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INVARIANT FUNDAMENTAL SOLUTIONS AND SOLVABILITY FOR $\mathrm{GL}(n, \mathbf{C})/U(p, q)$

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0. Introduction

Let G/H be a reductive symmetric space and let $D : C^\infty(G/H) \to C^\infty(G/H)$ be a non-trivial G-invariant differential operator. An invariant fundamental solution for D is a left-H-invariant distribution E on G/H solving the differential equation:

$$DE = \delta,$$

where δ is the Dirac measure at the origin of G/H.

Consider now the reductive symmetric space $G/H = \mathrm{GL}(n, \mathbf{C})/U(p, q)$. Let $\mathfrak{a}_q$ be a fundamental Cartan subspace for G/H (the 'most compact' Cartan subspace) and let A_q be the associated Cartan subset of G/H, identified with a real abelian subgroup of G. For every non-trivial G-invariant differential operator D we let $\gamma_q(D)$ be the differential operator with constant coefficients on A_q defined via the Harish-Chandra isomorphism. We use the Plancherel formula for $\mathrm{GL}(n, \mathbf{C})/U(p, q)$, obtained by Bopp and Harinck in [4], to construct invariant fundamental solutions for G-invariant differential operators D on G/H for which the differential operator $\gamma_q(D)$ has a fundamental solution, i.e. a distribution T_q on A_q solving the differential equation:

$$\gamma_q(D)T_q = \delta_q,$$

where δ_q is the Dirac measure at the origin of A_q.

This result is similar to the results obtained by Benabdallah and Rouvière for semisimple Lie groups, see [2, Théorème 1]. Their and our approach can be seen as a generalization of the method used by Hörmander to find fundamental solutions for non-zero differential operators with constant coefficients on $\mathbf{R}^n$, see [7, p.189f].

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We remark, since G/H is a split symmetric space, that the existence of an invariant fundamental solution for a G-invariant differential operator D on G/H implies solvability of D, in the sense that $DC^\infty(G/H) = C^\infty(G/H)$, see e.g. [1, p. 301f].

1. Structure of $\mathsf{X} = \mathrm{GL}(n, \mathsf{C})/U(p, q)$

Most of the contents of this section (and some of the next) are taken from [3] and [4]. Note though, that our notation may be different.

Let p and q be two integers such that $0 \le q \le p$ and let $n = p + q$. Let J be the diagonal matrix in $M_n(\mathsf{C})$ defined by:

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

where I_p (I_q) is the identity element of $M_p(\mathsf{C})$ $(M_q(\mathsf{C}))$. Let $G = \mathrm{GL}(n, \mathsf{C})$. Define an involution σ_G of G by:

$$\sigma_G(g) = J(g^*)^{-1}J, \; g \in G,$$

where g^* denotes the conjugated transpose of g. The classical Cartan involution is given by: $\theta(g) = (g^*)^{-1}$, and we observe that the two involutions commute. Let $H = U(p, q)$, respectively $K = U(n)$, be the subgroup of fixed elements of σ_G, respectively of θ. Then G/H is a reductive symmetric space of type $G_{\mathsf{C}}/G_{\mathsf{R}}$ (i.e. G complex and H a real form of G).

Define a map φ of G into G by:

$$\varphi(g) = g\sigma_G(g)^{-1} = gJg^*J, \; g \in G.$$

We deduce, since H is the subgroup of fixed elements of σ_G, that φ induces an injection, also denoted φ, from G/H into G. The image of φ, denoted by X, is a closed submanifold of G, see [8, p.402], and φ is seen to be a G-isomorphism from G/H onto X, equipped with the G-action: $g \cdot x = gx\sigma_G(g)^{-1}$, $x \in \mathsf{X}$, $g \in G$. We will in the following use this realization of G/H. We note that the action of H on X is given by the adjoint action of H on $\mathsf{X} \subset G$, since: $h \cdot x = hxh^{-1}$, $x \in \mathsf{X}$, $h \in H$.

Let $\mathfrak{g} = M_n(\mathsf{C})$ denote the Lie algebra of G and let $\sigma_{\mathfrak{g}}$ denote the involution on $\mathfrak{g}$ given by the differential of σ_G, i.e.:

$$\sigma_{\mathfrak{g}}(X) = -JX^*J, \; X \in \mathfrak{g}.$$

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into the ± 1-eigenspaces of $\sigma_{\mathfrak{g}}$, where:

$$\mathfrak{h} = \{X \in M_n(\mathsf{C}) | X = \sigma_{\mathfrak{g}}(X)\} = \left\{ \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \middle| \begin{array}{l} A \in M_p(\mathsf{C}) \text{ and } A^* = -A \\ B \in M_{p,q}(\mathsf{C}) \\ C \in M_q(\mathsf{C}) \text{ and } C^* = -C \end{array} \right\},$$

is the Lie algebra of H, and:

$$\mathfrak{q} = \{X \in M_n(\mathsf{C}) | X = -\sigma_{\mathfrak{g}}(X)\} = i\mathfrak{h}.$$

Let $x \in \mathsf{X}$, then $x = g\sigma_G(g)^{-1} = gJg^*J$ for some $g \in G$, whence $\sigma_{\mathfrak{g}}(x) = -J(gJg^*J)^*J = -gJg^*J = -x$, and we see that $\mathsf{X} \subset \mathfrak{q}$. We conclude by dimension considerations, that X is an open subset of $\mathfrak{q}$, and that we can consider $\mathsf{X} \subset \mathfrak{q}$ as an open submanifold of $\mathfrak{q}$, equipped with the inherited differential structure.

The classical Cartan involution θ on $\mathfrak{g}$ is given by: $\theta(X) = -X^*$, $X \in \mathfrak{g}$. The Cartan decomposition of $\mathfrak{g}$ into the ± 1-eigenspaces of θ is given by: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \{X \in \mathfrak{g} | \theta(X) = X\}$ is the Lie algebra of K, and $\mathfrak{p} = \{X \in \mathfrak{g} | \theta(X) = -X\}$.

Let exp denote the (matrix-)exponential map of $\mathfrak{g} = M_n(\mathsf{C})$ into $G = \mathrm{GL}(n, \mathsf{C})$.

Cartan subalgebras and Cartan subspaces

A Cartan subspace $\mathfrak{a}$ for X is defined (cf. [8, §1]) as a maximal abelian subspace of $\mathfrak{q}$ consisting of semisimple elements. We see, since $\mathfrak{h}$ is a real form of $\mathfrak{g}$, that $\mathfrak{a}$ is a Cartan subspace for X if and only if $i\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{h}$. The Cartan subset A of X associated to a Cartan subspace $\mathfrak{a}$ for X, is defined (cf. [8, §1]) as the centralizer of $\mathfrak{a}$ in X, $A = Z_{\mathsf{X}}(\mathfrak{a})$, under the adjoint action of X considered as a subset of G.

There exist $q + 1$ H-conjugacy classes of Cartan subspaces for X. A family of θ-stable representations hereof, $\{\mathfrak{a}_k\}_{k=0,\dots,q}$, is given by:

$$\mathfrak{a}_k = \left\{ H(t,u,\theta) = \begin{pmatrix} u_1 & & & & & & & \theta_1 \\ & \cdot & & & & & \cdot & \\ & & u_k & & & \theta_k & & \\ & & & t_1 & & & & \\ & & & & \cdot & & & \\ & & & & t_{n-2k} & & & \\ & & -\theta_k & & & u_k & & \\ & \cdot & & & & & \cdot & \\ -\theta_1 & & & & & & & u_1 \end{pmatrix} \right\},$$

where $t = (t_1, \dots, t_{n-2k}) \in \mathsf{R}^{n-2k}$, $u = (u_1, \dots, u_k) \in \mathsf{R}^k$ and $\theta = (\theta_1, \dots, \theta_k) \in \mathsf{R}^k$. We note that $\det H(t,u,\theta) = \prod_{j=1}^{n-2k} t_j^2 \prod_{l=1}^{k} (u_l^2 + \theta_l^2) \geq 0$, for $H(t,u,\theta) \in \mathfrak{a}_k$.

Remarks. We see that the (maximal split) Cartan subspace $\mathfrak{a}_0$ is contained in $\mathfrak{p} \cap \mathfrak{q}$, so $\mathsf{X} = G/H$ is a split symmetric space. The intersection $\mathfrak{a}_q \cap \mathfrak{k}$ is a maximal abelian subspace of $\mathfrak{k} \cap \mathfrak{q}$, hence $\mathfrak{a}_q$ is by definition a fundamental Cartan subspace for $\mathsf{X} = G/H$ (the 'most compact' Cartan subspace).

The Cartan subsets $A_k = Z_{\mathsf{X}}(\mathfrak{a}_k)$, $k \in \{0, \ldots, q\}$, are, since $\mathsf{X} \subset \mathfrak{q}$, given by: $A_k = \mathsf{X} \cap \mathfrak{a}_k$ (let $a \in \mathsf{X}$, then: $a \in Z_{\mathsf{X}}(\mathfrak{a}_k) \Leftrightarrow Ad(a)X = aXa^{-1} = X$, $\forall X \in \mathfrak{a}_k$ $\Leftrightarrow aX = Xa$, $\forall X \in \mathfrak{a}_k \Leftrightarrow a \in \mathfrak{a}_k$ (by maximality of $\mathfrak{a}_k$)) and are thus open subsets of $\mathfrak{a}_k$. It is easily seen that A_k is a closed subgroup of G, hence a real abelian Lie subgroup of G with Lie algebra $\mathfrak{a}_k$. There are $\binom{p+q-2k}{q-k}$ connected components of A_k, see [4, p.51] for further details, with identity component given by $\exp \mathfrak{a}_k$.

We denote by $\Sigma_k = \Sigma(\mathfrak{g}, \mathfrak{a}_k)$ the root system of the pair $(\mathfrak{g}, \mathfrak{a}_{k,\mathsf{C}})$, where $\mathfrak{a}_{k,\mathsf{C}} = \mathfrak{a}_k + i\mathfrak{a}_k$. Let $H(t, u, \theta) \in \mathfrak{a}_k$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $H(t, u, \theta) \in M_n(\mathsf{C})$ ordered as below:

$$u_1 + i\theta_1, \ldots, u_k + i\theta_k, t_1, \ldots, t_{n-2k}, u_k - i\theta_k, \ldots, u_1 - i\theta_1.$$

The roots of Σ_k are given by the applications:

$$\Sigma_k = \{H(t, u, \theta) \mapsto \lambda_l - \lambda_j | l \neq j\}.$$

We define the positive roots, denoted by Σ_k^+, of Σ_k as:

$$\Sigma_k^+ = \{H(t, u, \theta) \mapsto \lambda_l - \lambda_j | l > j\}.$$

We say that a root $\alpha \in \Sigma_k$ is real, respectively imaginary or complex, if it is real-valued, respectively imaginary-valued, or neither real- nor imaginary-valued, on the Cartan subspace $\mathfrak{a}_k$. The set of real roots, positive real roots, imaginary roots, positive imaginary roots, complex roots and positive complex roots are denoted by $\Sigma_{k,\mathsf{R}}$, $\Sigma_{k,\mathsf{R}}^+$, $\Sigma_{k,I}$, $\Sigma_{k,I}^+$, $\Sigma_{k,\mathsf{C}}$ and $\Sigma_{k,\mathsf{C}}^+$ respectively. The positive real roots, $\Sigma_{k,\mathsf{R}}^+$, are given by the applications:

$$\Sigma_{k,\mathsf{R}}^+ = \{H(t, u, \theta) \mapsto t_l - t_j | 1 \leq j < l \leq n - 2k\}.$$

Let W_k denote the Weyl group associated to the root system Σ_k. We identify W_k with the permutation group $\mathfrak{S}_n$, acting on the n eigenvalues of elements in $\mathfrak{a}_k$. Let $\mathsf{D}(\mathsf{X})$ denote the algebra of G-invariant differential operators on X, let $S(\mathfrak{a}_k)$ be the symmetric algebra of the complexification of $\mathfrak{a}_k$ and let $I(\mathfrak{a}_k) = S(\mathfrak{a}_k)^{W_k}$ be the subalgebra of W_k-invariants hereof. The two algebras $\mathsf{D}(\mathsf{X})$ and $I(\mathfrak{a}_k)$ are isomorphic for all $k \in \{0, \ldots, q\}$, see [4, Théorème 2.1] for details. We let γ_k denote the isomorphism from $\mathsf{D}(\mathsf{X})$ onto $I(\mathfrak{a}_k)$ defined on [4, p.59].

The algebra $S(\mathfrak{a}_k)$ can be identified with the algebra of differential operators on the Lie group A_k with constant coefficients, by means of the action generated by:

$$Xf(a) = \frac{d}{dt} f(\exp tX \cdot a)_{|t=0},$$

for $X \in \mathfrak{a}_k$, where $f \in C^\infty(A_k)$ and $a \in A_k$.

We extend the Killing form B on $\mathfrak{s}l(n, \mathsf{C})$ to $\mathfrak{g}$ by: $B(X, Y) = 2nTrXY$, for $X, Y \in \mathfrak{g}$. This gives a canonical isomorphism between the algebra $\mathfrak{a}_{k,\mathsf{C}}$ and the complex dual $\mathfrak{a}^*_{k,\mathsf{C}}$ of $\mathfrak{a}_k$. For every root $\alpha \in \Sigma_k$, we let H_α be the element of $\mathfrak{a}_{k,\mathsf{C}}$ corresponding to the coroot $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Consider in particular the real root $\alpha \in \Sigma_{k,\mathsf{R}}$ given by the application $H(t, u, \theta) \mapsto t_l - t_j$. Then $H_\alpha = E_{k+l,k+l} - E_{k+j,k+j}$, where $E_{a,b} \in M_n(\mathsf{C})$ is the matrix with a 1 in the (a, b)'th entry and zeroes otherwise.

The Cartan decomposition of $\mathsf{X} = \mathrm{GL}(n, \mathsf{C})/U(p, q)$

Let $x \in \mathsf{X}$. The characteristic polynomial of the C-linear endomorphism $\mathrm{Ad}(x) - I$ on $\mathfrak{g} = \mathfrak{q}_{\mathsf{C}} = \mathfrak{h}_{\mathsf{C}}$ can be written as:

$$\det{}_{\mathsf{C}}((1 + z)I - \mathrm{Ad}(x)) \equiv z^n D_{\mathsf{X}}(x) \bmod z^{n+1},$$

for all $z \in \mathsf{C}$. The function D_{X} is an H-invariant analytic function on X. An element x in X is called regular (cf. [8, §1]) if $D_{\mathsf{X}}(x) \neq 0$, and the set of regular elements in any subset $U \subset \mathsf{X}$ will be denoted by U'.

Proposition 1.1 *Let $a \in A_k \subset \mathfrak{a}_k$, then:*

$$D_{\mathsf{X}}(a) = \prod_{\alpha \in \Sigma_k} \frac{\alpha(a)}{(\det a)^{n-1}}.$$

Proof. [4, p.55].

Put $H[U] = \bigcup_{h \in H} hUh^{-1}$ (the H-orbit of U) for any subset $U \subset \mathsf{X}$. We note that $Z_H(\mathfrak{a}_k) = Z_H(A_k)$ (and $N_H(\mathfrak{a}_k) = N_H(A_k)$), since $\exp \mathfrak{a}_k \subset A_k \subset \mathfrak{a}_k$ for all $k \in \{0, \ldots, q\}$. The quotients $N_H(\mathfrak{a}_k)/Z_H(\mathfrak{a}_k)$ and $N_H(A_k)/Z_H(A_k)$ are thus equal and finite. We also note that $Z_H(\mathfrak{a}_k) = Z_H(a)$ for $a \in A'_k$, since ia can be viewed as a ($\mathfrak{g}$-)regular element of the Cartan subalgebra $i\mathfrak{a}_k$ of $\mathfrak{h}$. The subgroup $Z_H(\mathfrak{a}_k) = Z_H(A_k)$ of H is in fact a Cartan subgroup of H.

Theorem 1.2 (The Cartan decomposition of $\mathsf{X} = \mathrm{GL}(n, \mathsf{C})/U(p, q)$). *The open and dense subset X' of regular elements of X is the disjoint union of the H-orbits of A'_k:*

$$\mathsf{X}' = \bigcup_{k=0}^{q} H[A'_k] = \bigcup_{k=0}^{q} \bigcup_{h \in H} hA'_k h^{-1}.$$

The map from $H/Z_H(A_k) \times A'_k$ into X defined by $(hZ_H(A_k), a) \mapsto hah^{-1}$ for $h \in H$ and $a \in A'_k$, is an everywhere regular $|N_H(A_k)/Z_H(A_k)|$-to-one map into X.

Proof. See [8, Theorem 2 (ii)].

It is well known, since $\mathsf{X} = G/H$ is a reductive symmetric space, that there exists, up to a constant, a unique G-invariant measure on X. Using the Cartan decomposition of X, we can express this measure by means of the invariant measures on A_k and $H/Z_H(\mathfrak{a}_k)$:

THEOREM 1.3. *There exist* $q+1$ *positive constants* C_k, *depending on the choice of the invariant measures* da *on* A_k *and* $d\dot{h}_k$ *on* $H/Z_H(\mathfrak{a}_k)$, $k \in \{0, \dots, q\}$, *such that:*

$$\int_{\mathsf{X}} f(x)dx = \sum_{k=0}^{q} C_k \int_{H/Z_H(\mathfrak{a}_k)} \int_{A_k} f(hah^{-1})|D_{\mathsf{X}}(a)|da d\dot{h}_k,$$

for all $f \in C_c(\mathsf{X})$.

PROOF. See [3, p.106-108] for details.

Orbital Integrals

Define a function $D_k(a)$ on A_k by:

$$D_k(a) = \frac{1}{(\det a)^{\frac{n-1}{2}}} \prod_{\alpha \in \Sigma_{k,I}^+} |\alpha(a)| \prod_{\alpha \in \Sigma_k^+ \setminus \Sigma_{k,I}^+} \alpha(a),$$

for $a \in A_k \subset \mathfrak{a}_k$. We note that $\det a > 0$ and that $D_k(a)^2 = |D_{\mathsf{X}}(a)|$, for $a \in A_k$.

DEFINITION 1.4. Let $k \in \{0, \dots, q\}$ and let $f \in C_c^\infty(\mathsf{X})$. The *orbital integral* K_f^k *of* f, *relative to the Cartan subspace* A_k, is the function defined on the regular elements $a \in A_k'$ by:

$$K_f^k(a) = D_k(a) \int_{H/Z_H(\mathfrak{a}_k)} f(hah^{-1}) d\dot{h}_k,$$

where $d\dot{h}_k$ is the invariant measure $H/Z_H(\mathfrak{a}_k)$ from above.

REMARKS. Let $k \in \{0, \dots, q\}$ and let $U \subset A_k'$ be a compact subset. Since D_{X} is an H-invariant continuous function, we conclude from regularity of the map $(hZ_H(A_k), a) \mapsto hah^{-1}$, $h \in H$, $a \in A_k'$, that the subset $H[U]$ is closed in X. We see in particular that the H-orbit $H[a]$ through any regular element $a \in A'$ is closed in X. So let $a \in A'$ and let $f \in C_c^\infty(\mathsf{X})$, then $\operatorname{supp} f \cap H[a] \subset \mathsf{X}$ is compact, and the above integral converges. We also easily see that $K_f \in C^\infty(A')$.

Let $U \subset \mathsf{X}$ and $V_k \subset A_k$, $k \in \{0, \dots, q\}$, be compact subspaces, and consider the Fréchet spaces: $C_U^\infty(\mathsf{X}) = \{f \in C_c^\infty(\mathsf{X}) |\ \operatorname{supp} f \subset U\}$ and:

$$C_{V_k}^\infty(A_k') = \left\{ F \in C^\infty(A_k') \;\middle|\; \begin{array}{c} \sup_{a \in V_k \cap A_k'} |XF(a)| < \infty, \ \forall X \in S(\mathfrak{a}_k) \quad \text{and} \\ F(a) \equiv 0 \quad \text{for} \quad a \in A_k' \setminus V_k \end{array} \right\}.$$

THEOREM 1.5. *Let $k \in \{0, \dots, q\}$ and let $U \subset \mathsf{X}$ be compact. There exists a compact subset $V_k \subset A_k$ such that $K_f^k(a) \equiv 0$ for $a \in A'_k \backslash V_k$ for all $f \in C_U^\infty(\mathsf{X})$; and the map: $f \mapsto K_f^k$ is a continuous map from $C_U^\infty(\mathsf{X})$ into $C_{V_k}^\infty(A'_k)$.*

PROOF. We can, since X is an open subset of $\mathfrak{q} = i\mathfrak{h}$, define a continuous injection $C_U^\infty(\mathsf{X}) \ni f \mapsto g \in C_c^\infty(\mathfrak{h})$, where the latter space is equipped with the Schwartz space topology, by:

$$g(X) = \begin{cases} f(-iX) & \text{if} \quad -iX \in \mathsf{X} \\ 0 & \text{otherwise} \end{cases},$$

for $X \in \mathfrak{h}$. We observe again that the algebra $i\mathfrak{a}_k$ is a Cartan algebra of $\mathfrak{h}$, and we identify the root systems of the pairs $(\mathfrak{h}_\mathsf{C}, (i\mathfrak{a}_k)_\mathsf{C})$ and $(\mathfrak{g}, \mathfrak{a}_{k,\mathsf{C}})$, also identifying the positive roots. Let $X \in \mathfrak{g}$. The characteristic polynomial of the C-linear endomorphism $\mathrm{ad}(X)$ on $\mathfrak{g} = \mathfrak{h}_\mathsf{C} = \mathfrak{q}_\mathsf{C}$ can be written as:

$$\det{}_\mathsf{C}(zI - \mathrm{ad}(X)) \equiv z^n D_\mathfrak{g}(X) \bmod z^{n+1},$$

for all $z \in \mathsf{C}$. An element X in $\mathfrak{g}$ is called $\mathfrak{g}$-regular if $D_\mathfrak{g}(X) \neq 0$, and the set of $\mathfrak{g}$-regular elements in any subset $u \subset \mathfrak{g}$ is denoted $u^{\mathfrak{g}-\mathrm{reg}}$. Let in particular $X \in \mathfrak{a}_{k,\mathsf{C}}$, then $D_\mathfrak{g}(X) = \prod_{\alpha \in \Sigma_k} \alpha(X)$, see [9, p.9], so $A'_k = A_k^{\mathfrak{g}-\mathrm{reg}} \subset \mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}}$.

The orbital integral Ψ_g^k of g, relative to the Cartan subalgebra $i\mathfrak{a}_k$ of $\mathfrak{h}$, is the function defined on the regular elements $X \in i\mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}}$ by:

$$\Psi_g^k(X) = d_k(X) \int_{H/Z_H(\mathfrak{a}_k)} g(\mathrm{Ad}(h)X) d\dot{h}_k,$$

where $d_k(X) = \mathrm{sign}(\prod_{\alpha \in \Sigma_{k,I}^+} \alpha(X)) \prod_{\alpha \in \Sigma_k^+} \alpha(X)$, see [9, p.35] for details. We thus see that:

$$K_f^k(a) = \frac{(-i)^{\left|\Sigma_k^+ \backslash \Sigma_{k,I}^+\right|}}{(\det a)^{\frac{n-1}{2}}} \Psi_g^k(ia),$$

for $a \in A'_k \subset \mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}}$.

Let $\mathscr{S}(i\mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}})$ be the Schwartz space on $i\mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}}$ (we can regard $\mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}}$ as an open subspace of $\mathfrak{a}_k \cong \mathsf{R}^n$). The map: $f \mapsto g \mapsto \Psi_g^k$, $C_U^\infty(\mathsf{X}) \to \mathscr{S}(i\mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}})$, is, by [9, Lemma I.3.6] and the remarks made on [9, p.40], continuous, and there exists a compact subset $W_k \subset \mathfrak{a}_k$, depending only on U, such that Ψ_g^k is identically zero on $i\mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}} \backslash iW_k$. We observe, since A'_k is an open subset of $\mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}}$, that $C_{V_k}^\infty(A'_k)$ is naturally embedded in $\mathscr{S}(i\mathfrak{a}_k^{\mathfrak{g}-\mathrm{reg}})$, so letting $V_k = W_k \cap A_k$ gives the result.

Using the notion of orbital integrals, we can now rewrite the integration formula introduced before. Let Φ be any locally integrable H-invariant function on X and let $f \in C_c^\infty(\mathsf{X})$, then we get by Fubini's Theorem:

$$\int_{\mathsf{X}} \Phi(x) f(x) dx = \sum_{k=0}^{q} C_k \int_{A'_k} K_f^k(a) D_k(a) \Phi(a) da. \tag{1}$$

2. Spherical distributions

Denote the the space of distributions on X, i.e. the continuous functionals on $C_c^\infty(\mathsf{X})$, by $D'(\mathsf{X})$. We note that a functional on $C_c^\infty(\mathsf{X})$ is in $D'(\mathsf{X})$ if and only if it is continuous on $C_U^\infty(\mathsf{X})$ for all compact subsets $U \subset \mathsf{X}$. The group G acts naturally on $D'(\mathsf{X})$ via the contragredient representation, and we denote the space of H-invariant distributions under this action by $D'(\mathsf{X})^H$.

Definition 2.1 An H-invariant distribution T on X is called a spherical distribution if and only if there exists a character χ of $\mathsf{D}(\mathsf{X})$ such that $DT = \chi(D)T$ for all $D \in \mathsf{D}(\mathsf{X})$.

The spherical distributions on X are characterized in [4, §2.3], they are in particular determined by locally integrable functions Φ on X, whose restrictions to X' are H-invariant analytic functions, [4, Théorème 2.8] (and satisfying some other conditions). The Dirac measure, $\delta \in D'(\mathsf{X})^H$, at the origin I of X can be decomposed as a direct integral of certain spherical distributions on X (The Plancherel formula for X, see Theorem 2.4), which will be constructed below.

Define a function $\widetilde{D}_k(a)$ on A_k as:

$$\widetilde{D}_k(a) = \frac{1}{(\det a)^{\frac{n-1}{2}}} \prod_{\alpha \in \Sigma_k^+} \alpha(a),$$

for $a \in A_k \subset \mathfrak{a}_k$. We note that $\widetilde{D}_k(a)^2 = (-1)^{|\Sigma_k^+|} D_{\mathsf{X}}(a)$, for $a \in A_k$.

Fix $k \in \{0, \ldots, q\}$. We define for all $(\mu, c, m) \in \mathsf{C}^{n-2k} \times \mathsf{C}^k \times \mathsf{Z}^k$ such that $\mu_l - \mu_j \notin \frac{i}{2}\mathsf{Z}$ for $1 \le j < l \le n-2k$, an H-invariant function $\phi^k(\mu, c, m)$ on X' by:

$$\phi^k(\mu, c, m)(a) = 0,$$

if $a \in A'_r$, $r < k$ or if $a \in A'_r \setminus \exp \mathfrak{a}_r$, $r \ge k$; and otherwise by:

$$\begin{aligned} \phi^k(\mu, c, m)(a) = {} & \frac{c_{k,r} \prod_{j=1}^k \ \mathrm{sign}\ \theta_j}{\widetilde{D}_r(a) \prod_{1 \le j < l \le n-2k} i(\mu_l - \mu_j)} \\ & \times \sum_{\sigma \in \mathfrak{S}_{n-2k}} \sum_{\tau \in \mathfrak{S}_r} \varepsilon(\sigma) \prod_{j=1}^{k} e^{ic_j u_{\tau(j)}} 2\cos(m_j \theta_{\tau(j)}) \prod_{j=1}^{r-k} e^{i(\mu_{\sigma(j)} + \mu_{\sigma(n+1-j-2k)}) u_{\tau(j+k)}} \\ & \times \prod_{j=r-k+1}^{n-k-r} e^{i\mu_{\sigma(j)} t_{j+k-r}} \prod_{j=1}^{r-k} \frac{\cosh((\mu_{\sigma(j)} - \mu_{\sigma(n+1-j-2k)})(|\theta_{\tau(j+k)}| - \pi))}{\sinh((\mu_{\sigma(j)} - \mu_{\sigma(n+1-j-2k)})\pi)}, \end{aligned} \tag{2}$$

for $a = \exp H(t,u,\theta) \in A'_r$, $r \geq k$, with $|\theta_j| < \pi$ for $j \in \{1,\ldots,r\}$, where $c_{k,r}$ is a constant given by: $c_{k,r} = \frac{i^k(-1)^{pq}}{(r-k)!}$.

Define for all $(\mu, c, m) \in \mathsf{C}^{n-2k} \times \mathsf{C}^k \times \mathsf{C}^k$ an element $Y_k = Y_k(\mu, c, m) \in \mathfrak{a}^*_{k,\mathsf{C}}$ by:

$$\langle Y_k, H(t,u,\theta)\rangle = \sum_{j=1}^{n-2k} \mu_j t_j + \sum_{j=1}^{k}(c_j u_j + m_j\theta_j),$$

for $H(t,u,\theta) \in \mathfrak{a}_k$. This defines an isomorphism between $\mathsf{C}^{n-2k} \times \mathsf{C}^k \times \mathsf{C}^k$ and $\mathfrak{a}^*_{k,\mathsf{C}}$, with which we will often identify the two spaces. The norm of Y_k is defined as the Euclidean norm of (μ, c, m):

$$|Y_k|^2 = \sum_{j=1}^{n-2k} |\mu_j|^2 + \sum_{j=1}^{k}\big(|c_j|^2 + |m_j|^2\big).$$

THEOREM 2.2. *Let $k \in \{0,\ldots,q\}$ and let $(\mu, c, m) \in \mathsf{R}^{n-2k} \times \mathsf{R}^k \times \mathsf{Z}^k$ such that $\mu_l \neq \mu_j$ for $1 \leq j < l \leq n-2k$. The function $\phi^k(\mu,c,m)$ defines a spherical distribution with character given by:*

$$D\phi^k(\mu,c,m) = \gamma_k(D)(iY_k)\phi^k(\mu,c,m),$$

for all $D \in \mathsf{D}(\mathsf{X})$.

PROOF. The function $\phi^k(\mu,c,m)$ is according to [4, p.86] the local expression for the spherical distribution on X defined in [4, Définition 4.6], satisfying the above.

Let $\varepsilon > 0$ and define the open tube $\Omega^k_\varepsilon \subset \mathsf{C}^{n-2k} \times \mathsf{C}^k \times \mathsf{Z}^k$ as: $\Omega^k_\varepsilon = \mathsf{R}^{n-2k}_\varepsilon \times \mathsf{R}^k_\varepsilon \times \mathsf{Z}^k$, where: $\mathsf{R}_\varepsilon = \mathsf{R} + i\,]-\varepsilon, \varepsilon[\subset \mathsf{C}$. By a holomorphic function in Ω^k_ε, we mean a function that is holomorphic in the $n-k$ first variables for all $m \in \mathsf{Z}^k$.

Fix again $k \in \{0,\ldots,q\}$ and define for all $(\mu,c,m) \in \Omega^k_{1/4}$ an H-invariant function on X' by (normalizing):

$$\phi^k_o(Y_k) = \phi^k_o(\mu,c,m) = \prod_{\alpha\in\Sigma^+_{k,\mathsf{R}}} i\langle Y_k, H_\alpha\rangle \prod_{\alpha\in\Sigma^+_k} i\langle Y_k, H_{-\alpha}\rangle \phi^k(\mu,c,m). \tag{3}$$

We note that $\prod_{\alpha\in\Sigma_{k,\mathsf{R}}}\langle Y_k, H_\alpha\rangle = \prod_{j\neq l}(\mu_l - \mu_j)$, so all the poles of $\phi^k(\mu,c,m)$ in the open tube $\Omega^k_{1/4}$ are cancelled by the normalization factor. The function $\phi^k_o(\mu,c,m)$ obviously defines a spherical distribution for all $(\mu,c,m) \in \mathsf{R}^{n-2k} \times \mathsf{R}^k \times \mathsf{Z}^k$ such that $\mu_l \neq \mu_j$ for $1 \leq j < l \leq n-2k$, with the same character as $\phi^k(\mu,c,m)$.

THEOREM 2.3. *Let $0 < \varepsilon < \frac{1}{4}$, let $k \in \{0,\ldots,q\}$ and let $f \in C^\infty_c(\mathsf{X})$. The* *functions $\phi_o^k(\mu, c, m)$ define spherical distributions for all (μ, c, m) in the open tube Ω_ε^k, with characters given by:*

$$D\phi_o^k(\mu, c, m) = \gamma_k(D)(iY_k)\phi_o^k(\mu, c, m),$$

for all $D \in \mathsf{D}(\mathsf{X})$. The map: $(\mu, c, m) \mapsto \langle \phi_o^k(\mu, c, m), f\rangle$ is a rapidly decreasing holomorphic function in the open tube Ω_ε^k.

PROOF. Let $k \in \{0, \dots, q\}$, assume that $r \geq k$ and let $a = \exp H(t, u, \theta) \in A'_r$ with $|\theta_j| < \pi$ for $j \in \{1, \dots, r\}$. We have the inequality:

$$\begin{aligned} \left|\phi_o^k(Y_k)(a)D_r(a)\right| &\leq \frac{2^k}{(r-k)!} \sum_{\sigma \in \mathfrak{S}_{n-2k}} \sum_{\tau \in \mathfrak{S}_r} \left| \prod_{\alpha \in \Sigma_k^+} \langle Y_k, H_\alpha \rangle \right| \\ &\times \prod_{j=1}^{r-k} \frac{|\cosh((\mu_{\sigma(j)} - \mu_{\sigma(n+1-j-2k)})(|\theta_{\tau(j+k)}| - \pi))|}{|\sinh((\mu_{\sigma(j)} - \mu_{\sigma(n+1-j-2k)})\pi)|} \\ &\times \prod_{j=1}^{k} e^{|Imc_j u_{\tau(j)}|} \prod_{j=1}^{r-k} e^{|Im(\mu_{\sigma(j)} + \mu_{\sigma(n+1-j-2k)})u_{\tau(j+k)}|} \prod_{j=r-k+1}^{n-k-r} e^{|Im\mu_{\sigma(j)} t_{j+k-r}|}. \end{aligned}$$

Fix $\sigma \in \mathfrak{S}_{n-2k}$ and $\tau \in \mathfrak{S}_r$. The fractions:

$$\prod_{j=1}^{r-k} \frac{|\mu_{\sigma(j)} - \mu_{\sigma(n+1-j-2k)}||\cosh((\mu_{\sigma(j)} - \mu_{\sigma(n+1-j-2k)})(|\theta_{\tau(j+k)}| - \pi))|}{(1 + |Y_k|)^{r-k}|\sinh((\mu_{\sigma(j)} - \mu_{\sigma(n+1-j-2k)})\pi)|},$$

and:

$$\frac{\left|\prod_{\alpha \in \Sigma_k^+} \langle Y_k, H_\alpha \rangle\right|}{\prod_{j=1}^{r-k} |\mu_{\sigma(j)} - \mu_{\sigma(n+1-j-2k)}|(1 + |Y_k|)^{n^2 - r + k}},$$

are bounded for all $\mu \in \mathsf{R}_\varepsilon^{n-2k}$ (note that $\left|\prod_{\alpha \in \Sigma_k^+} \langle Y_k, H_\alpha \rangle\right| \leq C(1 + |Y_k|)^{\left|\Sigma_k^+\right|}$, for some constant $C > 0$, and $\left|\Sigma_k^+\right| = n(n-1)/2 < n^2$), so there exists a constant $C > 0$, not depending on the choice of $a \in A'_r$ made before, such that:

$$\begin{aligned} |\phi_o^k(Y_k)(a)D_r(a)| &\leq C(1 + |Y_k|)^{n^2} \prod_{j=1}^{k} e^{|Imc_j u_{\tau(j)}|} \\ &\times \prod_{j=1}^{r-k} e^{|Im(\mu_{\sigma(j)} + \mu_{\sigma(n+1-j-2k)})u_{\tau(j+k)}|} \prod_{j=r-k+1}^{n-k-r} e^{|Im\mu_{\sigma(j)} t_{j+k-r}|}, \end{aligned} \tag{4}$$

for all $(\mu, c, m) \in \Omega_\varepsilon^k$.

Fix $(\mu, c, m) \in \Omega_\varepsilon^k$. We see, from (2) and (3), that $\phi_o^k(\mu, c, m)_{|A'_r} \in C^\infty(A'_r)$ and that $\phi_o^k(\mu, c, m) \cdot D_{r|A'_r}$ can be extended to a continuous function on each of the connected components of A_r (identically zero on $A_r \backslash \exp \mathfrak{a}_r$). Let

$0 \leq f \in C_c^\infty(\mathsf{X})$, then the integration formula (1), the estimate (4) and Theorem 1.5 yields:

$$\int_{\mathsf{X}} |\phi_o^k(Y_k)(x)| f(x)dx = \sum_{r=0}^{q} C_r \int_{A'_r} \left|K_f^r(a)\right| \left|\phi_o^k(Y_k)(a)D_r(a)\right| da < \infty.$$

We conclude, that the H-invariant analytic function $\phi_o^k(\mu, c, m)$ on X' is locally integrable on X and thus belongs to $D'(\mathsf{X})^H$.

Now fix $a \in A'_r$ and consider the map $(\mu, c, m) \mapsto \phi_o^k(\mu, c, m)(a)$ on Ω_ε^k. We see, again from (2) and (3), that this defines a holomorphic function in the open tube Ω_ε^k. Cauchy's Theorem, Fubini's Theorem and the estimate (4) then shows that the map:

$$(\mu, c, m) \mapsto \langle \phi_o^k(Y_k), f \rangle = \sum_{r=0}^{q} C_r \int_{A'_r} K_f^r(a) \phi_o^k(Y_k)(a) D_r(a) da,$$

is holomorphic in Ω_ε^k for fixed $f \in C_c^\infty(\mathsf{X})$.

Let $D \in \mathsf{D}(\mathsf{X})$ and assume that $f \in C_U^\infty(\mathsf{X})$ for some compact subset $U \subset \mathsf{X}$, then Theorem 2.2 yields:

$$\begin{aligned}\gamma_k(D)(iY_k)\langle \phi_o^k(Y_k), f \rangle &= \langle D\phi_o^k(Y_k), f \rangle = \langle \phi_o^k(Y_k), {}^t Df \rangle \\ &= \sum_{r=0}^{q} C_r \int_{A'_r} K_{{}^t Df}^r(a) \phi_o^k(Y_k)(a) D_r(a) da,\end{aligned}$$

for all $(\mu, c, m) \in \mathsf{R}^{n-2k} \times \mathsf{R}^k \times \mathsf{Z}^k$ such that $\mu_l \neq \mu_j$ for $1 \leq j < l \leq n - 2k$. This equality extends by holomorphy to the open tube Ω_ε^k, and we thus conclude that $\phi_o^k(\mu, c, m)$ is a spherical distribution for all (μ, c, m) in the open tube Ω_ε^k, with the given character. This shows, together with the estimate (4) and Theorem 1.5, that there exists a constant $C > 0$, only depending on U, such that:

$$|\gamma_k(D)(iY_k)\langle \phi_o^k(Y_k), f \rangle| \leq C(1 + |Y_k|)^{n^2} \sum_{r=0}^{q} \left\| K_{{}^t Df}^r \right\|_\infty < \infty, \tag{5}$$

for all $(\mu, c, m) \in \Omega_\varepsilon^k$.

Claim. Let $k \in \{0, \dots, q\}$ and let $P_0, P_1, \dots, P_n$ be a generating set of homogeneous polynomials of $I(\mathfrak{a}_k)$, with $P_0 \equiv 1$. There exists a constant $C > 0$ such that:

$$|Y_k| \leq C \max_{j \in \{0, \dots, n\}} |P_j(iY_k)| \leq C \sum_{j=0}^{n} |P_j(iY_k)|,$$

for all $Y_k \in \mathfrak{a}_{k,\mathsf{C}}^*$.

Proof. The claim is obvious for $|Y_k| \leq 1$. The map $Y_k \mapsto \max_{j \in \{1, \dots, n\}} |P_j(iY_k)|$

is continuous, so there exists a constant $C>0$ such that $\max_{j\in\{1,\dots,n\}}|P_j(iY_k)| \geq 1/C$ for $|Y_k|=1$ (by compactness of the sphere, and since $P_j(Y_k)=0$, for all $j\in\{1,\dots,n\}$, implies $Y_k=0$, see e.g. [6,Chapter III, Corollary 3.5]). For $|Y_k|\geq 1$, we get:

$$\left|\frac{Y_k}{|Y_k|}\right| \leq C \max_{j\in\{1,\dots,n\}}\left|P_j\left(\frac{iY_k}{|Y_k|}\right)\right| \leq C \max_{j\in\{1,\dots,n\}}\frac{|P_j(iY_k)|}{|Y_k|},$$

since the degree of P_j, $j \geq 1$, is ≥ 1, thus proving the claim.

Combining this with (5) yields the following estimate: Let $U \subset \mathsf{X}$ be compact and let $N \in \mathsf{N} \cup \{0\}$. There exists a finite set of differential operators $\{D_j\}_j \subset \mathsf{D}(\mathsf{X})$, depending only on N, and a constant $C > 0$, only depending on U, such that:

$$(1 + |Y_k|)^N |\langle \phi_o^k(Y_k), f\rangle| \leq C \sum_{r=0}^{q} \sum_j \left\| K^r_{{}^tD_jf} \right\|_\infty < \infty, \tag{6}$$

for all $(\mu, c, m) \in \Omega_\varepsilon^k$ and all $f \in C_U^\infty(\mathsf{X})$, which by definition means that the function $(\mu, c, m) \mapsto \langle \phi_o^k(\mu, c, m), f\rangle$ is rapidly decreasing.

Remark. The proof of Theorem 2.3 actually shows that the functions $\phi_o^k(\mu, c, m)$ define spherical distributions for (μ, c, m) belonging to the complement of the set of hyperplanes: $\mu_l - \mu_j = i\frac{p}{2}$, $p \in \mathsf{Z}\backslash\{0\}$ for $1 \leq j < l \leq n-2k$. And hence also that the functions $\phi^k(\mu, c, m)$ define spherical distributions for (μ, c, m) belonging to the complement of the set of hyperplanes: $\mu_l - \mu_j = i\frac{p}{2}$, $p \in \mathsf{Z}$ for $1 \leq j < l \leq n-2k$.

We can now formulate the main theorem in [4]:

Theorem 2.4. (The Plancherel formula for $\mathsf{X} = \mathrm{GL}(n, \mathsf{C})/U(p, q)$) *There exists a constant* $C_{\mathrm{Pl}} > 0$ *such that:*

$$\langle \delta, f\rangle = f(I) = C_{\mathrm{Pl}} \sum_{k=0}^{q} \frac{1}{2^k k!} \sum_{m\in\mathsf{N}^k} \int_{c\in\mathsf{R}^k} \int_{\mu\in\mathsf{R}^{n-2k}} \langle \phi_o^k(\mu, c, m), f\rangle dc d\mu,$$

for all $f \in C_c^\infty(\mathsf{X})$.

Proof. The theorem is just a reformulation of [4, Théorème 5.6], since $\phi^k(\mu, c, m)$ and $\prod_{\alpha\in\Sigma_{k,\mathsf{R}}^+} i\langle Y_k, H_\alpha\rangle \prod_{\alpha\in\Sigma_k^+} i\langle Y_k, H_{-\alpha}\rangle = \left|\prod_{\alpha\in\Sigma_{k,\mathsf{R}}^+}\langle Y_k, H_\alpha\rangle \prod_{\alpha\in\Sigma_k^+} \langle Y_k, H_\alpha\rangle\right|$ are invariant under permutations, $\sigma \in \mathfrak{S}_{n-2k}$, of the μ-variable.

3. Invariant fundamental solutions

Fix $d \in \mathsf{N}$ and denote by $Pol(d)$ the complex vector space of polynomials of degree $\leq d$ in N variables, and by $Pol^o(d)$ the vector space with the origin removed (the zero polynomial). We define a norm in $Pol(d)$ by $Q \mapsto \widetilde{Q}(0)$, where:

$$\widetilde{Q}(\xi) = \left(\sum_{\alpha} \left| Q^{(\alpha)}(\xi) \right|^2 \right)^{\frac{1}{2}},$$

for $\xi \in \mathsf{C}^N$, with $Q(\xi) = \sum_{\alpha} Q^{(\alpha)}(\xi)$ written in the multi index notation. We have, using the equality:

$$\frac{1}{\alpha!} Q^{(\alpha)}(0) = \sum_{\beta} \frac{(\alpha+\beta)!}{\alpha!\beta!} (-\xi)^{\beta} \frac{1}{(\alpha+\beta)!} Q^{(\alpha+\beta)}(\xi),$$

for all $Q \in Pol(d)$ and for all $\xi \in \mathsf{C}^N$, the following important inequality:

$$\widetilde{Q}(0) \leq C(1+|\xi|)^d \widetilde{Q}(\xi), \tag{7}$$

where $C > 0$ is a constant depending only on the degree and the number of variables.

Lemma 3.1. *Let $N \in$. There exists for every closed ball $B \subset \mathsf{C}^N$ with center 0 a non-negative function $\Phi \in C^{\infty}\left(Pol^o(d) \times \mathsf{C}^N\right)$ such that:*

(i) $\Phi(Q, \xi)$ vanishes for $\xi \notin B$ for all $Q \in Pol^o(d)$.

(ii) Fix $Q \in Pol^o(d)$. Then:

$$\int_{\mathsf{C}^N} F(\xi)\Phi(Q, \xi)d\xi = F(0),$$

for all holomorphic functions F in B^o, the interior of B. Here $d\xi$ is the Lebesgue measure on C^N. In particular we get:

$$\int_{\mathsf{C}^N} \Phi(Q, \xi)d\xi = 1,$$

in the case $F \equiv 1$.

(iii) *There exists a constant $C > 0$ such that, for all $Q \in Pol^o(d)$ and for all $\xi \in \mathsf{C}^N$:*

$$\widetilde{Q}(0) \leq C|Q(\xi)|,$$

if $\Phi(Q, \xi) \neq 0$.

Proof. [7, Lemma 7.3.12].

Let $0 \neq D \in \mathsf{D}(\mathsf{X})$, let $0 < \varepsilon < \frac{1}{4}$ and fix $k \in \{0, \ldots, q\}$. We put, for notational purposes, $\phi_m^k(\mu, c) = \phi_o^k(\mu, c, m)$ and $\gamma_{k,m}(\mu, c) = \gamma_k(D)(iY_k)$. Then:

$$D\phi_m^k(\mu, c) = \gamma_{k,m}(\mu, c)\phi_m^k(\mu, c).$$

We will, for fixed m, consider $\gamma_{k,m}(\cdot, \cdot)$ as a polynomial in the $n-k$ variables

(μ, c), and furthermore, for fixed (μ, c), consider the (μ, c)-translated polynomial in $n-k$ variables defined by: $\gamma_{k,m}^{(\mu,c)}(\xi) = \gamma_{k,m}((\mu, c)+\xi)$, for $\xi \in \mathsf{C}^{n-k}$.

Let U be a compact subset of X and let $f \in C_U^\infty(\mathsf{X})$. Consider the application $f \mapsto \langle E_k, f\rangle$ formally defined by:

$$\langle E_k, f\rangle = \sum_{m\in\mathsf{N}^k}\int_{c\in\mathsf{R}^k}\int_{\mu\in\mathsf{R}^{n-2k}}\int_B \frac{\langle \phi_m^k((\mu, c)+\xi), f\rangle}{\gamma_{k,m}((\mu, c)+\xi)}\Phi\left(\gamma_{k,m}^{(\mu,c)}, \xi\right)dc\,d\mu\,d\xi,$$

where the auxiliary function $\Phi \in C^\infty\left(Pol^o(d) \times \mathsf{C}^{n-k}\right)$ is defined as in Lemma 3.1, with d the degree of $\gamma_k(D)$ and $B = \{\xi \in \mathsf{C}^{n-k} \,|\, |\xi| \le \varepsilon\}$. There exists according to Lemma 3.1 and the estimate (7), constants $C > 0$, not depending on (μ, c, m) or ξ, such that:

$$\begin{aligned}|\gamma_{k,m}((\mu, c)+\xi)|^{-1} &= \left|\gamma_{k,m}^{(\mu,c)}(\xi)\right|^{-1} \le C\widetilde{\gamma}_{k,m}^{(\mu,c)}(0)^{-1}\\ &= C\widetilde{\gamma}_{k,m}(\mu, c)^{-1} \le C(1+|(\mu, c)|)^d\widetilde{\gamma}_{k,m}(0)^{-1},\end{aligned}$$

if $\Phi\left(\gamma_{k,m}^{(\mu,c)}, \xi\right) \neq 0$. This yields the estimate:

$$\left|\frac{\Phi\left(\gamma_{k,m}^{(\mu,c)}, \xi\right)}{\gamma_{k,m}((\mu, c)+\xi)}\right| \le C(1+|(\mu, c)|)^d\frac{\Phi\left(\gamma_{k,m}^{(\mu,c)}, \xi\right)}{\widetilde{\gamma}_{k,m}(0)},$$

for some constant $C > 0$, for all $(\mu, c, m) \in \mathsf{R}^{n-2k} \times \mathsf{R}^k \times \mathsf{Z}^k$ and $\xi \in \mathsf{C}^{n-k}$.

Let $N \in \mathsf{N} \cup \{0\}$. The estimate (6), appearing in the proof of Theorem 2.3, provides a finite set of differential operators, $\{D_j\}_j$, depending only on N, and a constant $C > 0$, only depending on U, such that:

$$(1+|(\mu, c, m)|)^N|\langle \phi_m^k(\mu, c), f\rangle| \le C\sum_{r=0}^{q}\sum_j\left\|K_{{}^tD_jf}^r\right\|_\infty < \infty,$$

for all $(\mu, c, m) \in \Omega_\varepsilon^k$. The seminorm $\sum_{r=0}^q\sum_j\left\|K_{{}^tD_jf}^r\right\|_\infty$ is according to Theorem 1.5 a continuous seminorm on $C_U^\infty(\mathsf{X})$, which we will denote $\sigma_{U,N}(f)$.

Assume there exists an integer $M \ge d \ge 0$ and a constant $C > 0$ such that:

$$\widetilde{\gamma}_{k,m}(0)^{-1} \le C(1+|m|)^M, \tag{8}$$

for all $m \in \mathsf{N}^k$, all the above then yields, with N large ($\gg M$):

$$\begin{aligned}|\langle E_k, f\rangle| &\le C\sum_{m\in\mathsf{N}^k}\int_{c\in\mathsf{R}^k}\int_{\mu\in\mathsf{R}^{n-2k}}\frac{\sigma_{U,N}(f)}{(1-\varepsilon+|(\mu, c, m)|)^{N-M^2}}\int_B\Phi\left(\gamma_{k,m}^{(\mu,c)}, \xi\right)dc\,d\mu\,d\xi\\ &= C\sum_{m\in\mathsf{N}^k}\int_{c\in\mathsf{R}^k}\int_{\mu\in\mathsf{R}^{n-2k}}\frac{\sigma_{U,N}(f)}{(1-\varepsilon+|(\mu, c, m)|)^{N-M^2}}dc\,d\mu \le C\sigma_{U,N}(f),\end{aligned}$$

for some constants $C > 0$, since $(1-\varepsilon+|(\mu, c, m)|)^{M^2-N}$ is integrable. The application $f \mapsto \langle E_k, f\rangle$ thus defines a H-invariant distribution, $E_k \in D'(\mathsf{X})^H$. We furthermore see, using Lemma 3.1 again, that:

$$\begin{aligned}\langle DE_k, f\rangle &= \sum_{m\in\mathsf{N}^k}\int_{c\in\mathsf{R}^k}\int_{\mu\in\mathsf{R}^{n-2k}}\int_B \frac{\langle D\phi_m^k((\mu,c)+\xi), f\rangle}{\gamma_{k,m}((\mu,c)+\xi)}\Phi\Big(\gamma_{k,m}^{(\mu,c)},\xi\Big)dc\,d\mu\,d\xi\\ &= \sum_{m\in\mathsf{N}^k}\int_{c\in\mathsf{R}^k}\int_{\mu\in\mathsf{R}^{n-2k}}\int_B \langle \phi_m^k((\mu,c)+\xi), f\rangle\Phi\Big(\gamma_{k,m}^{(\mu,c)},\xi\Big)dc\,d\mu\,d\xi\\ &= \sum_{m\in\mathsf{N}^k}\int_{c\in\mathsf{R}^k}\int_{\mu\in\mathsf{R}^{n-2k}} \langle \phi_m^k(\mu,c), f\rangle dc\,d\mu.\end{aligned}$$

Assume now, that all the norms $\widetilde{\gamma}_{k,m}(0)$, $k \in \{0,\ldots,q\}$ satisfy (8) for some integer M and some constant $C > 0$, we then define the H-invariant distribution:

$$E = C_{\mathrm{Pl}}\sum_{k=0}^{q}\frac{1}{2^k k!}E_k,$$

which satisfy:

$$\begin{aligned}\langle DE, f\rangle &= C_{\mathrm{Pl}}\sum_{k=0}^{q}\frac{1}{2^k k!}\langle DE_k, f\rangle\\ &= C_{\mathrm{Pl}}\sum_{k=0}^{q}\frac{1}{2^k k!}\sum_{m\in\mathsf{N}^k}\int_{c\in\mathsf{R}^k}\int_{\mu\in\mathsf{R}^{n-2k}}\langle \phi_m^k(\mu,c), f\rangle dc\,d\mu\\ &= f(I) = \langle\delta, f\rangle,\end{aligned}$$

according to the Plancherel formula, i.e. $E \in D'(\mathsf{X})^H$ is an invariant fundamental solution for D.

THEOREM 3.2. *Let $D \in \mathsf{D}(\mathsf{X})$ and let $\gamma_{q,m}(\mu,c) = \gamma_q(D)(iY_q)$. Assume there exists an integer $M \geq 0$ and a constant $C > 0$ such that:*

$$\widetilde{\gamma}_{q,m}(0)^{-1} \leq C(1+|m|)^M,$$

for all $m \in \mathsf{N}^q$. Then D has an invariant fundamental solution on X and is solvable, i.e. $DC^\infty(\mathsf{X}) = C^\infty(\mathsf{X})$.

PROOF. Let ν_k be the isomorphism of $\mathfrak{a}_{k,\mathsf{C}}$ onto $\mathfrak{a}_{k+1,\mathsf{C}}$, $k \in \{0,\ldots,q-1\}$, given by:

$$\nu_k(H) = Ad(g_k)H,$$

for $H \in \mathfrak{a}_{k,\mathsf{C}}$, where

$$g_k = \frac{1}{\sqrt{2}}(I + i(E_{k+1,n-k} + E_{n-k,k+1})).$$

The isomorphism ν_k prolongs to an isomorphism, also denoted by ν_k, from $I(\mathfrak{a}_k)$ onto $I(\mathfrak{a}_{k+1})$ for $k \in \{0, \dots q-1\}$. It can be shown, since ν_k acting on $\mathfrak{a}_{k,\mathsf{C}}$ is the restriction of an inner automorphism of $\mathfrak{h}_{\mathsf{C}}$, that $\gamma_{k+1} = \nu_k \circ \gamma_k$, see [3, Lemme 7.5], and so it also follows that $\gamma_k(D) = \nu_{k-1} \circ \dots \circ \nu_0 \circ \gamma_0(D)$. Let $Y_0(k) \in \mathfrak{a}^*_{0,\mathsf{C}}$ be defined by:

$$\langle Y_0(k), H(t,0,0)\rangle = \sum_{j=1}^{k} \frac{c_j - im_j}{2} t_j + \sum_{j=1}^{n-2k} \mu_j t_{k+j} + \sum_{j=1}^{k} \frac{c_j + im_j}{2} t_{n-k+j},$$

where $H(t,0,0) \in \mathfrak{a}_0$ with $t \in \mathsf{R}^n$, and $(\mu, c, m) \in \mathsf{C}^{n-2k} \times \mathsf{C}^k \times \mathsf{Z}^k$, then:

$$\gamma_k(D)(iY_k) = \gamma_0(D)(iY_0(k)),$$

since $\langle Y_0(k), H(t,0,0)\rangle = \langle Y_0, \nu_{k-1} \circ \dots \circ \nu_0(H(t,0,0))\rangle$. There exists a constant $C > 0$, by the inequality (7), such that:

$$\widetilde{\gamma}_{k-1,(m_1,\dots,m_{k-1})}(0)^{-1} \leq C(1+|m_k|)^{\deg D} \widetilde{\gamma}_{k,(m_1,\dots,m_{k-1},m_k)}(0)^{-1},$$

for $k \in \{0, \dots, q\}$, so there exists an integer $M_1 \geq 0$ and a constant $C > 0$ such that the following estimate holds for all $k \in \{0, \dots, q\}$:

$$\widetilde{\gamma}_{k,m}(0)^{-1} \leq C(1+|m|)^{M_1},$$

for all $m \in \mathsf{N}^k$. We can thus define an invariant fundamental solution $E \in D'(\mathsf{X})^H$ by the construction before Theorem 3.2.

Since X is split, it follows from [1, Corollary 2] that X is D-convex for all non-zero G-invariant differential operators on X, and hence we conclude that D is solvable if it has an invariant fundamental solution on X, see [1, p.301f].

As mentioned before, we can consider the Cartan subsets A_k, $k \in \{0, \dots q\}$ as abelian Lie groups with Lie algebras $\mathfrak{a}_k$. Consider $X_k \in S(\mathfrak{a}_k)$ as a differential operator on A_k. A fundamental solution for X_k is a solution $T_k \in D'(A_k)$, the space of distributions on A_k, to the differential equation: $X_k T_k = \delta_k$, where δ_k denotes the Dirac measure on A_k at the origin I. We can then reformulate Theorem 3.2 as follows:

Theorem 3.3. *Let $D \in \mathsf{D}(\mathsf{X})$. Assume that $\gamma_q(D)$ has a fundamental solution on A_q. Then D has an invariant fundamental solution on* X *and is solvable.*

Proof. Let $D \in \mathsf{D}(\mathsf{X})$. It is easily seen, from e.g. [5, §7], that $\gamma_k(D)$ has a fundamental solution on A_k if and only if $\widetilde{\gamma}_{k,m}(0) \geq C(1+|m|)^{-M}$ for some integer $M \geq 0$ and some constant $C > 0$. We conclude from Theorem 3.2, that $D \in \mathsf{D}(\mathsf{X})$ has an invariant fundamental solution on X if $\gamma_q(D)$ has a fundamental solution on A_q.

4. Examples and further results

1) Let $D \in \mathsf{D}(\mathsf{X})$. Assume that $\gamma_q(D)$ has a fundamental solution on A_q. It follows from the proofs of Theorem 3.2 and Theorem 3.3, that the differential operators $\gamma_k(D)$ have fundamental solutions on A_k for all $k \in \{0, \dots, q\}$.
We define for $P_0 \in I(\mathfrak{a}_0)$ and $k \in \{0, \dots, q\}$ the polynomial: $P_{k,m}(\mu, c) = P_0(iY(k))$.

2) There exist polynomials $P_0 \in I(\mathfrak{a}_0)$ such that $\widetilde{P}_{q,m}(0)$ does not satisfy the condition in Theorem 3.2. Consider for example the invariant polynomial $I(\mathfrak{a}_0) \ni P_0 = \prod_{l>j}((X_l - X_j)^2 + a^2)$, with $a \in \mathsf{N}$ and $\{X_j\}_j$ the obvious basis of $\mathfrak{a}_0$. Then $P_{k,m}(\mu, c) = \prod_{j=1}^k((im_j)^2 + a^2)Q_k(\mu, c, m)$ for some polynomial Q_k, and $\widetilde{P}_{k,m}(0) = \widetilde{Q}(0)\prod_{j=1}^k |m_j^2 - a^2|$. There obviously exist integers $m \in \mathsf{N}^k$ such that $\widetilde{P}_{k,m}(0) = 0$, i.e. the differential operator $D \in \mathsf{D}(\mathsf{X})$ given by $\gamma_0^{-1}(P_0)$ does not satisfy the assumption in Theorem 3.2. We could also consider the polynomials $P_0 = \prod((X_{l_1} - X_{j_1})^2 - a(X_{l_2} - X_{j_2})^2 + b)$, where j_1, j_2, l_1, l_2 are different indices and a, b constants. It follows from [5, p.570] that $\widetilde{P}_{q,m}(0)^{-1}$, for certain values of a and b, grows faster than any exponent of $|m|$.

3) Consider for $d \in \mathsf{N}$ the polynomial $P_0^d = X_1^d + \ldots + X_n^d \in I(\mathfrak{a}_0)$. We note that the polynomials $\{P_0^d\}_{d=0,\dots,n}$, with $P_0^0 \equiv 1$, generate $I(\mathfrak{a}_0)$. Then:

$$
\begin{aligned}
P_{k,m}^d(\mu, c) &= P_0^d(iY(k)) \\
&= i^d\left(\sum_{j=1}^k\left(\frac{c_j - im_j}{2}\right)^d + \sum_{j=1}^k\left(\frac{c_j + im_j}{2}\right)^d + \sum_{j=1}^{n-2k}\mu_j^d\right) = \\
&= i^d\left(\left(\frac{1}{2}\right)^{d-1}\sum_{j=1}^k\sum_{e\,\text{even}}\binom{d}{e}c_j^{d-e}(im_j)^e + \sum_{j=1}^{n-2k}\mu_j^d\right),
\end{aligned}
$$

for $k \in \{0, \dots, q\}$. The norm of $P_{k,m}^d(\mu, c)$ is bounded away from zero for all m since $\widetilde{P}_{k,m}^d(0) \geq 2^{1-d}(n-k)^{1/2}d!$, i.e. the differential operators $D^d \in \mathsf{D}(\mathsf{X})$ given by $D^d = \gamma_0^{-1}(P_0^d)$ all have invariant fundamental solutions and are solvable.

4) Let $P_0, Q_0 \in I(\mathfrak{a}_0)$. The inequality $\widetilde{P_{k,m} \cdot Q_{k,m}}(0) \geq C\widetilde{P}_{k,m}(0) \cdot \widetilde{Q}_{k,m}(0)$, where C is a positive constant depending only on the degrees of the polynomials P_0 and Q_0, implies that if two differential operators $D_1, D_2 \in \mathsf{D}(\mathsf{X})$ satisfy the boundedness condition in Theorem 3.2, then so does the product $D_1 \cdot D_2$. It is also easily seen that the product of two solvable differential operators is a solvable differential operator.

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