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DUALITY FOR POSITIVE LINEAR MAPS IN MATRIX ALGEBRAS

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Abstract

We characterize extreme rays of the dual cone of the cone consisting of all s-positive (respectively t-copositive) linear maps between matrix algebras. This gives us a characterization of positive linear maps which are the sums of s-positive linear maps and t-copositive linear maps, which generalizes Størmer's characterization of decomposable positive linear maps in matrix algebras. With this duality, it is also easy to describe maximal faces of the cone consisting of all s-positive (respectively t-copositive) linear maps between matrix algebras.

1. Introduction

The structures of the convex cone of positive linear maps between C^*-algebras are turned out to be extremely complicated even when the domain and the range are low dimensional matrix algebras M_n. Several authors including [2], [4], [9], [10], [12], [14] and [15] have tried to decompose the cone into smaller cones consisting of more well-behaved positive linear maps such as completely positive and completely copositive linear maps. We denote by $\mathscr{B}(\mathscr{H})$ and $\mathcal{T}(\mathscr{H})$ the space of all bounded linear operators and trace class operators on a Hilbert space $\mathscr{H}$, respectively. One of the methods to examine the possibility of decomposition is to use the duality between the space $\mathscr{B}(A, \mathscr{B}(\mathscr{H}))$ of all bounded linear operators from a C^*-algebra A into $\mathscr{B}(\mathscr{H})$ and the projective tensor product $\mathcal{T}(\mathscr{H})\hat{\otimes}A$ given by

$$\langle x \otimes y, \phi \rangle = \mathrm{Tr}(\phi(y)x^{\mathrm{t}}), \qquad x \in \mathcal{T}(\mathscr{H}),\ y \in A,\ \phi \in \mathscr{B}(A, \mathscr{B}(\mathscr{H})),$$

where Tr and t denote the usual trace and the transpose, respectively. Using this duality, Woronowicz [15] has shown that every positive linear map from the matrix algebra M_2 into M_n is the sum of a completely positive linear map and a completely copositive linear map if and only if $n \leq 3$. The above duality is also useful to study extendibility of positive linear maps as was con-

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sidered by Størmer [13]. We denote by $\mathsf{P}_s[A,B]$ (respectively $\mathsf{P}^s[A,B]$) the convex cone of all s-positive (respectively s-copositive) linear maps from a C^*-algebra A into a C^*-algebra B. We also denote by $\mathsf{P}_\infty[A,B]$ (respectively $\mathsf{P}^\infty[A,B]$) the cone of all completely positive (respectively completely copositive) linear maps. The predual cones of $\mathsf{P}_s[A,\mathscr{B}(\mathscr{H})]$ and $\mathsf{P}^s[A,\mathscr{B}(\mathscr{H})]$ with respect to the above pairing has been determined by Itoh [3].

If we restrict ourselves to the cases of matrix algebras, then the above pairing may be expressed by

$$(1)\qquad \langle A,\phi\rangle = \mathrm{Tr}\left[\left(\sum_{i,j=1}^{m}\phi(e_{ij})\otimes e_{ij}\right)A^{\mathrm{t}}\right] = \sum_{i,j=1}^{m}\langle\phi(e_{ij}),a_{ij}\rangle,$$

for $A=\sum_{i,j=1}^m a_{ij}\otimes e_{ij}\in M_n\otimes M_m$ and a linear map $\phi: M_m\to M_n$, where $\{e_{ij}\}$ is the matrix units of M_m and the bilinear form in the right-side is given by $\langle X, Y\rangle = \mathrm{Tr}(YX^{\mathrm{t}})$ for $X, Y\in M_n$. Then (1) defines a bilinear pairing between the space $M_n\otimes M_m(=M_{nm})$ of all $nm\times nm$ matrices over the complex field and the space $\mathcal{L}(M_m,M_n)$ of all linear maps from M_m into M_n.

In this note, we show that the predual cone of $\mathsf{P}_s[M_m,M_n]$ with respect to the pairing (1) is generated by rank one matrices in M_{nm} whose range vectors in C^{nm} correspond to $m\times n$ matrices of ranks s. The predual cone of $\mathsf{P}^s[M_m,M_n]$ is obtained by block-transposing that of $\mathsf{P}_s[M_m,M_n]$. With this information, it is easy to characterize the predual cone of $\mathsf{P}_s[M_m,M_n]+\mathsf{P}^t[M_m,M_n]$. As an application, we extend Størmer's result [12] to give a characterization of linear maps which are sums of s-positive linear maps and t-copositive linear maps. We also show that Choi's examples [1] of non-decomposable positive linear maps are not the sum of 3-positive linear maps and 2-copositive linear maps. The second author [6], [7], [8] has modified the method in [11] to characterize maximal faces of the cones $\mathsf{P}_s[M_m,M_n]$ and $\mathsf{P}^s[M_m,M_n]$, and all faces of the cones $\mathsf{P}_\infty[M_m,M_n]$ and $\mathsf{P}^\infty[M_m,M_n]$. Generally, it turns out that every maximal face of a convex cone in a finite dimensional space corresponds to an extreme ray of the predual cone, whenever every extreme ray of the predual cone is exposed with respect to the pairing. This enables us to describe maximal faces of the cones $\mathsf{P}_s[M_m,M_n]$ and $\mathsf{P}^s[M_m,M_n]$ simultaneously. Compare with [7].

We develop in Section 2 some general aspects of dual cones how maximal faces of a cone correspond to extreme rays of the dual cone, and characterize extreme rays of the predual cones of $\mathsf{P}_s[M_m,M_n]$ and $\mathsf{P}^s[M_m,M_n]$ in Section 3. We also examine in Section 4 the Choi's example mentioned above. Throughout this note, we fix natural numbers m and n, and denote by just P_s (respectively P^s) for the cone $\mathsf{P}_s[M_m,M_n]$ (respectively $\mathsf{P}^s[M_m,M_n]$). Note that $\mathsf{P}_\infty=\mathsf{P}_{m\wedge n}$ and $\mathsf{P}^\infty=\mathsf{P}^{m\wedge n}$ in these notations, where $m\wedge n$ denotes the minimum of $\{m,n\}$.

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2. Duality of convex cones.

Let X and Y be finite dimensional normed space, which are dual each other with respect to a bilinear pairing $\langle\ ,\ \rangle$. For a subset C of X (respectively D of Y), we define the *dual cone* C° (respectively D°) by the set of all $y \in Y$ (respectively $x \in X$) such that $\langle x, y\rangle \geq 0$ for each $x \in C$ (respectively $y \in D$). It is clear that $C^{\circ\circ}$ is the closed convex cone of X generated by C. It is also easy to see that the dual cone of the intersection $C_1 \cap C_2$ of two cones C_1 and C_2 is nothing but the closed cone generated by C_1° and C_2°. In other words, we have the identities:

$$(C_1 \cap C_2)^\circ = (C_1^\circ \cup C_2^\circ)^{\circ\circ}, \qquad (C_1 \cup C_2)^\circ = C_1^\circ \cap C_2^\circ, \tag{2}$$

whenever C_1 and C_2 are closed convex cones of X. Indeed, we have $C_i = C_i^{\circ\circ} \supset (C_1^\circ \cup C_2^\circ)^\circ$ for $i = 1, 2$, and so $C_1 \cap C_2 \supset (C_1^\circ \cup C_2^\circ)^\circ$, which implies one direction of the first identity. On the other hand, $C_i^\circ \subset (C_1 \cap C_2)^\circ$ implies $(C_1^\circ \cup C_2^\circ)^{\circ\circ} \subset (C_1 \cap C_2)^\circ$. The second identity follows from the first one.

For a face F of a closed convex cone C of X, we define the subset F' of C° by

$$F' = \{y \in C^\circ : \langle x, y\rangle = 0 \ \text{ for each } x \in F\}.$$

It is then clear that F' is a closed face of C°. If we take an interior point x_0 of F then we see that

$$F' = \{y \in C^\circ : \langle x_0, y\rangle = 0\}.$$

Recall that a point x_0 of a convex set C is said to be an *interior point* if for any $x \in C$ there is $t > 1$ such that $(1-t)x + tx_0 \in C$. If C is a convex subset of a finite dimensional space then the set C of all interior points of C is nothing but the relative interior of C with respect to the affine manifold generated by C.

It is clear that $F \subset F''$ for any face F of C. Therefore, we have $F' \supset F''' \supset F'$, and so it follows that $F' = F'''$. We say that a face F of a closed convex cone C is *exposed* with respect to the pairing $\langle\ ,\ \rangle$ if there exists $y_0 \in C^\circ$ such that $F = \{x \in C : \langle x, y_0\rangle = 0\}$. If a face F is exposed by $y_0 \in C^\circ$ then take a face G of C° such that y_0 is an interior point of G. Then $F = G'$, and so $F'' = G''' = G' = F$. Therefore, we have the following:

Lemma 2.1. *Let F be a closed face of a closed convex cone C. Then F is exposed if and only if $F = F''$. The set F'' is the smallest exposed face containing F.*

If all closed faces of C and C° are exposed with respect to the pairing, then it is clear from Lemma 2.1 that the correspondence $F \mapsto F'$ is an order reversing one-to-one mapping from the complete lattice $\mathscr{F}(C)$ of all closed faces of C onto the complete lattice $\mathscr{F}(C^\circ)$. From this, it is easily seen that this map is an order reversing lattice isomorphism. Indeed, it is clear that

$$F_1' \vee F_2' \le (F_1 \wedge F_2)', \qquad F_1' \wedge F_2' \ge (F_1 \vee F_2)'.$$

Then it follows that

$$F_1 \vee F_2 = F_1'' \vee F_2'' \le (F_1' \wedge F_2')' \le (F_1 \vee F_2)'' = F_1 \vee F_2,$$

and so, we have

$$F_1' \vee F_2' = (F_1 \wedge F_2)', \qquad F_1' \wedge F_2' = (F_1 \vee F_2)'. \tag{3}$$

From now on throughout this section, we assume that C is a closed convex cone of X on which the pairing is *non-degenerate*, that is,

$$x \in C,\ \langle x, y \rangle = 0 \text{ for each } y \in C^\circ \Longrightarrow x = 0. \tag{4}$$

This assumption guarantees the existence of a point $\eta \in C^\circ$ with the property:

$$x \in C,\ x \ne 0 \Longrightarrow \langle x, \eta \rangle > 0, \tag{5}$$

which is seemingly stronger than (4). Indeed, we take for each $x \in C$ a neighborhood U_x of x and a point $y_x \in C^\circ$ such that $\langle z, y_x \rangle > 0$ for $z \in U_x$. Put $C_\epsilon = \{x \in C : \|x\| = \epsilon\}$ for $\epsilon > 0$. Then since C_1 is compact, we see that there exist $x_1, \ldots, x_r \in C_1$ such that $U_{x_1} \cup \ldots \cup U_{x_r} \supset C_1$. We may put $\eta = y_{x_1} + \cdots + y_{x_r}$. As an another immediate consequence of (4), we also have

$$F \in \mathscr{F}(C),\ F' = C^\circ \Longrightarrow F = \{0\}. \tag{6}$$

Lemma 2.2. *For a given point $y \in C^\circ$, the following are equivalent:*

(i) *y is an interior point of C°.*
(ii) *$\langle x, y \rangle > 0$ for each nonzero $x \in C$.*
(iii) *$\langle x, y \rangle > 0$ for each $x \in C$ which generates an extreme ray.*

Proof. If y is an interior point of C° then we may take $t < 1$ and $z \in C^\circ$ such that $y = (1-t)\eta + tz$, where $\eta \in C^\circ$ is a point with the property (5). Then we see that

$$\langle x, y \rangle = (1-t)\langle x, \eta \rangle + t\langle x, z \rangle > 0$$

for each nonzero $x \in C$. It is clear that (ii) and (iii) are equivalent. Now, we assume (ii), and take an arbitrary point $z \in C^\circ$. Then since C_1 is compact, $\alpha = \sup\{\langle x, z\rangle : x \in C_1\}$ is finite, and we see that $\langle x, z\rangle \le 1$ for each $x \in C_{1/\alpha}$. We also take δ with $0 < \delta < 1$ such that $\langle x, y\rangle \ge \delta$ for each $x \in C_{1/\alpha}$. Put

$$w = \left(1 - \frac{1}{1-\delta}\right) z + \frac{1}{1-\delta}\, y.$$

Then we see that $\langle x, w\rangle \ge 0$ for each $x \in C_{1/\alpha}$, and so $w \in C^\circ$. Since z was an arbitrary point of C° and $\frac{1}{1-\delta} > 1$, we see that y is an interior point of C°.

Lemma 2.3. *If F is a maximal face of C° then there is an extreme ray L of C such that $F = L'$.*

Proof. Note that F lies in the boundary of C°. If we take an interior point y_0 of F then there is $x_0 \in C$ which generates an extreme ray L such that $\langle x_0, y_0\rangle = 0$ by Lemma 2.2. Since x_0 is an interior point of L, we see that $y_0 \in L' \cap \operatorname{int} F$, from which we infer that $F \subset L'$. Because $L' \subsetneqq C^\circ$ by (6), we have $F = L'$.

Since $(L'')' = F$ in the above lemma, we see that every maximal face of C° is of the form G' for a unique nonzero exposed face G. Note that L'' need not be minimal even among nonzero exposed faces.

Lemma 2.4. *If L is an exposed face of C which is minimal among nonzero exposed faces, then L' is a maximal face of C°.*

Proof. Assume that L' is not maximal, and take a maximal face F of C° such that $L' \subsetneqq F$. Then there exists a nonzero exposed face M of C such that $F = M'$, and so $L = L'' \supset F' = M''$. Since L is minimal among nonzero exposed faces, we have $L = M$ and $L' = M' = F$, which is a contradiction.

Now, we summarize as follows:

Theorem 2.5. *Let X and Y be finite-dimensional normed spaces with a non-degenerate bilinear pairing $\langle\ ,\ \rangle$ on a closed convex cone C in X. Assume that every extreme ray of C is exposed with respect to the pairing. Then L' is a maximal face of C° for each extreme ray L of C. Conversely, every maximal face of C° is of the form L' for a unique extreme ray L of C.*

We will see that there is an extreme ray in $\mathsf{P}_1[M_3, M_3]$ which is not exposed, while every extreme ray of the dual cone of P_s (respectively P^s) is exposed.

3. Positive linear maps.

In this section, every vector in the space C^r will be considered as an $r \times 1$ matrix. The usual orthonormal basis of C^r and matrix unit M_r will be denoted by $\{e_i : i = 1, \dots, r\}$ and $\{e_{ij} : i,j = 1, \dots r\}$ respectively, regardless of the dimension r. For a matrix $A = \sum_{i,j=1} x_{ij} \otimes e_{ij} \in M_n \otimes M_m$, we denote by A^τ the *block-transpose* $\sum_{i,j=1}^m x_{ji} \otimes e_{ij}$ of A. Every vector $z \in \mathsf{C}^n \otimes \mathsf{C}^m$ may be written in a unique way as $z = \sum_{i=1}^m z_i \otimes e_i$ with $z_i \in \mathsf{C}^n$ for $i = 1, 2, \dots, m$. We say that z is an *s-simple vector* in $\mathsf{C}^n \otimes \mathsf{C}^m$ if the linear span of $\{z_1, \dots, z_m\}$ has the dimension $\le s$.

For an s-simple vector $z = \sum_{i=1}^m z_i \otimes e_i \in \mathsf{C}^n \otimes \mathsf{C}^m$, take a generator $\{u_1, u_2, \dots, u_s\}$ of the linear span of $\{z_1, z_2, \dots, z_m\}$ in C^n, and define $a_{ik} \in \mathsf{C}$, $a_k \in \mathsf{C}^m$, $u \in \mathsf{C}^n \otimes \mathsf{C}^s$ and $w \in \mathsf{C}^m \otimes \mathsf{C}^s$ by

$$
\begin{aligned}
z_i &= \sum_{k=1}^s a_{ik} u_k \in \mathsf{C}^n, \qquad i = 1, 2, \dots, m, \\
a_k &= \sum_{i=1}^m a_{ik} e_i \in \mathsf{C}^m, \qquad k = 1, 2, \dots, s, \\
u &= \sum_{k=1}^s \bar{u}_k \otimes e_k \in \mathsf{C}^n \otimes \mathsf{C}^s, \\
w &= \sum_{k=1}^s a_k \otimes e_k \in \mathsf{C}^m \otimes \mathsf{C}^s,
\end{aligned} \tag{7}
$$

where $\bar{u}_k$ denotes the vector whose entries are conjugates of those of the vector u_k. Then $zz^* = \sum_{i,j=1}^m z_i z_j^* \otimes e_{ij} \in M_n \otimes M_m$, $z_i z_j^* = \sum_{k,\ell=1}^s a_{ik} \bar{a}_{j\ell} u_k u_\ell^* \in M_n$, and so we have

$$
\begin{aligned}
\langle zz^*, \phi \rangle &= \sum_{i,j=1}^m \langle \phi(e_{ij}), z_i z_j^* \rangle \\
&= \sum_{i,j=1}^m \sum_{k,\ell=1}^s a_{ik} \bar{a}_{j\ell} \langle \phi(e_{ij}), u_k u_\ell^* \rangle \\
&= \sum_{i,j=1}^m \sum_{k,\ell=1}^s a_{ik} \bar{a}_{j\ell} \langle \phi(e_{ij}) \bar{u}_\ell, \bar{u}_k \rangle_{\mathsf{C}^n} \\
&= \sum_{i,j=1}^m \sum_{k,\ell=1}^s a_{ik} \bar{a}_{j\ell} \langle (\phi(e_{ij}) \otimes e_{k\ell}) u, u \rangle_{\mathsf{C}^n \otimes \mathsf{C}^s},
\end{aligned}
$$

for a linear map $\phi : M_m \to M_n$. We also have $ww^* = \sum_{k,\ell=1}^s a_k a_\ell^* \otimes e_{k\ell} \in M_m \otimes M_s$, and

$$(\phi \otimes \mathrm{id}_s)(ww^*) = \sum_{k,\ell=1}^{s} \phi(a_k a_\ell^*) \otimes e_{k\ell} = \sum_{k,\ell=1}^{s} \sum_{i,j=1}^{m} a_{ik}\bar{a}_{j\ell}\phi(e_{ij}) \otimes e_{k\ell}.$$

Therefore, it follows that

$$\langle zz^*, \phi \rangle = \langle (\phi \otimes \mathrm{id}_s)(ww^*)u, u \rangle_{\mathsf{C}^n \otimes \mathsf{C}^s}, \tag{8}$$

where id_s denotes the identity map of M_s.

With the exactly same calculation as above, we also have

$$\langle (zz^*)^\tau, \phi \rangle = \langle (\phi \otimes \mathrm{tp}_s)(\bar{w}\bar{w}^*)u, u \rangle_{\mathsf{C}^n \otimes \mathsf{C}^s}, \tag{9}$$

for an s-simple vector $z \in \mathsf{C}^n \otimes \mathsf{C}^m$ and a linear map $\phi : M_m \to M_n$, where tp_s denotes the transpose map of M_s.

Theorem 3.1. *For a linear map $\phi : M_m \to M_n$, we have the following:*

(i) *The map ϕ is s-positive if and only if $\langle zz^*, \phi \rangle \geq 0$ for each s-simple vector $z \in \mathsf{C}^n \otimes \mathsf{C}^m$.*

(ii) *The map ϕ is s-copositive if and only if $\langle (zz^*)^\tau, \phi \rangle \geq 0$ for each s-simple vector $z \in \mathsf{C}^n \otimes \mathsf{C}^m$.*

Proof. Assume that ϕ is s-positive and take an s-simple vector $z = \sum_{i=1}^m z_i \otimes e_i \in \mathsf{C}^n \otimes \mathsf{C}^m$. Then the identity (8) shows that $\langle zz^*, \phi \rangle \geq 0$. For the converse, assume that $\langle zz^*, \phi \rangle \geq 0$ for each s-simple vector $z \in \mathsf{C}^n \otimes \mathsf{C}^m$. For each $w \in \mathsf{C}^m \otimes \mathsf{C}^s$ and $u \in \mathsf{C}^n \otimes \mathsf{C}^s$, we take $a_k \in \mathsf{C}^m$ and $z_i \in \mathsf{C}^n$ as in the relations (7). Then we see that $(\phi \otimes \mathrm{id}_s)(ww^*)$ is positive semidefinite by (8), and so $\phi \otimes \mathrm{id}_s$ is a positive linear map. The exactly same argument may be applied for the second statement if we use the identity (9).

For $s = 1, 2, \ldots, m \wedge n$, we define convex cones V_s and V^s in $M_n \otimes M_m$ by

$$\mathsf{V}_s(M_n \otimes M_m) = \{zz^* : z \text{ is an } s\text{-simple vector in } \mathsf{C}^n \otimes \mathsf{C}^m\}^{\circ\circ},$$
$$\mathsf{V}^s(M_n \otimes M_m) = \{(zz^*)^\tau : z \text{ is an } s\text{−simple vector in } \mathsf{C}^n \otimes \mathsf{C}^m\}^{\circ\circ}.$$

Then Theorem 3.1 and the identity (2) say that the following pairs

$$(\mathsf{V}_s, \mathsf{P}_s), \qquad (\mathsf{V}^t, \mathsf{P}^t), \qquad (\mathsf{V}_s \cap \mathsf{V}^t, \mathsf{P}_s + \mathsf{P}^t) \tag{10}$$

are dual each other, for $s, t = 1, 2, \ldots, m \wedge n$. We note that $\mathsf{V}_{m\wedge n}(M_n \otimes M_m)$ is nothing but the cone $(M_n \otimes M_m)^+$ of all positive semi-definite matrices in $M_n \otimes M_m$.

Corollary 3.2. *A linear map $\phi : M_m \to M_n$ is the sum of an s-positive linear map and a t-copositive linear map if and only if $\langle A, \phi \rangle \geq 0$ for each $A \in \mathsf{V}_s \cap \mathsf{V}^t$.*

Størmer [12] characterized the decomposable positive maps among linear maps from a C^*-algebra into $\mathscr{B}(\mathscr{H})$. For a linear map $\phi : M_m \to M_n$, this tells us that ϕ is the sum of a completely positive linear map and a completely copositive linear map if and only if the following

$$(\phi\otimes \mathrm{id}_p)(\mathsf{V}_p\cap \mathsf{V}^p(M_m\otimes M_p))\subset (M_n\otimes M_p)^+. \tag{11}$$

holds for $p=1,2,\dots$. In order to generalize this result for the sums of s-positive and t-copositive linear maps, we use block-wise Hadamard product. For two block matrices $X=\sum_{k,\ell=1}^p x_{k\ell}\otimes e_{k\ell}\in M_n\otimes M_p$ and $Y=\sum_{k,\ell=1}^p y_{k\ell}\otimes e_{k\ell}\in M_m\otimes M_p$, we define the *block-wise Hadamard product* by

$$X\odot Y=\sum_{k,\ell=1}^p x_{k\ell}\otimes y_{k\ell}\in M_n\otimes M_m.$$

Then for every linear map $\phi: M_m\to M_n$, we see that the following identity

$$\begin{aligned}\langle(\phi\otimes \mathrm{id}_p)(Y),X\rangle &=\sum_{k,\ell=1}^p\langle\phi(y_{k\ell}),x_{k\ell}\rangle\\ &=\sum_{i,j=1}^m\left\langle\phi(e_{ij}),\sum_{k,\ell=1}^p\langle y_{k\ell},e_{ij}\rangle x_{k\ell}\right\rangle\\ &=\left\langle\sum_{i,j=1}^m\sum_{k,\ell=1}^p\langle y_{k\ell},e_{ij}\rangle x_{k\ell}\otimes e_{ij},\phi\right\rangle\\ &=\langle X\odot Y,\phi\rangle\end{aligned}\tag{12}$$

holds, using the relation $y_{k\ell}=\sum_{i,j=1}^m\langle y_{k\ell},e_{ij}\rangle e_{ij}$. For $A\in M_n\otimes M_m$, we denote by $A^\sigma\in M_m\otimes M_n$ the *shuffle* of A, that is, $(x\otimes y)^\sigma=y\otimes x$. Then it is easy to see that

$$\begin{aligned}&A\in \mathsf{V}_s(M_n\otimes M_m)\Longleftrightarrow A^\sigma\in \mathsf{V}_s(M_m\otimes M_n),\\ &A\in \mathsf{V}^t(M_n\otimes M_m)\Longleftrightarrow A^\sigma\in \mathsf{V}^t(M_m\otimes M_n).\end{aligned}\tag{13}$$

Let $y=\sum_{k=1}^p y_k\otimes e_k\in \mathsf{C}^m\otimes\mathsf{C}^p$ be an s-simple vector with $y_k=\sum_{\alpha=1}^s b_{k\alpha}u_\in\mathsf{C}^m$ for $k=1,2,\dots,p$. Then we have

$$\begin{aligned}y_ky_\ell^*&=\sum_{\alpha,\beta=1}^s b_{k\alpha}\bar b_{\ell\beta}u_\alpha u_\beta^*\\ &=\sum_{\alpha,\beta=1}^s\sum_{i,j=1}^m b_{k\alpha}\bar b_{\ell\beta}\langle u_\alpha u_\beta^*,e_{ij}\rangle e_{ij}\\ &=\sum_{\alpha,\beta=1}^s\sum_{i,j=1}^m b_{k\alpha}\bar b_{\ell\beta}\langle u_\alpha,e_i\rangle\overline{\langle u_\beta,e_j\rangle}e_{ij}.\end{aligned}$$

For an arbitrary given $x = \sum_{k=1}^{p} x_k \otimes e_k \in \mathsf{C}^n \otimes \mathsf{C}^p$, put

$$\begin{aligned} z_\alpha &= \sum_{k=1}^{p} b_{k\alpha} x_k \in \mathsf{C}^n, \qquad \alpha = 1, 2, \ldots, s, \\ w_i &= \sum_{\alpha=1}^{s} \langle u_\alpha, e_i \rangle z_\alpha \in \mathsf{C}^n, \qquad i = 1, 2, \ldots, m, \\ w &= \sum_{i=1}^{m} w_i \otimes e_i \in \mathsf{C}^n \otimes \mathsf{C}^m. \end{aligned}$$

Then we have

$$\begin{aligned} xx^* \odot yy^* &= \sum_{k,\ell=1}^{p} x_k x_\ell^* \otimes y_k y_\ell^* \\ &= \sum_{k,\ell=1}^{p} \sum_{\alpha,\beta=1}^{s} \sum_{i,j=1}^{m} b_{k\alpha} \bar{b}_{\ell\beta} \langle u_\alpha, e_i \rangle \overline{\langle u_\beta, e_j \rangle} x_k x_\ell^* \otimes e_{ij} \\ &= \sum_{i,j=1}^{m} w_i w_j^* \otimes e_{ij} = ww^*, \end{aligned}$$

which belongs to $\mathsf{V}_s(M_n \otimes M_m)$, since w is an s-simple vector of $\mathsf{C}^n \otimes \mathsf{C}^m$. Therefore, we see that

$$X \in (M_n \otimes M_p)^+, \; Y \in \mathsf{V}_s(M_m \otimes M_p) \Longrightarrow X \odot Y \in \mathsf{V}_s(M_n \otimes M_m). \tag{14}$$

By the same argument, we also have

$$X \in (M_n \otimes M_p)^+, \; Y \in \mathsf{V}^t(M_m \otimes M_p) \Longrightarrow X \odot Y \in \mathsf{V}^t(M_n \otimes M_m). \tag{15}$$

THEOREM 3.3. *For a linear map $\phi : M_m \to M_n$, we have the following:*

(i) *ϕ is s-positive if and only if $(\phi \otimes \mathrm{id}_n)(\mathsf{V}_s(M_m \otimes M_n)) \subset (M_n \otimes M_n)^+$.*

(ii) *ϕ is t-copositive if and only if $(\phi \otimes \mathrm{id}_n)(\mathsf{V}^t(M_m \otimes M_n)) \subset (M_n \otimes M_n)^+$.*

(iii) *ϕ is the sum of an s-positive linear map and a t-copositive linear map if and only if $(\phi \otimes \mathrm{id}_n)(\mathsf{V}_s \cap \mathsf{V}^t(M_m \otimes M_n)) \subset (M_n \otimes M_n)^+$.*

PROOF. If ϕ is s-positive and $Y \in \mathsf{V}_s(M_m \otimes M_p)$ with $p = 1, 2, \ldots$, then we have

$$\langle (\phi \otimes \mathrm{id}_p)(Y), X \rangle = \langle X \odot Y, \phi \rangle \geq 0$$

for each $X \in (M_n \otimes M_p)^+$ by (14) and the duality between V_s and P_s. Therefore, $(\phi \otimes \mathrm{id}_p)(Y) \in (M_n \otimes M_p)^+$. For the converse, note that every $A \in M_n \otimes M_m$ is written by

$$A = A \odot J_m = J_n \odot A^\sigma,$$

where $J_r = \sum_{i,j=1}^{r} e_{ij} \otimes e_{ij} \in M_r \otimes M_r$ for $r = 1, 2, \ldots$. Therefore, for each $A \in \mathsf{V}_s(M_n \otimes M_m)$, we have

$$\langle A, \phi \rangle = \langle J_n \odot A^{\sigma}, \phi \rangle = \langle (\phi \otimes \mathrm{id}_n)(A^{\sigma}), J_n \rangle \geq 0$$

by (13). This proves (i). The exactly same argument also proves (ii) and (iii) if we use (15) and (13).

We note that the trace map $X \mapsto \mathrm{Tr}(X)I$ (respectively the identity matrix) is a typical interior point of the cones $\mathsf{P}_{m \wedge n}$ and $\mathsf{P}^{m \wedge n}$ (respectively V_1). It is also easy to see that these play the rôles of η in (5) for any pairs of dual cones in (10).

We also note that every face of $\mathsf{V}_{m \wedge n}$ is exposed with respect to the pairing. To see this, take a face F of $\mathsf{V}_{m \wedge n}$ and an interior point A of F. Then F consists of all positive semi-definite matrices whose range spaces are contained in the range space of A. If we take a positive semi-definite matrix B whose range space is orthogonal to that of A and a linear map $\phi : M_n \to M_m$ such that $\sum_{i,j=1}^{m} \phi(e_{ij}) \otimes e_{ij} = B$, then we see that ϕ is completely positive, and F is exposed by ϕ. In this way, we see that every face of $\mathsf{P}_{m \wedge n}$ corresponds to a face of $\mathsf{V}_{m \wedge n}$, which is determined by the range space of an interior point. Therefore, every face of $\mathsf{P}_{m \wedge n}$ corresponds to a subspace of $\mathsf{C}^n \otimes \mathsf{C}^m$, in the order-reversing way. For an order preserving lattice isomorphism between faces of $\mathsf{P}_{m \wedge n}$ and subspaces of $\mathsf{C}^n \otimes \mathsf{C}^m = M_{m,n}$, we refer to [8].

Since every extreme ray of V_s is an extreme ray of $\mathsf{V}_{m \wedge n}$ and P_s is larger than $\mathsf{P}_{m \wedge n}$, it follows that every extreme ray of V_s is exposed with respect to the pairing. The same argument holds for the pair $(\mathsf{V}^s, \mathsf{P}^s)$, because the block transpose map $A \mapsto A^{\tau}$ is linear. Therefore, we may apply Theorem 2.5 to get the following:

THEOREM 3.4. *Let* P_s (*respectively* P^s) *be the convex cone of all s-positive* (*respectively s-copositive*) *linear maps from* M_m *into* M_n. *For each s-simple vector* $z \in \mathsf{C}^n \otimes \mathsf{C}^m$, *the set*

$$\{\phi \in \mathsf{P}_s : \langle zz^*, \phi \rangle = 0\} \qquad (\text{respectively } \{\phi \in \mathsf{P}^s : \langle (zz^*)^{\tau}, \phi \rangle = 0\})$$

is a maximal face of P_s (*respectively* P^s). *Conversely, every maximal face of* P_s (*respectively* P^s) *arises in this form for an s-simple vector* $z \in \mathsf{C}^n \otimes \mathsf{C}^m$.

4. Examples

The first example of an indecomposable positive linear map between M_3 was given by Choi [1] by considering a positive semi-definite biquadratic form which is not the sum of the squares of bilinear forms. This example $\phi : M_3 \to M_3$ is defined by

$$\phi : \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} \end{pmatrix} + \mu \begin{pmatrix} x_{33} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & x_{22} \end{pmatrix},$$

where $\mu \geq 1$. Later, Størmer [12] showed that the above map is not decomposable by the condition (11). In order to apply Corollary 3.2, we modify the matrix in [12] to define

$$A = \begin{pmatrix} \alpha e_{11} + \alpha^2 e_{22} + e_{33} & \alpha e_{12} & \alpha e_{13} \\ \alpha e_{21} & e_{11} + \alpha e_{22} + \alpha^2 e_{33} & \alpha e_{23} \\ \alpha e_{31} & \alpha e_{32} & \alpha^2 e_{11} + e_{22} + \alpha e_{33} \end{pmatrix} \in M_3 \otimes M_3,$$

where α is a nonnegative real number. It is easy to see that A is positive semi-definite whenever $\alpha \geq 0$, and $A \in \mathsf{V}_3$. If we put

$$z_1 = \alpha e_2 + e_4, \quad z_2 = \alpha e_6 + e_8, \quad z_3 = \alpha e_7 + e_3$$

then we see that

$$A^\tau = \sum_{i=1}^{3} z_i z_i^* + \alpha(e_1 e_1^* + e_5 e_5^* + e_9 e_9^*),$$

and so $A \in \mathsf{V}^2$, whenever $\alpha \geq 0$. A direct calculation shows that

$$\langle A, \phi \rangle = 3\alpha(\mu\alpha - 1),$$

which is negative if $\alpha = 1/2\mu$ for example. Therefore, we see that the map ϕ is not the sum of a 3-positive linear map and a 2-copositive linear map. The authors were not able to determine whether the above matrix A belongs to V_2. If this is the case then we may conclude that ϕ is not the sum of a 2-positive linear map and a 2-copositive linear map. See [14] for the case of $\mu = 1$. Actually, we could not find an explicit example of $3^2 \times 3^2$ matrix which lies in $\mathsf{V}_2 \cap \mathsf{V}^2 \setminus \mathsf{V}_1$, although we know that this set is nonempty since there are examples of positive linear maps between M_3 which are not the sums of 2-positive linear maps and 2-copositive linear maps. See [5], [9] and [14]. The following proposition says that we must consider matrices whose ranks are at least two, in order to find examples in $\mathsf{V}_2 \cap \mathsf{V}^2 \setminus \mathsf{V}_1$.

Proposition 4.1. *Let* $x \in \mathsf{C}^n \otimes \mathsf{C}^m$. *Then the rank one matrix* $xx^* \in M_n \otimes M_m$ lies in $\mathsf{V}_{m\wedge n} \cap \mathsf{V}^{m\wedge n}$ if and only if it lies in V_1.

Proof. Put $x = \sum_{i=1}^m x_i \otimes e_i \in \mathsf{C}^n \otimes \mathsf{C}^m$. If $xx^* \in \mathsf{V}_1$ then x is a 1-simple vector, and so we may write $x = \sum_{i=1}^m \lambda_i y \otimes e_i \in \mathsf{C}^n \otimes \mathsf{C}^m$. If we put $\hat{x} = \sum_{i=1}^m \overline{\lambda_i} y \otimes e_i$ then we have $(xx^*)^\tau = \hat{x}\hat{x}^*$, and so $xx^* \in \mathsf{V}^{m\wedge n}$. For the converse, we may assume that $x_1 \neq 0$ without loss of generality. For $a = \sum_{i=1}^m a_i \otimes e_i \in \mathsf{C}^n \otimes \mathsf{C}^m$, note that

$$a^*(xx^*)^\tau a = \sum_{i,j=1}^m a_i^* x_j x_i^* a_j = \sum_{i,j=1}^m \langle x_i, a_j\rangle\langle x_j, a_i\rangle.$$

If we take $a = x_k \otimes e_1 - x_1 \otimes e_k$, for $k = 2, 3, \ldots, m$, then

$$a^*(xx^*)^\tau a = 2(|\langle x_1, x_k\rangle|^2 - \|x_1\|^2\|x_k\|^2),$$

which should be nonnegative. Therefore, we see that x_k is a scalar multiple of x_1 for each $k = 2, 3, \ldots, m$, and so x is a 1-simple vector.

Note that the map ϕ with $\mu = 1$ generates an extreme ray as was shown in [2]. We remark that this ray is not exposed. To see this, first note that if $\eta \otimes \xi \in \mathsf{C}^3 \otimes \mathsf{C}^3$ is a 1-simple vector then

$$\langle(\eta\otimes\xi)(\eta\otimes\xi)^*, \phi\rangle = \mathrm{Tr}[\phi(\xi\xi^*)(\eta\eta^*)^{\mathrm{t}}] = \langle\phi(\xi\xi^*)\bar\eta, \bar\eta\rangle.$$

So, if $\langle(\eta\otimes\xi)(\eta\otimes\xi)^*, \phi\rangle = 0$ then by a direct calculation we see that the pair (ξ, η) is one of the following:

$$(e_1, e_3), \quad (e_2, e_1), \quad (e_3, e_2), \quad (\xi_\alpha, \eta_\alpha), \tag{16}$$

where $\xi_\alpha = (e^{ia}, e^{ib}, e^{ic})$, $\eta_\alpha = (e^{-ia}, e^{-ib}, e^{-ic})$ and $\alpha = (a, b, c)$ runs through R^3. Therefore, if we denote by L the extreme ray generated by ϕ then we have

$$L' = \{xx^* \in \mathsf{V}_1(M_3 \otimes M_3) : x = e_1\otimes e_2,\ e_2\otimes e_3,\ e_3\otimes e_1,\ \eta_\alpha\otimes\xi_\alpha\ (\alpha\in\mathsf{R}^3)\}^{\circ\circ}.$$

By the arguments in Section 5 of [6], we see that $L \subsetneq L''$.

Added in proof. It was shown in the paper [16] by Kil-Chan Ha that the map ϕ in section 4 is not the sum of a 2-positive map and a 2-copositive map.

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