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ON SOME DEFORMATIONS OF RIEMANN SURFACES. I

RYSZARD L. RUBINSZTEIN

Abstract

We define a family of infinitesimal deformations of compact Riemann surfaces of genus $g \geq 2$ which generalizes the Fenchel-Nielsen deformations. Those new deformations are associated to smooth vector fields on the circle. We compute a representation of the deformations in terms of Poincaré series and determine the corresponding Eichler cohomology classes.

Let R be a compact Riemann surface (a complex manifold of complex dimension 1) of genus $g \geq 2$. Let C be a simple closed geodesic on R (with respect to the hyperbolic metric).The Fenchel-Nielsen deformation of R is obtained by cutting R along the geodesic C, rotating one side of the cut by some angle α and then regluing both sides of the cut in their new position. When the angle α is allowed to converge to 0, one obtains the infinitesimal Fenchel-Nielsen deformation. This deformation has been extensively studied, see e.g. [5], [6].

In this paper we introduce a new family of infinitesimal deformations of R generalizing that of Fenchel-Nielsen.

Let $\mathfrak{X}$ be a smooth vector field on the circle S^1. Let C_0 and C_1 be a pair of geodesics on R which intersect in one point. Given those data, we construct an infinitesimal deformation $\varphi_{(C_0,C_1)}(\mathfrak{X})$ of R. The geometric meaning of the deformation is as follows: the vector field $\mathfrak{X}$ on S^1 generates a 1-parameter group of diffeomorphisms f_t of S^1 (the flow of $\mathfrak{X}$). Identify the geodesic C_0 with S^1 (the intersection point of C_0 with C_1 is identified with $1 \in S^1$). Cut the surface R along C_0, change the position of one side of the cut by the diffeomorphism f_t and reglue both sides of the cut in their new position. When t converges to 0 one obtains an infinitesimal deformation $\varphi_{(C_0,C_1)}(\mathfrak{X})$ of the surface R.

In the special case when the vector field $\mathfrak{X}$ on S^1 is the constant one, $\mathfrak{X} = \widehat{\dfrac{d}{dx}}$ (see Sec.1), the 1-parameter group of diffeomorphisms f_t is the group

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of rotations of the circle and our construction gives the infinitesimal Fenchel-Nielsen deformation based on the geodesic C_0.

The contents of the paper are as follows: in Section 2 we construct the infinitesimal deformation $\varphi_{(C_0,C_1)}(\mathfrak{X})$ and compute the Beltrami differential $\nu = \nu(\mathfrak{X})$ which represents it. In Section 3 we describe the deformation in terms of quadratic differentials in the lower half-plane H^*. This is done for the case when $\mathfrak{X}$ has a finite Fourier expansion. The quadratic differential is given by a Poincaré series. The main result is Theorem 3.7. In Section 4 we give a description of the Eichler cohomology class which corresponds to our deformation (again for $\mathfrak{X}$ with a finite Fourier expansion). Results of Sections 3 and 4 generalize some of the results of S. Wolpert, [5], for the Fenchel-Nielsen deformation. Finally, in Section 5 we point out that the infinitesimal deformation $\varphi_{(C_0,C_1)}(\mathfrak{X})$ defines a vector field $\Phi_{(C_0,C_1)}(\mathfrak{X})$ on the Teichmüller space $T(R)$ of R.

We construct our deformations in the context of quasiconformal mappings. For the background material on quasiconformal mappings and Teichmüller spaces we refer to [2].

1. Vector fields on S^1

Let S^1 be a circle. We look upon S^1 as the unit circle in the complex plane,

$$S^1 = \{z \in \mathsf{C} \mid |z| = 1 \}.$$

Let $\mathfrak{X}$ be a smooth tangent vector field on S^1. $\mathfrak{X}$ determines a 1-parameter group of diffeomorphisms of S^1,

$$f_t : S^1 \longrightarrow S^1, \qquad t \in \mathsf{R},$$

with $f_t \circ f_s = f_{t+s}$, $f_0 = \mathrm{id}_{S^1}$ and such that $\left.\frac{d}{dt} f_t(z)\right|_{t=0} = \mathfrak{X}(z)$ for $z \in S^1$.

Let $p : \mathsf{R} \longrightarrow S^1$, $p(x) = e^{2\pi i x}$. The map p is a universal covering of S^1. By the Covering Homotopy Property of p there exists a unique lifting of $\{f_t\}_{t\in\mathsf{R}}$ to a 1-parameter family of smooth maps

$$\tilde{f}_t : \mathsf{R} \longrightarrow \mathsf{R}$$

satisfying $p \circ \tilde{f}_t = f_t \circ p$ for $t \in \mathsf{R}$ and $\tilde{f}_0 = \mathrm{id}_{\mathsf{R}}$.

By a standard unique path lifting argument it follows then that $\tilde{f}_t \circ \tilde{f}_s = \tilde{f}_{t+s}$ for all $t, s \in \mathsf{R}$, hence the lifting $\tilde{f}_t : \mathsf{R} \longrightarrow \mathsf{R}$ is a 1-parameter group of diffeomorphisms of R.

Since $p : \mathsf{R} \longrightarrow S^1$ is a local diffeomorphism, there exists a unique tangent vector field $\tilde{\mathfrak{X}}$ on R such that $dp_x(\tilde{\mathfrak{X}}(x)) = \mathfrak{X}(p(x))$ for all $x \in \mathsf{R}$. It is clear that

$$\frac{d}{dt}(\tilde{f}_t(x))\bigg|_{t=0} = \tilde{\mathfrak{X}}(x), \qquad x \in \mathsf{R}.$$

Hence $\{\tilde{f}_t\}_{t\in\mathsf{R}}$ is the 1-parameter group of diffeomorphisms of R generated by the vector field $\tilde{\mathfrak{X}}$.

For every $t \in \mathsf{R}$ the map $f_t : S^1 \longrightarrow S^1$ is homotopic to identity, hence $deg(f_t) = 1$. It follows that

$$\tilde{f}_t(x+1) = \tilde{f}_t(x) + 1 \tag{1.1}$$

for all $t, x \in \mathsf{R}$.

Moreover, for every $t \geq 0$ there are real constants $\alpha_t', \alpha_t'' > 0$ such that

$$\alpha_t' \leq \left|\frac{d}{dx}\tilde{f}_s(x)\right| \leq \alpha_t'' \tag{1.2}$$

for all $x \in \mathsf{R}$ and all $s \in \mathsf{R}$ with $|s| \leq t$. There is also a real constant $M > 0$ such that

$$\left|\frac{d}{dt}\tilde{f}_t(x)\right| \leq M \tag{1.3}$$

for all $x, t \in \mathsf{R}$.

$\frac{d}{dx}$ is a nowhere vanishing vector field on R. Via the map p it descends to a vector field on S^1 which we denote by $\widehat{\frac{d}{dx}}$. Hence, in our notation

$$\widetilde{\widehat{\frac{d}{dx}}} = \frac{d}{dx}.$$

Every smooth vector field $\mathfrak{X}$ on S^1 can now be written as $\mathfrak{X} = h\widehat{\frac{d}{dx}}$, with $h : S^1 \longrightarrow \mathsf{R}$ being a smooth function. Then $\tilde{\mathfrak{X}} = \tilde{h}\frac{d}{dx}$ with $\tilde{h} : \mathsf{R} \longrightarrow \mathsf{R}$, $\tilde{h} = h \circ p$. Note that $\tilde{h}(x+1) = \tilde{h}(x)$.

Remark 1.4. Let $\mathrm{Diff}_+(S^1)$ be the group of orientation preserving diffeomorphisms of S^1. Considered as a topological space (with a suitable topology, see [4]) $\mathrm{Diff}_+(S^1)$ is not simply-connected. Let $\mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R})$ be the space of all diffeomorphisms $g : \mathsf{R} \longrightarrow \mathsf{R}$ satisfying

$$g(x+1) = g(x) + 1 \qquad \text{for all } x \in \mathsf{R}.$$

$\mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R})$ is a group with respect to composition. There is a continuous map

$$\pi : \mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R}) \longrightarrow \mathrm{Diff}_+(S^1)$$

given by $\pi(g)(z) = p(g(x))$ for any $z \in S^1$ and $x \in p^{-1}(z)$. The map π is a group homomorphism and a covering map.

Moreover, as a topological space $\mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R})$ is contractible. Indeed, a contraction of $\mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R})$ to a point is given by

$$H : \mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R}) \times I \longrightarrow \mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R}),$$

where $H(g,s)(x) = (1-s)g(x) + sx$ for $x \in \mathsf{R}$, $s \in I = [0,1]$.
Hence $\pi : \mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R}) \longrightarrow \mathrm{Diff}_+(S^1)$ is a universal covering space of $\mathrm{Diff}_+(S^1)$. Its group of covering transformations is the additive group of integers Z acting on $\mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R})$ by $n(g)(x) = g(x) + n$ for $n \in \mathsf{Z}, x \in \mathsf{R}$.

If a 1-parameter group $\{f_t\}_{t\in\mathsf{R}}$ of diffeomorphisms of S^1 is given, we can look upon it as a curve in $\mathrm{Diff}_+(S^1)$. Then the 1-parameter group $\{\tilde{f}_t\}_{t\in\mathsf{R}}$ of diffeomorphisms of R constructed above is just the lifting of this curve to $\mathrm{Diff}_1^{\mathrm{per}}(\mathsf{R})$ with the starting point $(t = 0)$ at id_{R}.

2. Construction of a deformation

2.1. Let R be a compact Riemann surface of genus $g \geq 2$. By "Riemann surface" we mean a compact complex manifold of complex dimension 1.

By the Uniformization Theorem R can be described as a quotient of the complex upper half-plane H by a Fuchsian group Γ acting freely and properly discontinuously on H, $R = \mathsf{H}/\Gamma$. The hyperbolic Poincaré metric on H induces then a Riemannian metric on R.

Let C_0 and C_1 be two simple closed oriented geodesics on R.

Definition 2.1. The pair of geodesics (C_0, C_1) is called a 1-*pair* if C_0 and C_1 intersect in exactly one point.

Given a smooth vector field $\mathfrak{X}$ on S^1, a compact Riemann surface R of genus $g \geq 2$ and a 1-pair of geodesics (C_0, C_1) on R, we shall construct an infinitesimal deformation of R.

If the vector field $\mathfrak{X}$ is constant i.e. if $\mathfrak{X} = a\widehat{\dfrac{d}{dx}}$ for some constant $a \in \mathsf{R}$, then the resulting deformation does not depend on the choice of C_1 but only on the geodesic C_0 and it represents the infinitesimal Fenchel-Nielsen deformation of R along C_0 (as described in [5]) with the speed depending on a. In this sense our construction generalizes the Fenchel-Nielsen deformation.

2.2. Let (C_0, C_1) be a 1-pair of geodesics on the Riemann surface $R, R = \mathsf{H}/\Gamma$. There is an element $\gamma_0 \in \Gamma$ such that C_0 is the projection to R

of the axis of γ_0 in H. Conjugating Γ with a Möbius transformation if necessary, we can assume that 0 and ∞ are the repelling respectively the attracting fixed point of γ_0. It follows that

$$\gamma_0(z) = \lambda z, \qquad z \in \mathsf{H},$$

with λ being a real number > 1. The axis of γ_0 is the positive imaginary half-axis.

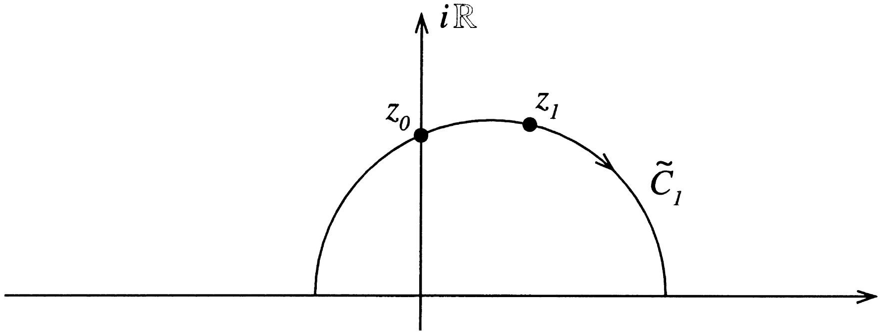

Figure 1.

Let $x_0 \in R$ be the intersection point of C_0 and C_1. Let C_1 be parametrized by its arc-length, $C_1 = C_1(t)$, in such a way that $x_0 = C_1(0) = C_1(q)$, where q is the length of C_1. Choose a point $z_0 = si \in \mathsf{H},\ s > 0$, lying on the axis of γ_0, which projects to x_0. Let $\tilde{C}_1 = \tilde{C}_1(t)$ be the lifting of C_1 to H with $\tilde{C}_1(0) = z_0$. $\tilde{C}_1$ is a geodesic in H. Let $z_1 = \tilde{C}_1(q) \in \mathsf{H}$. Then z_1 projects to x_0 in R and, hence, there is an element $\gamma_1 \in \Gamma$ such that $z_1 = \gamma_1(z_0)$. It follows that the geodesic $\tilde{C}_1$ is the axis of the hyperbolic Möbius transformation γ_1.

By cojugating Γ again, if necessary, with a Möbius transformation γ of the form $\gamma(z) = \mu z,\ \mu > 0$, we can assume that $s = 1$ i.e. that $z_0 = i \in \mathsf{H}$.

The only role the geodesic C_1 is playing in our construction is to distinguish a point on the geodesic C_0 (the point of intersection). This point allows us then to identify C_0 with the circle S^1.

This way to distinguish the point on C_0 depends only on the free homotopy classes of the curves in R represented by C_0 and C_1. Therefore, it allows the construction to be performed on the Teichmüller space $T(R)$ of R (see Section 5).

2.3. Let (C_0, C_1) be a 1-pair of geodesics on the Riemann surface $R,\ R = \mathsf{H}/\Gamma$. As explained above there are two elements $\gamma_0, \gamma_1 \in \Gamma$ with the axes $\tilde{C}_0$ and $\tilde{C}_1$ respectively, such that $\tilde{C}_i$ projects to $C_i,\ i = 0, 1$. Moreover we can assume that $\gamma_0(z) = \lambda z,\ z \in \mathsf{H}$, for some $\lambda > 1$ and that the intersection point of $\tilde{C}_0$ and $\tilde{C}_1$ is $z_0 = i \in \mathsf{H}$.

The length of the geodesic C_0 is equal to $l = \log \lambda$. By the Collar Theorem, [1; Thm 4.1.1, p. 94], there is a real number $\varepsilon = \varepsilon(l)$, $0 < \varepsilon < \pi/2$, depending only on l such that the sector

$$\tilde{W} = \left\{ z \in \mathsf{H} \;\middle|\; \frac{\pi}{2} - \varepsilon < \arg z < \frac{\pi}{2} + \varepsilon \right\}$$

of the upper half-plane H projects to a tubular neighbourhood of C_0 in R.

Choose a smooth non-decreasing function $s : [0, \pi] \longrightarrow \mathsf{R}$ such that $s(\theta) = 0$ for $\theta \leq \frac{\pi}{2} - \frac{\varepsilon}{2}$ and $s(\theta) = 1$ for $\theta \geq \frac{\pi}{2} - \frac{\varepsilon}{4}$.

Let $\mathfrak{X}$ be a smooth vector field on the circle S^1. $\mathfrak{X}$ generates a 1-parameter group $f_t : S^1 \longrightarrow S^1$, $t \in \mathsf{R}$, of diffeomorphisms of S^1. As explained in Section 1, this group lifts to a 1-parameter group $\tilde{f}_t : \mathsf{R} \longrightarrow \mathsf{R}$ of diffeomorphisms of R satisfying

$$\tilde{f}_t(x+1) = \tilde{f}_t(x) + 1 \qquad \text{for all } x, t \in \mathsf{R}. \tag{2.2}$$

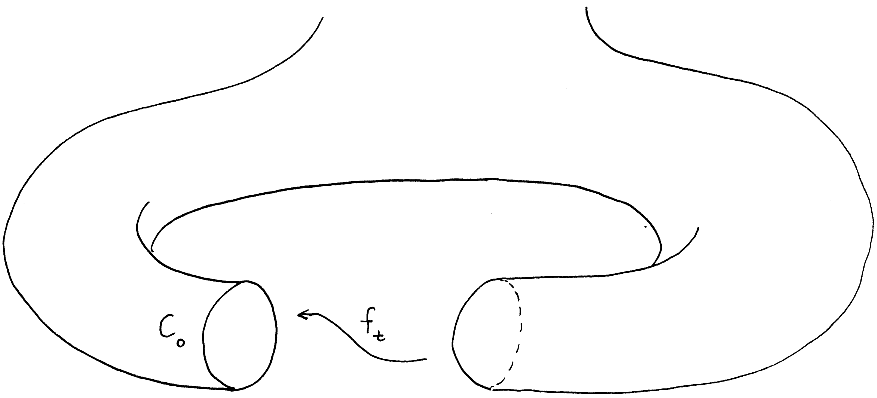

Figure 2.

The geometric meaning of the deformation which we are going to construct is as follows: cut the surface R along the geodesic C_0, change the position of *one side* of the cut by the diffeomorphism f_t and then reglue both sides of the cut in their new position.

That, however, requires an identification of C_0 with the circle S^1. Such an identification is obtained by identifying the intersection point of C_0 and C_1 with $1 \in S^1$ and by the standard parametrization of the oriented closed geodesic C_0. (Observe that as long as only the Fenchel-Nielsen deformation was considered, the identification of the point was not necessary since in that case the diffeomorphisms f_t were just rotations of the circle and these are rotation-invariant).

We shall now describe our construction.

Define a 1-parameter family of functions $\psi_t : \mathsf{H} \longrightarrow \mathsf{R}$, $t \in \mathsf{R}$, by

$$\psi_t(z) = \psi_t(re^{i\theta}) = l\tilde{f}_{ts(\theta)}\left(\frac{1}{l}\log(r)\right) - \log(r) \tag{2.3}$$

for $z = re^{i\theta} \in \mathsf{H}$.

Then define a 1-parameter family of mappings $F_t : \mathsf{H} \longrightarrow \mathsf{H}$, $t \in \mathsf{R}$, by

$$F_t(z) = e^{\psi_t(z)} \cdot z. \tag{2.4}$$

Observe that $\arg F_t(z) = \arg z$ for all $z \in \mathsf{H}$, $t \in \mathsf{R}$. It follows then immediately from (2.3) and (2.4) that

$$F_{t_1} \circ F_{t_2} = F_{t_1+t_2} \quad \text{and} \quad F_0 = \mathrm{id}. \tag{2.5}$$

Both ψ_t and F_t are C^∞-functions of variables t and z. Hence, for every $t \in \mathsf{R}$, the map F_t is a smooth diffeomorphism of H.

Observe also that, because of (2.2), we have

$$F_t \circ \gamma_0 = \gamma_0 \circ F_t, \qquad t \in \mathsf{R}. \tag{2.6}$$

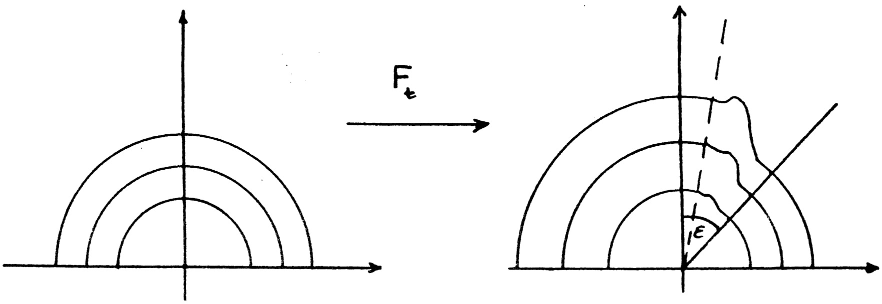

Figure 3.

The geometric meaning of the maps F_t is as follows: identify the oriented geodesic C_0 with the circle S^1 in the way described above. Then we can look upon the 1-parameter group of diffeomorphisms f_t as acting on C_0. Choose some $t \in \mathsf{R}$. We want to describe the displacement in the collar neighbourhood of C_0 which starts with the identity on one side of the collar and then gradually maps the consecutive layers of the collar into themselves by the maps f_s with varying parameter s until it arrives at the value $s = t$. From that layer on the mapping is done by f_t with constant t. The map F_t describes the lifting of such a displacement to the universal cover $\tilde{W}$ of the collar. Actually, $\tilde{W}$ is a sector in H and the map F_t is extended to the whole upper halfplane H.

We shall now compute the complex dilatation of F_t. Using $2\log r = \log(z\bar{z})$, we get $(\log r)_z = \frac{1}{2z}$ and $(\log r)_{\bar{z}} = \frac{1}{2\bar{z}}$. Similarily, using $\theta = -i(\log z - \log r)$, we get $(\theta)_z = \frac{1}{2iz}$ and $(\theta)_{\bar{z}} = -\frac{1}{2i\bar{z}}$. Therefore

$$(\psi_t(z))_z = \left(l\tilde{f}_{ts(\theta)}\left(\frac{1}{l}\log(r)\right) - \log(r)\right)_z$$

$$= lts'(\theta)\frac{1}{2iz}\left(\frac{d}{dt}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right) + \frac{1}{2z}\left(\frac{d}{dx}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right) - \frac{1}{2z}$$

and

$$(F_t(z))_z = (e^{\psi_t(z)}\cdot z)_z = e^{\psi_t(z)}(1 + z(\psi_t(z))_z)$$

$$= \frac{1}{2}e^{\psi_t(z)}\left[1 - ilts'(\theta)\left(\frac{d}{dt}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right) + \left(\frac{d}{dx}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right)\right].$$

Similarily we obtain

$$(F_t(z))_{\bar{z}} = (e^{\psi_t(z)}\cdot z)_{\bar{z}} = ze^{\psi_t(z)}(\psi_t(z))_{\bar{z}}$$

$$= -\frac{z}{2\bar{z}}e^{\psi_t(z)}\left[1 - ilts'(\theta)\left(\frac{d}{dt}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right)\right.$$

$$\left. - \left(\frac{d}{dx}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right)\right].$$

Hence the complex dilatation of the mapping $F_t : \mathsf{H} \longrightarrow \mathsf{H}$ is

$$\mu(F_t)(z) = \frac{(F_t(z))_{\bar{z}}}{(F_t(z))_z} \tag{2.7}$$

$$= -\frac{z}{\bar{z}}\left[1 - \frac{2\left(\frac{d}{dx}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right)}{1 - ilts'(\theta)\left(\frac{d}{dt}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right) + \left(\frac{d}{dx}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right)}\right].$$

Remark 2.8. 1) Observe that since $s(\theta) = 0$ for $0 \le \theta \le \frac{\pi - \varepsilon}{2}$ and since

$\left(\frac{d}{dx}\tilde{f}\right)_0(x) \equiv 1$, then for all vector fields $\mathfrak{X}$ we have $\mu(F_t)(z) = 0$ for z such that $0 \le \arg(z) \le \frac{\pi - \varepsilon}{2}$.

2) Observe also that for the Fenchel-Nielsen deformation which corresponds to the case when the vector field $\mathfrak{X} = c\widehat{\frac{d}{dx}}$, c-constant, we have $\tilde{f}_t(x) = x + ct$. Then $\left(\frac{d}{dx}\tilde{f}\right)_s(x) \equiv 1$, $\left(\frac{d}{dt}\tilde{f}\right)_s(x) \equiv c$ and

$$\mu(F_t)(z) = -\frac{z}{\bar{z}}\left(1 - \frac{2}{2 - iltcs'(\theta)}\right), \qquad z \in \mathsf{H}.$$

(Compare with [5; p. 503] or [2; p. 220].) Since $s'(\theta) = 0$ also for $\frac{\pi}{2} - \frac{\varepsilon}{4} \le \theta \le \pi$, the Beltrami coefficients $\mu(F_t)$ for the Fenchel-Nielsen deformation are supported in the sector $\frac{\pi}{2} - \frac{\varepsilon}{2} \le \theta \le \frac{\pi}{2} - \frac{\varepsilon}{4}$. This is however not the case if we consider more general deformations.

Denote

$$a = a(t,z) = \left(\frac{d}{dx}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right),$$

$$b = b(t,z) = lts'(\theta)\left(\frac{d}{dt}\tilde{f}\right)_{ts(\theta)}\left(\frac{1}{l}\log(r)\right).$$

According to (1.2) and (1.3) there exist real constants $\alpha_t^1, \alpha_t^2, B > 0$ such that $\alpha_t^1 \le a \le \alpha_t^2$ and $|b| \le B$ for all $z \in \mathsf{H}$. Then

$$\begin{aligned} |\mu(F_t)(z)| = \left|\frac{1 - a - ib}{1 + a - ib}\right| &= \left(1 - \frac{4a}{(1+a)^2 + b^2}\right)^{1/2} \\ &\le \left(1 - \frac{4a}{(1+a^2)(1+B^2)}\right)^{1/2} \\ &\le \max_{j=1,2}\left(1 - \frac{4\alpha_t^j}{(1+\alpha_t^j)^2(1+B^2)}\right)^{1/2} \\ &= k < 1, \end{aligned} \tag{2.9}$$

for all $z \in \mathsf{H}$. Therefore, for every $t \in \mathsf{R}$, $F_t : \mathsf{H} \longrightarrow \mathsf{H}$ is a quasiconformal mapping.

Let $\langle\gamma_0\rangle$ be the subgroup of Γ generated by the transformation γ_0 and let $B(\mathsf{H}, \langle\gamma_0\rangle)$ be the space of Beltrami differentials on H with respect to the group $\langle\gamma_0\rangle$ (see [2; p. 124]). Let $B(\mathsf{H}, \langle\gamma_0\rangle)_1 = \{\mu \in B(\mathsf{H}, \langle\gamma_0\rangle) \mid \|\mu\|_\infty < 1\}$ be the corresponding space of Beltrami coefficients.

It follows from (2.6) and (2.9) that

$$\mu(F_t) \in B(\mathsf{H}, \langle\gamma_0\rangle)_1, \qquad t \in \mathsf{R}. \tag{2.10}$$

Since $F_0 = \mathrm{id}$, we have $\mu(F_0) = 0$. $\{\mu(F_t) \mid t \in \mathsf{R}\}$ is a curve in the space of Beltrami coefficients $B(\mathsf{H}, \langle\gamma_0\rangle)_1$. The tangent vector to this curve at $t = 0$ is

$$\begin{aligned} \frac{\partial}{\partial t}\mu(F_t)\bigg|_{t=0}(z) &= \frac{z}{2\bar{z}}\left[s(\theta)\left(\frac{\partial^2\tilde{f}}{\partial t\partial x}\right)_0\left(\frac{1}{l}\log(r)\right)\right. \\ &\qquad \left. + ils'(\theta)\left(\frac{\partial\tilde{f}}{\partial t}\right)_0\left(\frac{1}{l}\log(r)\right)\right] \\ &= \frac{z}{2\bar{z}}\left[s(\theta)\tilde{h}'\left(\frac{1}{l}\log(r)\right) + ils'(\theta)\tilde{h}\left(\frac{1}{l}\log(r)\right)\right], \end{aligned} \tag{2.11}$$

where $\tilde{\mathfrak{X}}$ is the lifting to R of the vector field $\mathfrak{X}$ on S^1 and the function $\tilde{h} : \mathsf{R} \longrightarrow \mathsf{R}$ is given by

$$\tilde{\mathfrak{X}} = \tilde{h} \cdot \frac{d}{dx}.$$

Observe again that the infinitesimal Beltrami differential $\frac{\partial}{\partial t}\mu(F_t)\Big|_{t=0}(z)$ vanishes for z with $0 < \arg z < (\pi - \varepsilon)/2$.

The 1-parameter family of deformations of the complex structure of the Riemann surface R which we want to associate with the vector field $\mathfrak{X}$ on S^1 is obtained by cutting R along the goedesic C_0, moving *one side* of the cut by the diffeomorphism f_t and then regluing both sides of the cut in the new position.

We shall describe *only the infinitesimal deformation* of the complex structure of R obtained in this way.

To this end, let us first define a Beltrami differential ν_o on H by

$$\nu_o(z) = \begin{cases} \dfrac{\partial}{\partial t}\mu(F_t)|_{t=0}(z) & \text{if } \mathrm{Re}(z) \geq 0, \\ 0 & \text{if } \mathrm{Re}(z) < 0, \end{cases} \tag{2.12}$$

i.e.

$$\nu_o(z) = \frac{z}{2\bar{z}}\left[s(\theta)\tilde{h}'\left(\frac{1}{l}\log(r)\right) + ils'(\theta)\tilde{h}\left(\frac{1}{l}\log(r)\right)\right]$$

if $z = re^{i\theta}$ with $0 < \theta \leq \pi/2$ and $\nu_o(z) = 0$ otherwise.

By our construction ν_o vanishes outside the sector $(\pi - \varepsilon)/2 \leq \arg z \leq \pi/2$. Moreover, we have

$$\nu_o(\gamma_o(z))\frac{\overline{\gamma_o'(z)}}{\gamma_o'(z)} = \nu_o(\lambda z) = \nu_o(z). \tag{2.13}$$

This follows from (2.10) or can be checked directly (recall that $\tilde{h}(x+1) = \tilde{h}(x)$).

Now define a Beltrami differential $\nu(\mathfrak{X})$ on H by

$$\nu(\mathfrak{X})(z) = \sum_{\gamma \in \langle\gamma_0\rangle \backslash \Gamma} \nu_o(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)}, \qquad z \in \mathsf{H}. \tag{2.14}$$

Concerning convergence of this series: our choice of ε garanties that for every $z \in \mathsf{H}$ there is at most one term in the series which does not vanish at z.

It follows from our construction that $\nu(\mathfrak{X})$ is a Beltrami differential on H with respect to Γ, $\nu(\mathfrak{X}) \in B(\mathsf{H}, \Gamma)$.

It is the Beltrami differential $\nu(\mathfrak{X})$ which describes our infinitesimal deformation of the complex structure of the Riemann surface R induced by the vector field $\mathfrak{X}$ on S^1.

To be exact: let $T_B(\Gamma)$ be the Teichmüller space of the Fuchsian group Γ. (For the definitions and notations concerning Teichmüller spaces, see [2; Chap. 6]. We follow the notations used in that book).

Let $\Phi : B(\mathsf{H}, \Gamma)_1 \longrightarrow T_B(\Gamma)$ be the Bers projection, [2; p. 150]. We consider $\nu(\mathfrak{X}) \in B(\mathsf{H}, \Gamma)$ as a tangent vector to $B(\mathsf{H}, \Gamma)_1$ at 0. Then

$$\varphi_{(C_0,C_1)}(\mathfrak{X}) := (d\Phi)_o(\nu(\mathfrak{X}))$$

is a tangent vector to the Teichmüller space $T_B(\Gamma)$ at the base point. Every such a vector represents an infinitesimal deformation of the complex structure of R. The infinitesimal deformation of R induced by the vector field $\mathfrak{X}$ is, by definition, the one represented by $\varphi_{(C_0,C_1)}(\mathfrak{X})$.

Let $A_2(\mathsf{H}^*, \Gamma)$ be the space of holomorphic quadratic differentials on the lower half-plane H^* with respect to Γ. Let $B : T_B(\Gamma) \longrightarrow A_2(\mathsf{H}^*, \Gamma)$ be the Bers embedding. The Bers embedding identifies the tangent space to $T_B(\Gamma)$

at the base point with the complex vector space $A_2(\mathsf{H}^*, \Gamma)$. We shall now proceed to describe the tangent vector $\varphi_{(C_0,C_1)}(\mathfrak{X})$ as an element of $A_2(\mathsf{H}^*, \Gamma)$.

3. Description of the deformations by quadratic differentials

Let $\mathfrak{X}$ be a smooth vector field on S^1.

Let R be a compact Riemann surface of genus $g \geq 2$ and suppose that $R = \mathsf{H}/\Gamma$, where Γ is a Fuchsian group.

Finally, let (C_0, C_1) be a 1-pair of geodesics on R (see Definition 2.1).

In Section 2, given such data, we have constructed a Beltrami differential $\nu = \nu(\mathfrak{X}) \in B(\mathsf{H}, \Gamma)$. We look upon ν as a tangent vector to the space of Beltrami coefficients $B(\mathsf{H}, \Gamma)_1$ at 0. Let $\Phi : B(\mathsf{H}, \Gamma)_1 \longrightarrow T_B(\Gamma)$ be the Bers projection. Then $\varphi_{(C_0,C_1)}(\mathfrak{X}) = (d\Phi)_o(\nu)$ is a tangent vector to the Teichmüller space $T_B(\Gamma)$ at the base point and represents a deformation of the Riemann surface R.

The Bers embedding $B : T_B(\Gamma) \longrightarrow A_2(\mathsf{H}^*, \Gamma)$ gives an identification of $\varphi_{(C_0,C_1)}(\mathfrak{X})$ with a quadratic differential on H^* with respect to Γ. We shall now compute this quadratic differential in case when the vector field $\mathfrak{X}$ has a *finite* Fourier expansion.

First of all observe that the Beltrami differential $\nu = \nu(\mathfrak{X})$ and, hence, the quadratic differential $\varphi_{(C_0,C_1)}(\mathfrak{X})$ depends linearly on $\mathfrak{X}$,

$$\varphi_{(C_0,C_1)}(a_1\mathfrak{X}_1 + a_2\mathfrak{X}_2) = a_1\varphi_{(C_0,C_1)}(\mathfrak{X}_1) + a_2\varphi_{(C_0,C_1)}(\mathfrak{X}_2), \tag{3.1}$$

where $\mathfrak{X}_1, \mathfrak{X}_2$ are smooth vector fields on S^1 and $a_1, a_2 \in \mathsf{R}$. This follows immediately from (2.12). Moreover, since $A_2(\mathsf{H}^*, \Gamma)$ is a vector space over complex numbers, we can extend the definition of $\varphi_{(C_0,C_1)}(\mathfrak{X})$ in an obvious way to the case when $\mathfrak{X}$ is a *complexified* vector field on S^1 i.e. when $\mathfrak{X} = h\widehat{\frac{d}{dx}}$ with $h : S^1 \longrightarrow \mathsf{C}$ being a smooth function. Then (3.1) holds with arbitrary $a_1, a_2 \in \mathsf{C}$ and arbitrary complexified vector fields $\mathfrak{X}_1, \mathfrak{X}_2$ on S^1.

According to [2; Thm 6.10, p. 157] the quadratic differential $\varphi_{(C_0,C_1)}(\mathfrak{X}) \in A_2(\mathsf{H}^*, \Gamma)$ is given by

$$\begin{aligned} \varphi_{(C_0,C_1)}(\mathfrak{X})(z) &= (d\Phi)_o(\nu)(z) \\ &= -\frac{6}{\pi}\iint\limits_{\mathsf{H}} \frac{\nu(\zeta)}{(\zeta - z)^4}\, d\xi d\eta \end{aligned} \tag{3.2}$$

for $z \in \mathsf{H}^*$. (Here $\zeta = \xi + i\eta$ and the integration is with respect to the Lebesgue measure on H.)

We shall now compute the integral in (3.2) in the case when $\mathfrak{X} = \mathfrak{X}_n$ is the complexified vector field on S^1 such that its lifting to R is given by

$$\tilde{\mathfrak{X}} = \tilde{\mathfrak{X}}_n = \tilde{h}_n \cdot \frac{d}{dx}$$

with

$$\tilde{h}_n(x) = e^{2\pi inx}, \qquad x \in \mathsf{R},$$

n being an integer, $n \in \mathsf{Z}$. The result will be given as a Poincaré series.

From now on $\log(z)$ is the branch of $\log$ given by $0 \le \arg z < 2\pi$.

Lemma 3.3. *Let n be an integer, $n \neq 0$, and let $\tilde{h}_n(x) = e^{2\pi inx}$. Let $\nu_o \in B(\mathsf{H}, \langle\gamma_0\rangle)$ be the Beltrami differential defined in* (2.12) *for the vector field $\tilde{\mathfrak{X}}_n = \tilde{h}_n \cdot \frac{d}{dx}$. Let*

$$I(\nu_o)(z) = -\frac{6}{\pi} \iint_{\mathsf{H}} \frac{\nu_o(\zeta)}{(\zeta - z)^4} d\xi d\eta \qquad \text{for } z \in \mathsf{H}^*.$$

Then

$$I(\nu_o)(z) = \mathscr{B} \cdot \frac{1}{z^2} e^{2\pi in \log(z)/l}, \qquad z \in \mathsf{H}^*,$$

where $\mathscr{B}$ is a constant,

$$\mathscr{B} = \mathscr{B}(l, n) = 2\pi in\left(e^{-4\pi^2 n/l} - 1\right)^{-1} e^{\pi^2 n/l}\left(1 + \frac{4\pi^2 n^2}{l^2}\right).$$

Proof. By integrating in the polar coordinates we have

$$I(\nu_o)(z) = -\frac{6}{\pi} \int_{t=0}^{\pi/2} \int_{r=0}^{\infty} \frac{re^{i2t}\left[s(t)\tilde{h}_n'\left(\frac{1}{l}\log(r)\right) + ils'(t)\tilde{h}_n\left(\frac{1}{l}\log(r)\right)\right]}{2(re^{it} - z)^4} drdt$$

$$= -\frac{3}{\pi} \int_0^{\pi/2} e^{-i2t} \left(s(t) \int_0^{\infty} \frac{r\tilde{h}_n'\left(\frac{1}{l}\log(r)\right)}{(r - ze^{-it})^4} dr \right.$$

$$\left. + ils'(t) \int_0^{\infty} \frac{r\tilde{h}_n\left(\frac{1}{l}\log(r)\right)}{(r - ze^{-it})^4} dr \right) dt.$$

The integral $I_1 = \int_0^\infty \frac{r\tilde{h}_n(\frac{1}{l}\log(r))}{(r - ze^{-it})^4} dr$ is computed by the calculus of residues. We integrate the function $f(\zeta) = \frac{\zeta e^{2\pi i n \log(\zeta)/l}}{(\zeta - ze^{-it})^4}$ along a contour of type

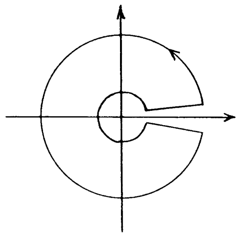

Figure 4.

Observe that the factor $e^{2\pi i n \log(\zeta)/l}$ is bounded in $\mathsf{C} - \{0\}$. $f(\zeta)$ has one singularity at $\zeta_o = ze^{-it}$ with a residue

$$\begin{aligned}\mathrm{Res}(f, \zeta_o) &= \frac{1}{3!}\frac{d^3}{d\zeta^3}\left(\zeta e^{2\pi i n \log(\zeta)/l}\right)\bigg|_{\zeta = ze^{-it}} \\ &= -\frac{\pi i n}{3l}\left(1 + \frac{4\pi^2 n^2}{l^2}\right) \cdot \frac{1}{z^2} e^{2\pi i n \log(z)/l} \cdot e^{(2\pi n t/l) + i2t}\end{aligned}$$

Since $\tilde{h}_n\left(\frac{\log(r) + 2\pi i}{l}\right) = \tilde{h}_n\left(\frac{\log(r)}{l}\right) \cdot e^{-4\pi^2 n/l}$, we get

$$I_1 = (1 - e^{-4\pi^2 n/l})^{-1} \cdot 2\pi i \cdot \mathrm{Res}(f, \zeta_o) = \mathscr{A} \cdot e^{(2\pi n t/l) + i2t},$$

where $\mathscr{A} = (1 - e^{-4\pi^2 n/l})^{-1} \cdot \frac{2\pi^2 n}{3l}\left(1 + \frac{4\pi^2 n^2}{l^2}\right) \cdot \frac{1}{z^2} e^{2\pi i n \log(z)/l}$ is independent of t.

Since $\tilde{h}'_n(x) = 2\pi i n \tilde{h}_n(x)$, we obtain

$$\begin{aligned}I(\nu_o)(z) &= -\frac{3}{\pi}\mathscr{A}\int_0^{\pi/2} e^{-i2t}(2\pi i n s(t) + i l s'(t)) e^{(2\pi n t/l) + i2t} dt \\ &= -\frac{3}{\pi}\mathscr{A} i l \int_0^{\pi/2} \frac{d}{dt}\left(s(t) e^{2\pi n t/l}\right) dt \\ &= -\frac{3\mathscr{A} i l}{\pi} \cdot e^{\pi^2 n/l},\end{aligned}$$

which gives the result of Lemma 3.3.

In the case $n = 0$ (which gives the Fenchel-Nielsen deformation) we have

LEMMA 3.4. *Let* $\nu_o \in B(\mathsf{H}, \langle\gamma_0\rangle)$ *be the Beltrami differential defined in* (2.12) *for the vector field* $\tilde{\mathfrak{x}}_0 = \frac{d}{dx}$ (*i.e.* $\tilde{h}(x) \equiv 1$). *Let*

$$I(\nu_o)(z) = -\frac{6}{\pi}\iint_{\mathsf{H}} \frac{\nu_o(\zeta)}{(\zeta - z)^4} d\xi d\eta \qquad \text{for } z \in \mathsf{H}^*.$$

Then

$$I(\nu_o)(z) = -\frac{il}{2\pi}\cdot\frac{1}{z^2}, \qquad z \in \mathsf{H}^*.$$

PROOF. By integrating in polar coordinates (since $\tilde{h}' \equiv 0$):

$$I(\nu_o)(z) = -\frac{6}{\pi}\int_0^{\pi/2} \tfrac{1}{2} e^{i2t} ils'(t)\left(\int_0^\infty \frac{r}{(re^{it} - z)^4} dr\right) dt$$

$$= -\frac{3il}{\pi}\int_0^{\pi/2} e^{i2t}s'(t)\cdot\frac{1}{6e^{i2t}z^2} dt = -\frac{il}{2\pi z^2}\int_0^{\pi/2} s'(t)dt$$

$$= -\frac{il}{2\pi}\cdot\frac{1}{z^2}.$$

Let us recall that the Bergman kernel for the upper half-plane $\mathscr{K}_{\mathsf{H}}(z,\zeta) = \frac{12}{\pi}\cdot\frac{1}{(\zeta - z)^4}$, $\zeta \in \mathsf{H}, z \in \mathsf{H}^*$, has the following invariance property:

$$\mathscr{K}_{\mathsf{H}}(z,\zeta) = \mathscr{K}_{\mathsf{H}}(\gamma(z),\gamma(\zeta))\cdot\gamma'(z)^2\cdot\gamma'(\zeta)^2, \qquad \zeta \in \mathsf{H}, z \in \mathsf{H}^*,$$

for all $\gamma \in \mathrm{PSL}(2,\mathsf{R})$.

It follows that

$$(3.5)\qquad -\frac{6}{\pi}\iint_{\mathsf{H}} \frac{\nu_o(\gamma(\zeta))\dfrac{\overline{\gamma'(\zeta)}}{\gamma'(\zeta)}}{(\zeta - z)^4}\, d\xi d\eta = -\frac{6}{\pi}\iint_{\mathsf{H}} \frac{\nu_o(\gamma(\zeta))|\gamma'(\zeta)|^2}{(\zeta - z)^4\gamma'(\zeta)^2}\, d\xi d\eta$$
$$= -\frac{6}{\pi}\iint_{\mathsf{H}} \frac{\nu_o(\gamma(\zeta))|\gamma'(\zeta)|^2}{(\gamma(\zeta) - \gamma(z))^4}\cdot \gamma'(z)^2 d\xi d\eta$$
$$= -\frac{6}{\pi}\gamma'(z)^2 \iint_{\mathsf{H}} \frac{\nu_o(\zeta)}{(\zeta - \gamma(z))^4}\, d\xi d\eta$$
$$= I(\nu_o)(\gamma(z))\cdot \gamma'(z)^2.$$

Since all the partial sums of the Beltrami differential

$$\nu(z) = \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma} \nu_o(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)}$$

are bounded by $\| \nu \|_\infty$ and since for every $z \in \mathsf{H}^*$ the function $g(\zeta) = \dfrac{1}{(\zeta - z)^4}$ is absolutly integrable in H, it follows from (3.2) and (3.5) that

$$(3.6)\qquad \varphi_{(C_0,C_1)}(\mathfrak{X})(z) = (d\Phi)_o(\nu)(z) = -\frac{6}{\pi}\iint_{\mathsf{H}} \frac{\nu(\zeta)}{(\zeta - z)^4}\, d\xi d\eta$$
$$= \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma} -\frac{6}{\pi}\iint_{\mathsf{H}} \frac{\nu_o(\gamma(\zeta))\dfrac{\overline{\gamma'(\zeta)}}{\gamma'(\zeta)}}{(\zeta - z)^4}\, d\xi d\eta$$
$$= \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma} I(\nu_o)(\gamma(z))\cdot \gamma'(z)^2.$$

Finally, we have

THEOREM 3.7. *If n is an integer and the complexified vector field $\mathfrak{X}_n$ on S^1 is given by the function $\tilde{h}_n : \mathsf{R} \longrightarrow \mathsf{C}, \tilde{h}_n(x) = e^{2\pi i n x}$, then the quadratic differential $\varphi_{(C_0,C_1)}(\mathfrak{X}_n) \in A_2(\mathsf{H}^*, \Gamma)$ is given by the Poincaré series*

$$\varphi_{(C_0,C_1)}(\mathfrak{X}_n) = \mathscr{B}_n \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma} e^{2\pi i n \log(\gamma)/l}\left(\frac{\gamma'}{\gamma}\right)^2,$$

with $\mathscr{B}_n$ being a constant,

$$\mathscr{B}_n = \mathscr{B}(n,l) = \begin{cases} -\dfrac{il}{2\pi} & \text{if } n = 0, \\ 2\pi i n e^{\pi^2 n/l}\left(1 + \dfrac{4\pi^2 n^2}{l^2}\right)(e^{-4\pi^2 n/l} - 1)^{-1} & \text{if } n \neq 0, \end{cases}$$

and the series converges absolutely and locally uniformly in H^*.

Proof. The formula for $\varphi_{(C_0,C_1)}(\mathfrak{X}_n)$ follows from (3.6) together with Lemmas 3.3 and 3.4. For the statement about convergence: since $\pi < \operatorname{Im}\log(z) < 2\pi$ for $z \in \mathsf{H}^*$ so for any given n the function $e^{2\pi i n \log(z)/l}$ is bounded in H^*. Hence the claim about convergence follows from [2; Thm 7.2, p. 186].

Remarks 3.8. 1) Theorem 3.7 generalizes a result of S. Wolpert, [5; Thm 2.7, p. 516], [2; Thm 8.2, p. 223], describing the Poincaré series corresponding to the Fenchel-Nielsen deformation. It is the case $n = 0$ of our Theorem 3.7. Observe that compared to Wolpert's original version, [5], our constant $\mathscr{B}_0$ has an extra factor l, the length of the geodesic C_0. This is due to the fact that our deformations are done by identifying C_0 with the circle S^1 and, hence, when looked upon in the Riemann surface R these deformations are done with speed l while Wolpert's deformation was done with speed 1. There is also a difference of sign when compared to [2]. This is because the Fenchel-Nielsen deformation in [2] is done in opposite direction when compared to Wolpert's one and ours.

2) It follows from Theorem 3.7 that the infinitesimal deformations associated to the vector fields $\mathfrak{X}_n$ are independent of the choice of the auxiliary function $s(\theta)$ and depend only on n, the Riemann surface R and the 1-pair of geodesics (C_0, C_1). Consequently, the same holds for any vector field $\mathfrak{X}$ with finite Fourier expansion – the infinitesimal deformation depends only on $\mathfrak{X}, R$ and (C_0, C_1). Actually, this can be proven for any C^∞ vector field $\mathfrak{X}$ on S^1. That will be shown in another paper.

3) In Section 4 we shall give another, independent proof of Theorem 3.7 closer to the one given in [5] for the Fenchel-Nielsen deformation. This second proof is somewhat more complicated, but it gives at the same time a description of the Eichler cohomology classes corresponding to the deformations induced by the vector fields $\mathfrak{X}_n$.

4. Description of the deformations by Eichler cohomology classes

Let $\nu = \nu(\mathfrak{X})$ be the Beltrami differential on H defined in (2.14), $\nu \in B(\mathsf{H}, \Gamma)$. ν determines an infinitesimal deformation $\varphi_{(C_0,C_1)}(\mathfrak{X})$ of the Riemann surface $R = \mathsf{H}/\Gamma$. In Section 3, assuming that $\mathfrak{X}$ had a finite Fourier expansion, we have given a description of $\varphi_{(C_0,C_1)}(\mathfrak{X})$ as a Poincaré series. In this Section we shall describe the Eichler cohomology class corresponding to $\varphi_{(C_0,C_1)}(\mathfrak{X})$ (again for $\mathfrak{X}$ with finite Fourier expansion).

We extend ν to a Beltrami differential $\hat{\nu}$ on C by

$$\hat{\nu}(z) = \begin{cases} \nu(z), & z \in \mathsf{H}, \\ 0, & z \in \mathsf{C} - \mathsf{H}. \end{cases}$$

Let us consider a potential function $F_\nu : \mathsf{C} \longrightarrow \mathsf{C}$ for $\hat{\nu}$ given by

$$F_\nu(z) = -\frac{z(z-1)}{\pi} \iint_{\mathsf{H}} \frac{\nu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta, \tag{4.1}$$

(see [2; p. 197] and [3; Chap. IV, Lemma 1.4, p. 136]).

Let Π_2 be the space of polynomials in one complex variable of degree ≤ 2. The group Γ acts on Π_2 via

$$\gamma_*(P) = \frac{P \circ \gamma}{\gamma'}, \qquad \gamma \in \Gamma, P \in \Pi_2.$$

The space of infinitesimal deformations of R is identified with a subspace of the first Eichler cohomology group $H^1(\Gamma, \Pi_2)$.

The Eichler cohomology class $[\chi_\nu] \in H^1(\Gamma, \Pi_2)$ corresponding to the infinitesimal deformation $\varphi_{(C_0,C_1)}(\mathfrak{X})$ is given by the cocycle

$$\chi_\nu : \Gamma \longrightarrow \Pi_2,$$

where

$$\chi_\nu(\gamma) = \frac{F_\nu \circ \gamma}{\gamma'} - F_\nu. \tag{4.2}$$

(See [2; p. 197].)

We shall now determine the cocycle χ_ν.

Let n be an integer, $n \in \mathsf{Z}$, and let $\mathfrak{X} = \mathfrak{X}_n$ be the (complexified) vector field on S^1 given by the function $\tilde{h}_n(x) = e^{2\pi i n x}$, $\tilde{\mathfrak{X}}_n = \tilde{h}_n \cdot \dfrac{d}{dx}$. Let $\nu_o^n \in B(\mathsf{H}, \langle\gamma_0\rangle)$ be the Beltrami differential defined in (2.12) for $\tilde{h} = \tilde{h}_n$. Finally, let $I_n(z)$ be the potential function for ν_o^n,

$$I_n(z) = -\frac{z(z-1)}{\pi} \iint_{\mathsf{H}} \frac{\nu_o^n(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta, \qquad z \in \mathsf{C}. \tag{4.3}$$

LEMMA 4.4. *If* $n \neq 0$ *then for all* $z \in \mathsf{C} - \{0\}$ *satisfying* $\dfrac{\pi}{2} \leq \arg z < 2\pi$ *one has*

$$I_n(z) = C_n\, z\,(e^{2\pi i n(\log(z)-2\pi i)/l} - 1),$$

where C_n is a constant,

$$C_n = C(n,l) = l e^{\pi^2 n/l} \cdot (e^{4\pi^2 n/l} - 1)^{-1}.$$

PROOF. The proof is very similar to that of Lemma 3.3. Since

$$\nu_o^n(z) = \frac{z}{2\bar{z}}\left[s(\theta)\tilde{h}_n'\left(\frac{1}{l}\log(r)\right) + ils'(\theta)\tilde{h}_n\left(\frac{1}{l}\log(r)\right)\right]$$

if $z = re^{i\theta}$ with $0 < \theta \le \dfrac{\pi}{2}$ and $\nu_o^n(z) = 0$ otherwise, integrating in polar coordinates we obtain

$$I_n(z) = -\frac{z(z-1)}{2\pi}\int_{t=0}^{\pi/2}\int_{r=0}^{\infty}\frac{s(t)\tilde{h}_n'\left(\frac{1}{l}\log(r)\right) + ils'(t)\tilde{h}_n\left(\frac{1}{l}\log(r)\right)}{e^{-it}(re^{it}-1)(re^{it}-z)}\,drdt$$

$$= -\frac{z(z-1)}{2\pi}\int_{t=0}^{\pi/2} e^{-it}\left[s(t)\int_{r=0}^{\infty}\frac{\tilde{h}_n'\left(\frac{1}{l}\log(r)\right)}{(r-e^{-it})(r-ze^{-it})}\,dr\right.$$

$$\left. + ils'(t)\int_{r=0}^{\infty}\frac{\tilde{h}_n\left(\frac{1}{l}\log(r)\right)}{(r-e^{-it})(r-ze^{-it})}\,dr\right]dt.$$

The integral $J_n(z,t) = \displaystyle\int_{r=0}^{\infty}\frac{\tilde{h}_n(\frac{1}{l}\log(r))}{(r-e^{-it})(r-ze^{-it})}dr$ is evaluated by the calculus of residues. We integrate the function $g(\zeta) = \dfrac{e^{2\pi i n\log(\zeta)/l}}{(\zeta-e^{-it})(\zeta-ze^{-it})}$ along a contour of type

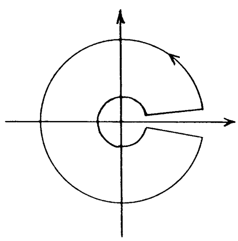

Figure 5.

Observe that we are interested in the case when $z \neq 0, 1$ and that the nominator of $g(\zeta)$ is bounded in $\mathsf{C} - \{0\}$. The function $g(\zeta)$ has two singular points (both simple poles): one at $\zeta_1 = e^{-it}$ with a residue

$$\operatorname{Res}(g, \zeta_1) = e^{it} \cdot \frac{e^{-2\pi n(2\pi - t)/l}}{1 - z}$$

and one at $\zeta_2 = ze^{-it}$ with a residue

$$\operatorname{Res}(g, \zeta_2) = e^{it} \cdot \frac{e^{2\pi in \log(ze^{-it})/l}}{z - 1}.$$

Since $\tilde{h}_n((2\pi i + \log(r))/l) = \tilde{h}_n(\log(r)/l) \cdot e^{-4\pi^2 n/l}$, we get

$$\begin{aligned} J_n(z, t) &= (1 - e^{-4\pi^2 n/l})^{-1} 2\pi i (\operatorname{Res}(g, \zeta_1) + \operatorname{Res}(g, \zeta_2)) \\ &= 2\pi i (1 - e^{-4\pi^2 n/l})^{-1} \cdot \frac{e^{it}}{z - 1} \cdot (e^{2\pi in \log(ze^{-it})/l} - e^{-2\pi n(2\pi - t)/l}). \end{aligned}$$

If $\frac{\pi}{2} \leq \arg z < 2\pi$ then $\log(ze^{-it}) = \log(z) - it$ for all $0 \leq t \leq \frac{\pi}{2}$ and therefore

$$J_n(z, t) = 2\pi i (1 - e^{-4\pi^2 n/l})^{-1} e^{-4\pi^2 n/l} \frac{e^{it}}{z - 1} \cdot e^{2\pi nt/l} \cdot (e^{2\pi in(\log(z) - 2\pi i)/l} - 1).$$

Let $\mathscr{C}(z) = i(e^{4\pi^2 n/l} - 1)^{-1} z\,(e^{2\pi in(\log(z) - 2\pi i)/l} - 1)$. Then

$$\begin{aligned} I_n(z) &= -\mathscr{C}(z) \int_{t=0}^{\pi/2} e^{2\pi nt/l} (2\pi ins(t) + ils'(t))\, dt \\ &= -il\mathscr{C}(z) \Big[e^{2\pi nt/l} s(t) \Big]_{t=0}^{t=\pi/2} \\ &= -il\mathscr{C}(z) e^{\pi^2 n/l}, \end{aligned}$$

which proves Lemma 4.4.

For the sake of completeness let us also consider the case $n = 0$. This has been solved by Wolpert in [5; Sec. 2].

Lemma 4.5. *$I_0(z) = \frac{il}{2\pi} z(\log(z) - 2\pi i)$ for all $z \in \mathsf{C} - \{0\}$ satisfying $\frac{\pi}{2} \leq \arg(z) < 2\pi$.*

Proof. By integrating in the polar coordinates again we get

$$I_0(z) = -\frac{z(z-1)}{2\pi}\int_{t=0}^{\pi/2}\int_{r=0}^{\infty}\frac{ils'(t)}{e^{-it}(re^{it}-1)(re^{it}-z)}drdt$$

$$= -\frac{z(z-1)il}{2\pi}\int_{t=0}^{\pi/2} s'(t)e^{-it}\left(\int_{r=0}^{\infty}\frac{dr}{(r-e^{-it})(r-ze^{-it})}\right)dt$$

$$= -\frac{z(z-1)il}{2\pi}\int_{t=0}^{\pi/2} s'(t)\cdot\frac{1}{z-1}(2\pi i-\log(z))\,dt.$$

The last equality holds provided $\frac{\pi}{2} \leq \arg(z) < 2\pi$. Hence

$$I_0(z) = -\frac{zil}{2\pi}(2\pi i - \log(z))\Big[s(t)\Big]_{t=0}^{t=\pi/2}$$

$$= \frac{il}{2\pi}\,z\,(\log(z) - 2\pi i).$$

Let us now recall that if μ is any bounded measurable function on C and if $F_\mu(z)$ is defined by

$$F_\mu(z) = -\frac{z(z-1)}{\pi}\iint_{\mathsf{C}}\frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)}d\xi d\eta$$

then

(4.6)
(i) F_μ is a continuous function on C,
(ii) $F_\mu(0) = F_\mu(1) = 0$,
(iii) $(F_\mu)_{\bar z} = \mu$ in the sense of distributions,
(iv) $F_\mu(z) = o(|z|^2)$ as $z \longrightarrow \infty$.

See [3; Lemma 1.4, p. 136].

Applying (4.6) (iii) to $I_n(z)$ we see that $(I_n)_{\bar z} = \nu_o^n$ in the sense of distributions and, hence,

$$(I_n \circ \gamma)_{\bar z} = (\nu_o^n \circ \gamma)\cdot\overline{\gamma'}, \qquad \gamma \in \Gamma.$$

It follows that for every $\gamma \in \Gamma$

$$(4.7) \qquad P_\gamma^n(z) = \frac{(I_n\circ\gamma)(z)}{\gamma'(z)} - \left(-\frac{z(z-1)}{\pi}\iint_{\mathsf{H}}\frac{\nu_o^n(\gamma(\zeta))\overline{\gamma'(\zeta)}}{\zeta(\zeta-1)(\zeta-z)\gamma'(\zeta)}d\xi d\eta\right)$$

is a holomorphic function on C and $P_\gamma^n(z) = O(|z|^2)$ as $z \longrightarrow \infty$. Therefore $P_\gamma^n(z)$ is a polynomial in z of degree at most 2. Observe that $P_e^n(z) \equiv 0$, where e is the identity element of Γ.

Let $C_n = C(n, l)$, $n \neq 0$, be the constant of Lemma 4.4 and let $\operatorname{Log}(z)$ be the branch of logarithm given by $-\pi \leq \arg(z) < \pi$.

LEMMA 4.8. *Let n be an integer,* $n \neq 0$.
(i) *If* $\gamma_1, \gamma_2 \in \Gamma$ *represent the same coset in* $\langle \gamma_0 \rangle \backslash \Gamma$ *then* $P^n_{\gamma_1} = P^n_{\gamma_2}$.
(ii) *If* $\gamma \in \langle \gamma_0 \rangle$ *then* $P^n_\gamma = 0$.
(iii) *If* $\gamma \in \Gamma - \langle \gamma_0 \rangle$ *and* $\gamma(z) = \dfrac{az+b}{cz+d}$, $ad - bc = 1$, *then* $P^n_\gamma(z) = a^n_{0\gamma} + a^n_{1\gamma} z + a^n_{2\gamma} z^2$ *with*

$$a^n_{2\gamma} = ac(e^{2\pi i n \operatorname{Log}(\frac{a}{c})/l} - 1) \cdot C_n,$$
$$a^n_{0\gamma} = bd(e^{2\pi i n \operatorname{Log}(\frac{b}{d})/l} - 1) \cdot C_n \qquad \text{and with}$$
$$P^n_\gamma(1) = (a+b)(c+d)(e^{2\pi i n \operatorname{Log}(\frac{a+b}{c+d})/l} - 1) \cdot C_n.$$

REMARK. The expressions in (iii) are well defined: if $\gamma \in \Gamma - \langle \gamma_0 \rangle$ and $\gamma(z) = \dfrac{az+b}{cz+d}$ then $a, b, c, d, a+b, c+d \neq 0$, see [5; p. 514].

PROOF. (i) It follows from Lemma 4.4 that $(I_n \circ \gamma_o)(z) = I_n(\lambda z) = \lambda I_n(z)$ for all $z \in \mathsf{H}^*$, $n \neq 0$. Hence

$$\frac{(I_n \circ (\gamma_o \circ \gamma))(z)}{(\gamma_o \circ \gamma)'(z)} = \frac{(I_n \circ \gamma)(z)}{\gamma'(z)}, \qquad z \in \mathsf{H}^*, n \neq 0.$$

The integrand in the second term on the right hand side of (4.7) will remain unchanged if γ is replaced by $\gamma_o \circ \gamma$ since, by (2.13), $\nu^n_o(\gamma_o(\zeta)) = \nu^n_o(\zeta), \zeta \in \mathsf{H}$. Thus

$$P^n_{\gamma_o \circ \gamma} = P^n_\gamma, \qquad \gamma \in \Gamma,\ n \in \mathsf{Z},\ n \neq 0.$$

(ii) If $\gamma \in \langle \gamma_0 \rangle$ then, by (i), $P^n_\gamma = P^n_e = 0$.

(iii) Suppose that $\gamma \in \Gamma - \langle \gamma_0 \rangle$ and $\gamma(z) = \dfrac{az+b}{cz+d}$, $ad - bc = 1$. Let

$$G^n_\gamma(z) = -\frac{z(z-1)}{\pi} \iint_{\mathsf{H}} \frac{\nu^n_o(\gamma(\zeta))\overline{\gamma'(\zeta)}}{\zeta(\zeta-1)(\zeta-z)\gamma'(\zeta)} d\xi d\eta.$$

According to (4.6), G^n_γ is continuous in C, $G^n_\gamma(0) = G^n_\gamma(1) = 0$ and $G^n_\gamma(z) = o(|z|^2)$ as $z \longrightarrow \infty$. On the other hand, from Lemma 4.4

$$\frac{I_n(\gamma(z))}{\gamma'(z)} = C_n \frac{\gamma(z)}{\gamma'(z)} \left(e^{2\pi i n (\log(\gamma(z)) - 2\pi i)/l} - 1\right)$$
$$= C_n(az+b)(cz+d)\left(e^{2\pi i n (\log(\frac{az+b}{cz+d}) - 2\pi i)/l} - 1\right), \qquad z \in \mathsf{H}^*.$$

Hence

$$P_\gamma^n(0) = \lim_{\substack{z\to 0\\ z\in \mathsf{H}^*}} P_\gamma^n(z) = \lim_{\substack{z\to 0\\ z\in \mathsf{H}^*}} \frac{I_n(\gamma(z))}{\gamma'(z)} = C_n bd\left(e^{2\pi in \mathrm{Log}(\frac{b}{d})/l} - 1\right),$$

$$P_\gamma^n(1) = \lim_{\substack{z\to 1\\ z\in \mathsf{H}^*}} \frac{I_n(\gamma(z))}{\gamma'(z)} = C_n(a+b)(c+d)\left(e^{2\pi in \mathrm{Log}(\frac{a+b}{c+d})/l} - 1\right),$$

and

$$\lim_{z\to\infty} \frac{1}{z^2} P_\gamma^n(z) = \lim_{\substack{z\to \infty\\ z\in \mathsf{H}^*}} \frac{I_n(\gamma(z))}{\gamma'(z)} = C_n ac\left(e^{2\pi in \mathrm{Log}(\frac{a}{c})/l} - 1\right).$$

This proves part (iii) of Lemma 4.8.

Again, for the sake of completeness let us recall the corresponding result for $n = 0$ from [5; p. 513–514].

Let $C_0 = \dfrac{il}{2\pi}$.

LEMMA 4.9.

(i) *For every* $\gamma \in \Gamma$

$$P^0_{\gamma_o\circ\gamma}(z) = P^0_\gamma(z) + \frac{il^2}{2\pi}\cdot\frac{\gamma(z)}{\gamma'(z)}, \qquad z \in \mathsf{C}.$$

(ii) *If* $\gamma = (\gamma_o)^m$, $m \in \mathsf{Z}$, *then* $P^0_\gamma(z) = m\,\dfrac{il^2}{2\pi}\,z$.

(iii) *If* $\gamma \in \Gamma - \langle\gamma_0\rangle$ *and* $\gamma(z) = \dfrac{az+b}{cz+d}$, $ad - bc = 1$, *then* $P^0_\gamma(z) = a^0_{0\gamma} + a^0_{1\gamma}z + a^0_{2\gamma}z^2$ with

$$a^0_{2\gamma} = ac\,\mathrm{Log}\left(\frac{a}{c}\right)\cdot C_0,$$

$$a^0_{0\gamma} = bd\,\mathrm{Log}\left(\frac{b}{d}\right)\cdot C_0 \qquad \text{and}$$

$$P^0_\gamma(1) = (a+b)(c+d)\,\mathrm{Log}\left(\frac{a+b}{c+d}\right)\cdot C_0.$$

PROOF. (i) From the explicit description of $I_0(z)$ in Lemma 4.5 one has

$$\frac{I_0(\gamma_o(z))}{\gamma_o'(z)} = I_0(z) + \frac{il^2}{2\pi}z.$$

Hence

$$\frac{I_0((\gamma_o \circ \gamma)(z))}{(\gamma_o \circ \gamma)'(z)} = \frac{I_0(\gamma(z))}{\gamma'(z)} + \frac{il^2\gamma(z)}{2\pi\gamma'(z)}, \qquad \gamma \in \Gamma.$$

Since the second term on the right hand side of (4.7) is unchanged when one replaces γ with $\gamma_o \circ \gamma$ (see the proof of Lemma 4.8), we obtain (i).

(ii) follows directly from (i).

(iii) is proven in the same way as in Lemma 4.8.

LEMMA 4.10. *Let $\mathfrak{X} = \mathfrak{X}_n$ be the complexified vector field on S^1 given by the function $\tilde{h}_n(x) = e^{2\pi i n x}$, $\tilde{\mathfrak{X}}_n = \tilde{h}_n \frac{d}{dx}$, n-an integer. Let $\nu^n \in B(\mathsf{H}, \Gamma)$ be the Beltrami differential defined in* (2.14) *for $\mathfrak{X}_n$ and let $F_{\nu^n} : \mathsf{C} \longrightarrow \mathsf{C}$ be the potential function for $\widehat{\nu^n}$ defined in* (4.1). *Then*

$$F_{\nu^n} = \sum_{\gamma \in \langle\gamma_0\rangle\backslash\Gamma} \left(\frac{I_n \circ \gamma}{\gamma'} - P^n_\gamma \right),$$

and the series converges absolutely and locally uniformly on C.

PROOF. Let $R(\zeta, z) = \dfrac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}$. We have

$$F_{\nu^n}(z) = -\frac{1}{\pi} \iint_{\mathsf{H}} \nu^n(\zeta) R(\zeta, z)\, d\xi d\eta.$$

Since $\nu^n(z) = \sum_{\gamma \in \langle\gamma_0\rangle\backslash\Gamma} \nu^n_o(\gamma(z)) \dfrac{\overline{\gamma'(z)}}{\gamma'(z)}$ and all the partial sums of this series are bounded by $\| \nu^n_o \|_\infty$, we can integrate term by term and obtain

$$\begin{aligned} F_{\nu^n}(z) &= \sum_{\gamma \in \langle\gamma_0\rangle\backslash\Gamma} -\frac{1}{\pi} \iint_{\mathsf{H}} \nu^n_o(\gamma(\zeta)) \frac{\overline{\gamma'(\zeta)}}{\gamma'(\zeta)} R(\zeta, z) d\xi d\eta \\ &= \sum_{\gamma \in \langle\gamma_0\rangle\backslash\Gamma} \left(\frac{I_n(\gamma(z))}{\gamma'(z)} - P^n_\gamma(z) \right). \end{aligned}$$

THEOREM 4.11. *The Eichler cohomology class $[\chi_{\nu^n}] \in H^1(\Gamma, \Pi_2)$ corresponding to the infinitesimal deformation $\varphi_{(C_0, C_1)}(\mathfrak{X}_n)$ of the Riemann surface $R = \mathsf{H}/\Gamma$ is given by the cocycle χ_{ν^n}, where*

$$\chi_{\nu^n}(\omega) = \frac{F_{\nu^n} \circ \omega}{\omega'} - F_{\nu^n} = \sum_{\gamma \in \langle\gamma_0\rangle\backslash\Gamma} \left(P^n_{\gamma\circ\omega} - \frac{P^n_\gamma \circ \omega}{\omega'} \right)$$

for $\omega \in \Gamma$. (The series converges absolutely and locally uniformly on C.)

PROOF. The function

$$f^n_\gamma(z) = \left(\frac{I_n \circ \gamma}{\gamma'} - P^n_\gamma\right)(z) = -\frac{1}{\pi}\iint_{\mathsf{H}} \nu^n_o(\gamma(\zeta))\frac{\overline{\gamma'(\zeta)}}{\gamma'(\zeta)} R(\zeta, z)d\xi d\eta$$

depends only on the right coset of γ in $\langle\gamma_0\rangle\backslash\Gamma$. Hence, for $\omega \in \Gamma$,

$$\begin{aligned} F_{\nu^n} &= \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma} f^n_\gamma = \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma} f^n_{\gamma\circ\omega} \\ &= \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma} \left(\frac{I_n \circ (\gamma\circ\omega)}{(\gamma\circ\omega)'} - P^n_{\gamma\circ\omega}\right). \end{aligned}$$

Since

$$\frac{F_{\nu^n}\circ\omega}{\omega'} = \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma} \frac{f^n_\gamma\circ\omega}{\omega'} = \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma}\left(\frac{I_n\circ(\gamma\circ\omega)}{(\gamma\circ\omega)'} - \frac{P^n_\gamma\circ\omega}{\omega'}\right),$$

we obtain Theorem 4.11.

REMARK 4.12. According to [2; Thm 6.10] the quadratic differential $\varphi_{(C_0,C_1)}(\mathfrak{X}_n) \in A_2(\mathsf{H}^*, \Gamma)$ is equal to $(F_{\nu^n})'''$. Thus it follows from Lemma 4.10 that $\varphi_{(C_0,C_1)}(\mathfrak{X}_n)$ can be obtained by differentiating term by term three times the series $\sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma}\left(\frac{I_n\circ\gamma}{\gamma'} - P^n_\gamma\right)$. Using Bol's formula (see e.g. [5; p. 515]), we get as a result

$$\varphi_{(C_0,C_1)}(\mathfrak{X}_n) = \sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma} (I'''_n\circ\gamma)\cdot(\gamma')^2.$$

From the explicit description of I_n given in Lemmas 4.4 and 4.5 we get then another proof of Theorem 3.7.

5. Vector fields on Teichmüller spaces

Let R be a compact Riemann surface of genus $g \geq 2$ and let (C_0, C_1) be a 1-pair of geodesics on R (see Definition 2.1).

Let $T(R)$ be the Teichmüller space of R. We represent points of $T(R)$ by equivalence classes of quasiconformal maps $f : R \longrightarrow S$, S being a compact Riemann surface of genus g (see [2; p. 14]). Denote by $[S,f]$ the equivalence class of f, $[S,f] \in T(R)$.

Given $[S,f] \in T(R)$, consider the simple closed curves $f(C_0)$ and $f(C_1)$ in S. Let C_0^f, C_1^f be the simple closed geodesics in S in the free homotopy classes of $f(C_0), f(C_1)$ respectively. Then (C_0^f, C_1^f) is a 1-pair of geodesics in S. Indeed, since C_0 and C_1 have just one intersection point in R, so do the curves $f(C_0)$ and $f(C_1)$ in S as well. Hence, by [1; Thm 1.6.7, p. 23], the geodesics C_0^f, C_1^f intersect in at most one point. On the other hand the homological intersection number $\langle C_0, C_1 \rangle$ of C_0 and C_1 is ± 1. Since f is an orientation preserving homeomorphism, we have

$$\langle C_0^f, C_1^f \rangle = \langle f(C_0), f(C_1) \rangle = \langle C_0, C_1 \rangle = \pm 1.$$

Therefore C_0^f and C_1^f cannot be disjoint. Thus C_0^f and C_1^f intersect in exactly one point and, hence, (C_0^f, C_1^f) is a 1-pair of geodesics. It is also clear that the pair (C_0^f, C_1^f) is independent of the choice of f within the equivalence class $[S,f]$.

Suppose $\mathfrak{X}$ is a smooth vector field on the circle S^1. Given a point $[S,f] \in T(R)$ let us consider the infinitesimal deformation $\varphi_{(C_0^f, C_1^f)}(\mathfrak{X})$ of the Riemann surface S constructed in Section 2. We look now upon $\varphi_{(C_0^f, C_1^f)}(\mathfrak{X})$ as a tangent vector to the Teichmüller space $T(R)$ at the point $[S,f]$. In that way every smooth vector field $\mathfrak{X}$ on the circle S^1 induces a vector field $\Phi_{(C_0, C_1)}(\mathfrak{X})$ on the Teichmüller space $T(R)$.

In the special case when $\mathfrak{X} = \widehat{\dfrac{d}{dx}}$ is the constant vector field on S^1, $\Phi_{(C_0, C_1)}(\mathfrak{X})$ is equal to the Fenchel-Nielsen vector field with respect to the geodesic C_0 (multiplied by C_0:s length l),

$$\Phi_{(C_0, C_1)}\left(\widehat{\frac{d}{dx}}\right) = l \frac{\partial}{\partial \tau_{C_0}},$$

which has been introduced and studied in [5] and [6]. See also [2; Sec. 8.3].

Geometry of the vector fields $\Phi_{(C_0, C_1)}(\mathfrak{X})$ is a subject of a work in progress.

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DEPARTMENT OF MATHEMATICS
UPPSALA UNIVERSITY
BOX 480
S-751 06 UPPSALA
SWEDEN.
E-mail: ryszard@math.uu.se