MATH. SCAND. 86 (2000), 244–248

NONSTANDARD CRITERIA FOR BOREL-MEASURABILITY

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Abstract

Let $(X, \mathscr{A})$ be a measurable space, (Y, δ) be a metric space with Borel-σ-algebra $\mathscr{B}$ and $f : X \to Y$ be a function. If (Y, δ) is a σ-compact space, then it is shown that the $\mathscr{A}, \mathscr{B}$-measurability of f is equivalent to the fact that the standard part of ${}^*f$ is constant on $\mathscr{A}$-monads, a result which is not true any more if we replace "σ-compact" by "locally compact". We moreover prove nonstandard criteria for special classes of $\mathscr{A}, \mathscr{B}$-measurable functions with values in an arbitrary metric space.

1. The results

In this paper we consider a superstructure containing two given sets X, Y and the set R of real numbers, and we work with a polysaturated nonstandard model for this superstructure.

Let $\mathscr{A}$ be a σ-algebra on X and $\mathscr{B}$ be the Borel-σ-algebra of a metric space (Y, δ). For $x_1, x_2 \in {}^*X$ we write $x_1 \underset{\mathscr{A}}{\approx} x_2$ iff for all $A \in \mathscr{A}$ there holds: $x_1 \in {}^*A \Longleftrightarrow x_2 \in {}^*A$.

According to Ross [5] the following two conditions are equivalent for a function $f : X \to Y$:

(I) f is $\mathscr{A}, \mathscr{B}$-measurable.

(II) $x_1 \underset{\mathscr{A}}{\approx} x_2 \Rightarrow {}^*f(x_1) \underset{\mathscr{B}}{\approx} {}^*f(x_2) \quad (x_1, x_2 \in {}^*X)$.

As the referee has pointed out this equivalence can also be shown with similar methods as those in the proof of Lemma 6.

In ${}^*Y$ we have furthermore an equivalence relation $\underset{\delta}{\approx}$ derived from the metric δ, namely $y_1 \underset{\delta}{\approx} y_2 \Longleftrightarrow {}^*\delta(y_1, y_2) \approx 0$. If $Y = \mathsf{R}$ it is known that for *bounded* f the $\mathscr{A}, \mathscr{B}$-measurability of f is furthermore equivalent to

(III) $x_1 \underset{\mathscr{A}}{\approx} x_2 \Rightarrow {}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2) \quad (x_1, x_2 \in {}^*X)$,

see Ross [5]. This result can also be derived from Loeb [4], Theorem 1.3, p. 67.

Received December 16, 1996; in revised form October 15, 1997.

The following Theorem 1 extends this result to metric spaces Y, and it shows furthermore that condition (III) already implies that $f(X)$ is totally bounded.

1. Theorem. *For a function $f : X \to Y$ the following conditions are equivalent:*

(i) *f is $\mathscr{A}, \mathscr{B}$-measurable and $f(X)$ is totally bounded.*

(ii) $x_1 \underset{\mathscr{A}}{\approx} x_2 \Rightarrow {}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2)$.

Now we give results which, for the case $Y = \mathsf{R}^n$, are nonstandard criteria for the $\mathscr{A}, \mathscr{B}$-measurability of arbitrary not necessarily bounded functions.

Let $\mathrm{fin}({}^*Y) := \{z \in {}^*Y : {}^*\delta(z, {}^*y)$ is finite for all $y \in Y\}$ and $\mathrm{cpt}({}^*Y) := \cup\{{}^*C : C \subset Y$ compact$\}$. Let furthermore $\mathrm{ns}({}^*Y)$ and $\mathrm{pns}({}^*Y)$ be the systems of nearstandard-points and prenearstandard-points of ${}^*Y$.

2. Theorem. *For a function $f : X \to Y$ the following conditions are equivalent:*

(i) *f is $\mathscr{A}, \mathscr{B}$-measurable and bounded subsets of $f(X)$ are totally bounded.*

(ii) $(x_1 \underset{\mathscr{A}}{\approx} x_2 \wedge {}^*f(x_1) \in \mathrm{fin}({}^*Y)) \Rightarrow {}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2)$.

If $\mathscr{C}$ is a system of subsets of Y, $\sigma(\mathscr{C})$ denotes the smallest σ-algebra on Y containing $\mathscr{C}$.

3. Theorem. *Let $\mathscr{C}$ be the system of all compact subsets of Y. For a function $f : X \to Y$ the following conditions are equivalent:*

(i) *f is $\mathscr{A}, \sigma(\mathscr{C})$-measurable.*

(ii) $(x_1 \underset{\mathscr{A}}{\approx} x_2 \wedge {}^*f(x_1) \in \mathrm{cpt}({}^*Y)) \Rightarrow {}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2)$.

For σ-compact spaces Y, we obtain by Theorem 3 two equivalences for the $\mathscr{A}, \mathscr{B}$-measurability. Both equivalences do not hold for arbitrary metric spaces (see Remark 5).

4. Corollary. *Let (Y, δ) be a σ-compact metric space. For a function $f : X \to Y$ the following conditions are equivalent:*

(i) *f is $\mathscr{A}, \mathscr{B}$-measurable.*

(ii) $(x_1 \underset{\mathscr{A}}{\approx} x_2 \wedge {}^*f(x_1) \in \mathrm{ns}({}^*Y)) \Rightarrow {}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2)$.

(iii) $(x_1 \underset{\mathscr{A}}{\approx} x_2 \wedge {}^*f(x_1) \in \mathrm{cpt}({}^*Y)) \Rightarrow {}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2)$.

Proof. Direct consequence of $\mathscr{B} = \sigma(\mathscr{C})$, Theorem 3 and (I) $\Longleftrightarrow$ (II).

One can apply Corollary 4 to $Y = \mathsf{R}^n$. Observe that in this case $\mathrm{fin}({}^*\mathsf{R}^n) = \mathrm{ns}({}^*\mathsf{R}^n) = \mathrm{cpt}({}^*\mathsf{R}^n)$.

5. Remark. In Corollary 4 condition (i) always implies (ii) and (iii) for arbitrary metric spaces. However, even for locally-compact metric spaces, (ii) (and hence (iii)) do not imply (i): Consider $X := Y := \mathsf{R}$ and let δ be the discrete metric on R. Let f be the identity function on R and $\mathscr{A} := \{A \subset \mathsf{R} : A \text{ or } \mathsf{R} \setminus A \text{ is countable}\}$. Then $\mathscr{B} = \mathscr{P}(\mathsf{R}), \mathrm{ns}({}^*Y) = \mathsf{R}$ and f is not $\mathscr{A},\mathscr{B}$-measurable. However (ii) holds, as $x_1 \underset{\mathscr{A}}{\approx} x_2$ and ${}^*f(x_1) = x_1 \in \mathsf{R}$ implies $x_1 = x_2$.

2. Proof of the Results

The following Lemma is the crucial tool for the proof of our main results. In the proof of Lemma 6 and Theorem 1 we use that there exists a hyperfinite ${}^*\mathscr{A}$-partition of ${}^*X$, say P_h, which refines each finite standard ${}^*\mathscr{A}$-partition (see [4]). Then $x_1 \underset{\mathscr{A}}{\approx} x_2$ if $x_1, x_2 \in E \in \mathsf{P}_h$.

6. Lemma. *Let $C \subset Y$ be closed. Then $f^{-1}(C) \in \mathscr{A}$ if:*

$$\text{(C)} \qquad (x_1 \underset{\mathscr{A}}{\approx} x_2 \wedge {}^*f(x_1) \in {}^*C) \Rightarrow {}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2).$$

Proof. Let $n \in \mathsf{N}$. Then, using (C), the existence of P_h and backwards transfer, it follows that there exists a finite $\mathscr{A}$-measurable partition $\{E_1^{(n)}, \ldots, E_{k(n)}^{(n)}\}$ of X with

$$(\forall x_1, x_2 \in E_i^{(n)})(f(x_1) \in C \Rightarrow \delta(f(x_1), f(x_2)) \leq 1/n).$$

Hence there exists $I_n \subset \{1, \ldots, k(n)\}$ with

$$\text{(1)} \qquad f^{-1}(C) \subset \bigcup_{i \in I_n} E_i^{(n)} \subset \{x \in X : \delta(f(x), C) \leq 1/n\}.$$

As C is closed, we have $f^{-1}(C) = \bigcap_{n \in \mathsf{N}} \{x \in X : \delta(f(x), C) \leq 1/n\}$; since $\bigcup_{i \in I_n} E_i^{(n)} \in \mathscr{A}$, we obtain $f^{-1}(C) \in \mathscr{A}$ by (1).

Proof of Theorem 1. (i) $\Rightarrow$ (ii): Let $x_1 \underset{\mathscr{A}}{\approx} x_2$ and $\varepsilon \in \mathsf{R}_+$. As f is $\mathscr{A},\mathscr{B}$-measurable we have ${}^*f(x_1) \underset{\mathscr{B}}{\approx} {}^*f(x_2)$ (use (I) $\Longleftrightarrow$ (II)). Since $f(X)$ is totally bounded, there exists $y \in Y$ with ${}^*f(x_1) \in {}^*(U_\varepsilon(y))$, where $U_\varepsilon(y) = \{z \in Y : \delta(z, y) < \varepsilon\}$ (see e.g. 24.9(ii) of [3]). As $U_\varepsilon(y) \in \mathscr{B}$, we obtain ${}^*f(x_2) \in {}^*(U_\varepsilon(y))$. Hence ${}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2)$.

(ii) $\Rightarrow$ (i): Let $C \subset Y$ be a closed set. Then (ii) implies that condition (C) of Lemma 6 is fulfilled. Hence $f^{-1}(C) \in \mathscr{A}$, whence f is $\mathscr{A},\mathscr{B}$-measurable.

Let $\varepsilon \in \mathsf{R}_+$. Then, using (ii), the existence of P_h and backwards transfer, it follows that there exists a finite $\mathscr{A}$-measurable partition $\{E_1, \ldots, E_k\}$ of X with

$$(x_1, x_2 \in E_i) \Rightarrow \delta(f(x_1), f(x_2)) \leq \varepsilon.$$

Hence $f(X)$ ist totally bounded.

PROOF OF THEOREM 2. (i) $\Rightarrow$ (ii): Let $x_1 \underset{\mathscr{A}}{\approx} x_2$ and ${}^*f(x_1) \in \mathrm{fin}({}^*Y)$. Then there exists a bounded set $B \subset f(X)$ with ${}^*f(x_1) \in {}^*B$. By (i) B is totally bounded and hence ${}^*f(x_1) \in \mathrm{pns}({}^*Y)$ (see 24.9(ii) of [3]). Let $\varepsilon \in \mathsf{R}_+$. Then there exists $y_1 \in Y$ with ${}^*f(x_1) \in {}^*(U_\varepsilon(y_1))$. As f is $\mathscr{A},\mathscr{B}$-measurable and $x_1 \underset{\mathscr{A}}{\approx} x_2$ we obtain ${}^*f(x_2) \in {}^*(U_\varepsilon(y_1))$ and hence ${}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2)$.

(ii) $\Rightarrow$ (i): The $\mathscr{A},\mathscr{B}$-measurability follows from Lemma 6 applied to all bounded and closed sets C.

Let $B \subset f(X)$ be a bounded set. It remains to show that B is totally bounded. There exists $X_0 \subset X$ with $B = f(X_0)$. Let $f_0 := f|X_0$. Then we have for $x_1, x_2 \in {}^*X_0$:

$$x_1 \underset{\mathscr{A}\cap X_0}{\approx} x_2 \Rightarrow x_1 \underset{\mathscr{A}}{\approx} x_2 \underset{\text{(ii)}}{\Rightarrow} {}^*\delta({}^*f(x_1), {}^*f(x_2)) \approx 0 \Rightarrow {}^*\delta({}^*f_0(x_1), {}^*f_0(x_2)) \approx 0.$$

Hence by Theorem 1 the set $B = f(X_0) = f_0(X_0)$ is totally bounded.

PROOF OF THEOREM 3. (i) $\Rightarrow$ (ii): Let $x_1 \underset{\mathscr{A}}{\approx} x_2$ and ${}^*f(x_1) \in {}^*C$ for some $C \in \mathscr{C}$. Since ${}^*C \subset \mathrm{ns}({}^*Y)$, there exists $y_1 \in Y$ with ${}^*f(x_1) \underset{\delta}{\approx} {}^*y_1$. Let $\varepsilon \in \mathsf{R}_+$ and put $C_\varepsilon := \{y \in Y : \delta(y, y_1) \leq \varepsilon\}$. Then $C \cap C_\varepsilon$ is compact and ${}^*f(x_1) \in {}^*(C \cap C_\varepsilon)$. Since $x_1 \underset{\mathscr{A}}{\approx} x_2$ and f is $\mathscr{A},\sigma(\mathscr{C})$-measurable, we obtain that ${}^*f(x_2) \in {}^*(C \cap C_\varepsilon)$ (use (I) $\Longleftrightarrow$ (II) with $\sigma(\mathscr{C})$ instead of $\mathscr{B}$). Hence ${}^*\delta({}^*f(x_1), {}^*f(x_2)) \leq 2\varepsilon$, whence ${}^*f(x_1) \underset{\delta}{\approx} {}^*f(x_2)$.

(ii) $\Rightarrow$ (i): Apply Lemma 6 to all compact sets C.

We thank the referee for improving the presentation. The ideas of the referee led to clearer and much shorter proofs of our main Lemma 6 and Theorem 1.

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