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FRITZ CARLSON'S INEQUALITY AND ITS APPLICATION

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Abstract

A Carlson-type inequality is proved and it is applied to show a Babenko-Beckner type of the Hausdorff-Young inequality on n-dimensional torus.

Introduction

Fritz Carlson's inequality (1934) states, [4], that

$$\sum_{n=1}^{\infty} a_n < \sqrt{\pi}\left(\sum_{n=1}^{\infty} {a_n}^2\right)^{\frac{1}{4}}\left(\sum_{n=1}^{\infty} n^2 {a_n}^2\right)^{\frac{1}{4}}$$

holds for any positive sequence $(a_n)_{n=1}^{\infty}$ and not all a_n are 0. Let $a_n := \widehat{f}(n)$, for a periodic function f. Then, there can be equality only if f is a multiple of f', and therefor an exponential function $C_0 e^{bx}$. This is plainly impossible, [7].

Note that the sums $\sum_{n=1}^{\infty} {a_n}^2$ and $\sum_{n=1}^{\infty} n^2 {a_n}^2$ are supposed to be finite.

The corresponding integral inequality, [4], [7], is

$$\int_0^{\infty} f(x)dx \leq \sqrt{\pi}\left(\int_0^{\infty} f^2(x)dx\right)^{\frac{1}{4}}\left(\int_0^{\infty} x^2 f^2(x)dx\right)^{\frac{1}{4}}.$$

Here there is equality when $f(x) := \frac{1}{a+bx^2}$, for any positive a, b.
For $f \in A(\mathsf{T})$ and $\widehat{f}(0) = 0$, the other expression of Carlson's inequality is

$$\|f\|_{A(\mathsf{T})} \leq C\Big(\|f\|_2 \|f'\|_2\Big)^{\frac{1}{2}}. \tag{1}$$

Here $\|f\|_{A(\mathsf{T})} := \sum_{m \in \mathsf{Z}} |\widehat{f}(m)|$ and $A(\mathsf{T})$ is the space of continuous functions on T having an absolutely convergent Fourier series. The variety of the constant C in (1) depends on the definitions of T and the Fourier series of f.

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B. Kjellberg, [11], and D. Müller, [14] (Lemma 3.1) proved a multi-dimensional extension of Carlson's inequality of the integral type. By using the idea[1] of Theorem 2.7.6. in [15], Carlson's inequality can be carried over from R^n to T^n. Our proof of the multi-dimensional case of (1) (for the case R^n see [10]) is new and more direct.

The well-known classical Hausdorff-Young inequality (1912–1923) states that, for any complex-valued function g in the Banach space $L^p(\mathsf{T})$,

$$\|\widehat{g}\|_{p'} \leq \|g\|_p \tag{2}$$

holds for $1 \leq p \leq 2$. Here and throughout the paper, p' is the dual exponent of p. Also, $\|\widehat{g}\|_{p'} := \left(\sum_{n\in\mathsf{Z}} |\widehat{g}(n)|^{p'}\right)^{\frac{1}{p'}}$ and $\|g\|_p := \left(\int_{\mathsf{T}} |g(x)|^p dx\right)^{\frac{1}{p}}$ are supposed to be finite.

Titchmarsh, [18], proved (2) for the space $L^p(\mathsf{R})$ in 1924. In fact, (2) is true for locally compact unimodular groups , [13]. The result is due to R.A. Kunze (1957). Hardy and Littlewood, [8], showed that (2) is sharp and there is equality if and only if $g = C_0 e^{2\pi m i x}$ for $m \in \mathsf{Z}$.

For the space $L^p(\mathsf{R}^n)$ and for the even integer p', [2], the improvement is due to K.I. Babenko (1961) and for all p, [3], it is due to W. Beckner (1975). That is

$$\|\widehat{f}\|_{p'} \leq {B_p}^n \|f\|_p \tag{3}$$

holds for $p \in [1, 2]$. $B_p := \sqrt{\frac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}}$ is called the Babenko-Beckner constant. $\widehat{f}(\xi) := \int_{\mathsf{R}^n} f(x) e^{-2\pi i <\xi, x>} dx$ is the Fourier transform of f and $\langle \xi, x \rangle := \sum_{\nu=1}^n \xi_\nu x_\nu$.

B. Russo (1974), [16], and J.J.F. Fournier (1977), [6], proved (3) for certain classes of locally compact unimodular groups.

The extension of (3) is due to J. Inoue (1992), [9]. For certain classes of nilpotent Lie groups he improved (3) and obtained the constant

$$B_p^{\dim(G) - \frac{m}{2}}.$$

Here $G := \exp(\mathbf{g})$ and $\mathbf{g}$ is Lie algebras with the dual space $\widehat{\mathbf{g}}$. $\dim(G)$ is the dimension of nilpotent Lie groups G and m is the dimension of generic coadjoint orbits of G in $\widehat{\mathbf{g}}$.

For the even integer p', [1], M.E. Andersson (1994) and for all p, [17], P.

[1] The referee made kindly this idea clear to me. He also informed me of the references [11] and [14] and gave me valuable comments on this paper (see the remark).

Sjölin (1995) proved a Babenko-Beckner type inequality (3) for functions in the space $L^p(\mathsf{T})$, with small supports.

The purpose of this paper is to prove Carlson's inequality of type (1) on n-dimensional torus and applying it to prove a Babenko-Beckner type of the Hausdorff-Young inequality for periodic functions with small supports.

Theorems and Proofs

Let the multi-indices β and γ be vectors in R^n with components β_k and γ_k in N_0 such that $\gamma \le \beta$ is equivalent to $\gamma_k \le \beta_k$ for all $1 \le k \le n$. Define $m^\beta := \prod_{k=1}^n {m_k}^{\beta_k}$ for $m \in \mathsf{Z}^n$ and $0^0 := 1$.

Throughout this paper, $|\beta| := \sum_{k=1}^n \beta_k$ and $\beta\gamma := \prod_{k=1}^n \beta_k \gamma_k$. The operator $D^\beta := \prod_{k=1}^n \frac{\partial^{\beta_k}}{\partial {x_k}^{\beta_k}}$.

Let also

$$H_{p,a} := \sup\left\{ \frac{\|\widehat{g}\|_{p'}}{\|g\|_p} : \ g \in L^p(\mathsf{T}^n), \ \operatorname{supp} g \subset \overline{B}(0,a), \ \|g\|_p \ne 0 \right\}$$

and define $H_p := \lim_{a\to 0^+} H_{p,a}$. Here and everywhere in the paper a obeys the restriction $0 < a < \frac{1}{2}$ and $\overline{B}(0,a)$ is a closed ball of radius a, centered at the origin. Also, $\mathsf{T}^n := \left\{ x \in \mathsf{R}^n : |x_\nu| \le \frac{1}{2}, \ 1 \le \nu \le n \right\}$.

Assume $\varphi(x) := \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & |x| \ge 1 \end{cases}$ such that $\varphi \in C_0^\infty(\mathsf{R}^n)$, $0 \le \varphi \le 1$ and $\varphi_a := \varphi\left(\frac{x}{a}\right)$. Define $\Psi(x) .= \left(e^{-2\pi i<b,x>} - 1\right)\varphi_a(x)$. Here $b := (b_1, b_2, \cdots, b_n)$ and $|b_k| \le \frac{1}{2}$, $1 \le k \le n$.

With the previous notation, we prove the following:

THEOREM 1 (Generalisation of Carlson's inequality). *Let $f \in A(\mathsf{T}^n)$ and $\widehat{f}(0) = 0$. Let the absolute value of the multi-index β be equal to the positive integer α such that $\alpha \ge 1$ and $\alpha > \frac{n}{q}$ where $1 < q \le 2$. Then we get*

$$\|f\|_{A(\mathsf{T}^n)} \le K_{n,q}^{(\alpha)} {\|f\|_q}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|D^\beta f\|_q \right)^{\frac{n}{q\alpha}}.$$

In the case $\widehat{f}(0) \ne 0$, we obtain

$$\|f\|_{A(\mathsf{T}^n)} \le \|f\|_1 + K_{n,q}^{(\alpha)} {\|f\|_q}^{1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|D^\beta f\|_q \right)^{\frac{n}{q\alpha}}.$$

The positive constant $K_{n,q}^{(\alpha)}$ depends only on n, α and q.

PROOF OF THEOREM 1. The technique is analogous to the case $n = 1$, due to Hardy, [7]. Let $\widehat{f}(0) = 0$ and q' be the dual exponent of q. Define

$$S := \|\widehat{f}\|_{q'}^{q'}$$
$$T := \sum_{|\beta|=\alpha} \|\widehat{D^\beta f}\|_{q'}^{q'}.$$

For $t > 0$ we also define

$$P := \sum_{|\beta|=\alpha} (t + |(2\pi m)^\beta|^{q'})$$

Then $T \leq \left(\sum_{|\beta|=\alpha} \|\widehat{D^\beta f}\|_{q'}\right)^{q'}$.

By Hölder's inequality we get

$$\begin{aligned}
(4.1) \qquad \|f\|_{A(\mathsf{T}^n)} &= \sum_{|m|>0} |\widehat{f}(m)| P^{\frac{1}{q'}} P^{-\frac{1}{q'}} \\
&\leq \left(\sum_{|m|>0} |\widehat{f}(m)|^{q'} P\right)^{\frac{1}{q'}} \left(\sum_{|m|>0} P^{-\frac{q}{q'}}\right)^{\frac{1}{q}} \\
&\leq t^{-\frac{1}{q'}} \left(t c_{n,\alpha} S + T\right)^{\frac{1}{q'}} \left[\sum_{|m|>0} \left(1 + \frac{C_{n,\alpha}}{t} |m|^{q'\alpha}\right)^{-\frac{q}{q'}}\right]^{\frac{1}{q}}.
\end{aligned}$$

Because

$$\sum_{|\beta|=\alpha} \left(t + |(2\pi m)^\beta|^{q'}\right) = c_{n,\alpha} t + \sum_{|\beta|=\alpha} |(2\pi m)^\beta|^{q'} \geq t + t C_{n,\alpha} |m|^{q'\alpha}.$$

Here $c_{n,\alpha} := \sum_{|\beta|=\alpha} 1$ and $\widehat{D^\beta f}(m) = (2\pi i m)^\beta \widehat{f}(m)$. The positive constant $C_{n,\alpha}$ does depend on n and α.

It is not hard to see that the sum $\left[\sum_{|m|>0} \frac{1}{\left(1+|m|^{q'\alpha}\right)^{\frac{q}{q'}}}\right]^{\frac{1}{q}}$ is finite for $\alpha > \frac{n}{q}$ and

$$(4.2) \qquad \int_0^\infty \frac{dx}{(1 + x^{\frac{q'\alpha}{n}})^{\frac{q}{q'}}} = \frac{\Gamma\left(\frac{n(q-1)}{q\alpha}\right) \Gamma\left(\frac{(q-1)(q\alpha-n)}{q\alpha}\right)}{\frac{q\alpha}{n(q-1)} \Gamma(q-1)}.$$

Now, by (4.1) and (4.2) we obtain

$$\|f\|_{A(\mathrm{T}^n)} \le c_0 t^{-\frac{1}{q'}}\left(tc_{n,\alpha}S+T\right)^{\frac{1}{q'}}\left[\int_{\mathrm{R}^n}\frac{dx}{\left(1+\frac{C_{n,\alpha}}{t}|x|^{q'\alpha}\right)^{\frac{q}{q'}}}\right]^{\frac{1}{q}}$$

$$= c_0 t^{-\frac{1}{q'}}\left(\frac{t}{C_{n,\alpha}}\right)^{\frac{n}{qq'\alpha}}\left(tc_{n,\alpha}S+T\right)^{\frac{1}{q'}}\left(\int_{\mathrm{R}^n}\frac{dx}{(1+|x|^{q'\alpha})^{\frac{q}{q'}}}\right)^{\frac{1}{q}}$$

$$= c_0 t^{-\frac{1}{q'}}\left(\frac{t}{C_{n,\alpha}}\right)^{\frac{n}{qq'\alpha}}\left(tc_{n,\alpha}S+T\right)^{\frac{1}{q'}}\left(\int_0^\infty\int_{\{x\in\mathrm{R}^{n-1}:|x|=1\}}\frac{r^{n-1}drdx}{(1+r^{q'\alpha})^{\frac{q}{q'}}}\right)^{\frac{1}{q}}$$

$$= c_0\left(\frac{w_{n-1}}{n}\right)^{\frac{1}{q}} t^{-\frac{1}{q'}}\left(\frac{t}{C_{n,\alpha}}\right)^{\frac{n}{qq'\alpha}}(tc_{n,\alpha}S+T)^{\frac{1}{q'}}\left(\int_0^\infty\frac{dx}{\left(1+x^{\frac{q'\alpha}{n}}\right)^{\frac{q}{q'}}}\right)^{\frac{1}{q}}$$

$$= c_0 A_{n,q}^{(\alpha)} t^{\frac{nq'}{q\alpha}}\left(c_{n,\alpha}S+\frac{T}{t}\right)^{\frac{1}{q'}},$$

for a positive constant c_0. Here

$$A_{n,q}^{(\alpha)} := \sqrt[q]{\frac{(q-1)w_{n-1}\Gamma\left(\frac{n(q-1)}{q\alpha}\right)\Gamma\left(\frac{(q-1)(q\alpha-n)}{q\alpha}\right)}{q\alpha\Gamma(q-1)}\left(C_{n,\alpha}\right)^{\frac{n(1-q)}{q\alpha}}},$$

and w_{n-1} is the surface area of the unit sphere in R^{n-1}.

Choose $t=\frac{S}{T}$, then by using (two times) the classical Hausdorff-Young inequality (2) we get

$$\|f\|_{A(\mathrm{T}^n)} \le c_0 A_{n,q}^{(\alpha)}\sqrt[q]{\left(c_{n,\alpha}+1\right)^{q-1}}\|\widehat{f}\|_{q'}{}^{1-\frac{n}{q\alpha}}\left(\sum_{|\beta|=\alpha}\|\widehat{D^\beta f}\|_{q'}\right)^{\frac{n}{q\alpha}}$$

$$\le c_0 A_{n,q}^{(\alpha)}\sqrt[q]{\left(c_{n,\alpha}+1\right)^{q-1}}\|f\|_{q}{}^{1-\frac{n}{q\alpha}}\left(\sum_{|\beta|=\alpha}\|D^\beta f\|_{q}\right)^{\frac{n}{q\alpha}}$$

$$= K_{n,q}^{(\alpha)}\|f\|_{q}{}^{1-\frac{n}{q\alpha}}\left(\sum_{|\beta|=\alpha}\|D^\beta f\|_{q}\right)^{\frac{n}{q\alpha}}.$$

For the case $\widehat{f}(0)\neq 0$ the proof is similar and we know that $|\widehat{f}(0)|\le\|f\|_1$.

Application of Theorem 1 for estimating of the $A(\mathsf{T}^n)$-norm of Ψ and $H_{p,a}$

LEMMA (An upper bound for $\|\Psi\|_{A(\mathsf{T}^n)}$). *There exists a positive constant C_0, does not depend on a, such that*

$$\|\Psi\|_{A(\mathsf{T}^n)} \leq C_0 a.$$

PROOF OF LEMMA. It is obvious that $\Psi \in C_0{}^{\infty}(\mathsf{R}^n)$ and for $m \in \mathsf{Z}^n$ we get

$$|\widehat{\Psi}(m)| = |\int_{|x|\leq a} \Psi(x)e^{-2\pi i<m,x>}dx| \leq a^n \int_{|y|\leq 1} |e^{-2\pi ia<b,y>} - 1|dy$$
$$\leq \pi\sqrt{n}a^{n+1}\int_{|y|\leq 1} dx =_n a^{n+1},$$

because

$$|e^{-2\pi ia<b,y>} - 1| \leq 2\pi a| < b, y > \leq \pi\sqrt{n}a.$$

Here $\Omega_n := \sqrt{n}\pi w_n$ and $w_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the surface area of the unit sphere in R^n.

Furthermore, by Leibniz's formula, together with Minkowski's inequality we obtain

$$\text{(5)} \qquad \sum_{|\beta|=\alpha} \|D^\beta \Psi\|_q \leq \pi\sqrt{n}a \sum_{|\beta|=\alpha} \|D^\beta \varphi_a\|_q + \sum_{|\beta|=\alpha} \sum_{\substack{\gamma\leq\beta \\ |\gamma|\neq 0}} \binom{\beta}{\gamma} \pi^{|\gamma|} \|D^{\beta-\gamma}\varphi_a\|_q$$
$$\leq \pi\sqrt{n}a^{1-\alpha+\frac{n}{q}} \sum_{|\beta|=\alpha} \|D^\beta \varphi\|_q + \sum_{|\beta|=\alpha} \sum_{\substack{\gamma\leq\beta \\ |\gamma|\neq 0}} \binom{\beta}{\gamma} \pi^{|\gamma|} a^{|\gamma|-|\beta|+\frac{n}{q}} \|D^{\beta-\gamma}\varphi\|_q$$
$$\leq a^{1-\alpha+\frac{n}{q}} \left\{ \pi\sqrt{n} \sum_{|\beta|=\alpha} \|D^\beta \varphi\|_q + \sum_{|\beta|=\alpha} \sum_{\substack{\gamma\leq\beta \\ |\gamma|\neq 0}} \binom{\beta}{\gamma} \pi^{|\gamma|} \|D^{\beta-\gamma}\varphi\|_q \right\}$$
$$= A_{n,q,\alpha} a^{1-\alpha+\frac{n}{q}}, 5$$

because

$$\sum_{|\beta|=\alpha} \|D^\beta \varphi_a\|_q = a^{\frac{n}{q}-\alpha} \sum_{|\beta|=\alpha} \|D^\beta \varphi\|_q.$$

Now, by Theorem 1 and invoking (5), we get

$$\begin{aligned}\|\widehat{\Psi}\|_1 &= \sum_{m\in\mathsf{Z}^n} |\widehat{\Psi}(m)| \le \Omega_n a^{n+1} + \sum_{|m|>0} |\widehat{\Psi}(m)| \\ &\le \Omega_n a^{n+1} + K_{n,q}^{(\alpha)} \|\Psi\|_q^{\,1-\frac{n}{q\alpha}} \left(\sum_{|\beta|=\alpha} \|D^\beta \Psi\|_q \right)^{\frac{n}{q\alpha}} \\ &\le \Omega_n a^{n+1} + \left[K_{n,q}^{(\alpha)} \left(\pi\sqrt{n}\|\varphi\|_q \right)^{1-\frac{n}{q\alpha}} A_{n,q,\alpha}^{\frac{n}{q\alpha}} \right] a \\ &\le \left\{ \Omega_n + K_{n,q}^{(\alpha)} A_{n,q,\alpha}{}^{\frac{n}{q\alpha}} \left(\pi\sqrt{n}\|\varphi\|_q \right)^{1-\frac{n}{q\alpha}} \right\} a \\ &= C_0 a,\end{aligned}$$

because

$$\|\Psi\|_q \le \pi\sqrt{n}\|\varphi\|_q a^{1+\frac{n}{p}}.$$

Note that α is the positive integer defined in Theorem 1 and $\|\widehat{\Psi}\|_1 := \|\Psi\|_{A(\mathsf{T}^n)}$.

THEOREM 2 (An upper bound for $H_{p,a}$). *For a fixed $n \in \mathsf{N}$, there exists a positive constant C_0 which does not depend on a, such that*

$$H_{p,a} \le \Big(1 + C_0 a\Big) B_p{}^n, \ 1 \le \mathrm{p} \le 2.$$

PROOF OF THEOREM 2. The technique is analogous to the case $n = 1$, due to Y. Domar, [5]. Choose $f \in L^p(\mathsf{R}^n)$, $g \in L^p(\mathsf{T}^n)$, such that $f = g$ on the ball $\overline{B}(0, a)$ and zero outside of the ball. Define $g_b(x) := e^{-2\pi i<x,b>} g(x)$. Then

$$\begin{aligned}\widehat{g_b}(m) &= \widehat{g}(m + b) \\ \|f\|_p &= \|g\|_p \\ \widehat{f_b}(m) &= \widehat{g_b}(m).\end{aligned}$$

Also, we get

$$\begin{aligned}\widehat{g_b}(m) - \widehat{g}(m) &= \int e^{-2\pi i\langle m,x\rangle} \big(e^{-2\pi i\langle b,x\rangle} - 1\big) g(x) dx \\ &\le \int_{\overline{B}(0,a)} \Psi(x) g(x) e^{-2\pi i\langle m,x\rangle} dx \\ &= \int_{\overline{B}(0,a)} g(x) \left(\sum_{m'\in\mathsf{Z}^n} \widehat{\Psi}(m') e^{2\pi i\langle m',x\rangle} \right) e^{-2\pi i\langle m,x\rangle} dx \\ &= \sum_{m'\in\mathsf{Z}^n} \widehat{\Psi}(m')\widehat{g}(m - m').\end{aligned}$$

Thus, we obtain

$$\|\widehat{g}_b - \widehat{g}\|_{p'} \leq \left(\sum_{m'\in \mathsf{Z}^n} |\widehat{\Psi}(m')|\right)\left(\sum_{m\in \mathsf{Z}^n} |\widehat{g}(m)|^{p'}\right)^{\frac{1}{p'}} = \|\widehat{g}\|_{p'}\|\widehat{\Psi}\|_1.$$

By triangle inequality we have

$$\|\widehat{g}\|_{p'} - \|\widehat{g}_b\|_{p'} \leq \|\widehat{g}_b - \widehat{g}\|_{p'} \leq \|\widehat{g}\|_{p'}\|\widehat{\Psi}\|_1.$$

Similarly, for $t \in \mathsf{R}^n$, we obtain

$$\|\widehat{g}\|_{p'}\left(1 - \|\widehat{\Psi}\|_1\right) \leq \|\widehat{g}_b\|_{p'} = \left(\sum_{m\in \mathsf{Z}^n} |\widehat{g}_b(m)|^{p'}\right)^{\frac{1}{p'}} = \left(\sum_{m}\left|\widehat{f}_b(m)\right|^{p'}\right)^{\frac{1}{p'}}.$$

That is

$$\begin{aligned}\|\widehat{g}\|_{p'}\left(1 - \|\widehat{\Psi}\|_1\right) &\leq \left(\sum_{m}\int_{\{b:|b_k|\leq \frac{1}{2}\}}\left|\widehat{f}_b(m)\right|^{p'} db\right)^{\frac{1}{p'}} \\ &= \left(\sum_{m}\int_{\{t-m:|t_k-m_k|\leq \frac{1}{2}\}}\left|\widehat{f}(t)\right|^{p'} dt\right)^{\frac{1}{p'}} \\ &= \|\widehat{f}\|_{p'}.\end{aligned}$$

Now, by Lemma we get

$$\|\widehat{g}\|_{p'} \leq \frac{\|\widehat{f}\|_{p'}}{1 - \|\widehat{\Psi}\|_1} \leq \frac{\|\widehat{f}\|_{p'}}{1 - C_0 a}. \tag{6}$$

By (6), thus, we obtain

$$H_{p,a} = \sup_{g}\frac{\|\widehat{g}\|_{p'}}{\|g\|_p} \leq \frac{B_p{}^n}{1 - C_0 a}.$$

Because

$$\sup_{f}\frac{\|\widehat{f}\|_{p'}}{\|f\|_p} = B_p{}^n,$$

(see [3], p. 160). Choose a such that $C_0 a < \frac{1}{2}$, then we get

$$\frac{1}{1 - C_0 a} \approx 1 + C_0 a,$$

because $\frac{1}{1-C_0 a} = 1 + C_0 a + O\left(C_0{}^2 a^2\right)$. Hence

$$H_{p,a} \leq \left(1 + C_0 a\right) B_p{}^n.$$

REMARK. The arguments in this proof can be used to prove that the quotient of the norms $\widehat{g}$ and $\widehat{f}$ in $l^{p'}$ and $L^{p'}$, respectively, is $1 + O(a)$, as $a \to 0^+$.

The Babenko-Beckner type of the Hausdorff-Young inequality for periodic functions with small supports

THEOREM 3 $H_p \leq B_p{}^n$.
The proof is a consequence of Theorem 2.

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