Jordan C*-algebras,* Michigan Math. J. 24 (1977), 291–302.

THE UNIVERSITY OF READING
DEPARTMENT OF MATHEMATICS
WHITEKNIGHTS
READING RG6 6AX
UK

UNIVERSITY OF LONDON
GOLDSMITH'S COLLEGE
NEW CROSS
LONDON SE14 6NW
UK
Email address: maa01chc@gold.ac.uk

UNIVERSITY OF MARIBOR
FACULTY OF CIVIL ENGINEERING
DEPARTMENT OF BASIC SCIENCES
SMETANOVA 17
62000 MARIBOR
SLOVENIJA
E-mail address: Borut.Zalar@uni-mb.si or borut.zalar@uni-lj.si