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HOMOMORPHISMS FROM C^*-ALGEBRAS OF CONTINUOUS TRACE*

HUAXIN LIN

Abstract

Let A be a unital C^*-algebra of continuous trace, let B be a unital C^*-algebra and let $\phi, \psi : A \to B$ be two homomorphisms. We show that ϕ and ψ are stably approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(A, B)$. In the case that B is a purely infinite simple C^*-algebra, the above condition implies that ϕ and ψ are approximately unitarily equivalent.

0. Introduction

Earliest results about classifying homomorphisms from one given C^*-algebra to another is the Brown-Douglass-Fillmore theory ([BDF1] and [BDF2]) of 1970's which classifies monomorphisms from an abelian C^*-algebra $C(X)$ into the Calkin algebra (up to unitarily equivalence). The BDF-theory has a profound impact on operator theory, operator algebras, K-theory and other subjects of mathematics. More recently, there is a renew interest to classify homomorphisms from one given C^*-algebra A into another B. The question whether a C^*-algebra B of real rank zero has the so called (FN) property is in fact to ask when homomorphisms from $C(X)$ into B, where X is a compact subset of the plane, can be approximated by homomorphisms with finite dimensional range. Early results in this line are in [Ln1] and [Ln3]. For example, it is shown in [Ln1] that a unitary (corresponding to a homomorphism from $C(S^1)$ into A) $u \in B$ which is connected to the identity of B is approximated by unitaries with finite spectrum. More general and much better results about homomorphisms from $C(X)$ into a C^*-algebra B (of real rank zero) can be found in [EGLP], [Ln4], [D1], [GL1], [LP2], [EG] and [Ln7], etc. For the case that both A and B are purely infinite simple C^*-algebras, see [Ro1], [LP1], [LP2], [Ro2] and [Ph3]. These results play important roles in the theory classification of C^*-algebras. Some of these results are also used to solve a long standing problem in linear algebra: whether a pair

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of almost commuting selfadjoint matrices is close to a pair of commuting selfadjoint matrices (see [Ln5]).

Most of the above mentioned results (other than the case that A is purely infinite), A is assumed to be $C(X)$, or $PM_n(C(X))P$, where X is a compact metric space and P is a projection in $M_n(C(X))$ (or direct limits of these algebras). Consider $A = (A(t), \Gamma)$, a unital locally trivial continuous field of C^*-algebras over a compact Huasdorff space X with $A(t) \cong M_n$. If $A = PM_n(C(X))P$, then the so called Dixmier-Douady class $\delta(A) = 0$. Furthermore, A is just a corner (a unital hereditary C^*-subalgebra) of $M_n(C(X))$. As in previous results, the study the homomorphisms from $PM_n(C(X))P$ can alway be eventually reduced to the study homomorphisms from $M_n(C(X))$. Suppose that X is connected and each $A(t) \cong M_k$. Fix a point $\xi \in X$, set

$$I = \{a \in \Gamma : a(\xi) = 0\}.$$

We obtain a short exact sequence

$$0 \to I \to A \to M_k \to 0.$$

When $\delta(A) \neq 0$, the above sequence is not splitting. In fact, if $\delta(A) \neq 0$, it gives a nonzero element in $\mathrm{Ext}(M_k, I)$. On the other hand, if $A = PM_n(C(X))P$, then $A \otimes \mathcal{K} \cong C(X) \otimes \mathcal{K}$. So the six-term exact sequence breaks into

$$0 \to K_0(I) \to K_0(A) \to \mathbf{Z} \to 0 \quad \text{and} \quad 0 \to K_1(I) \to K_1(A) \to 0.$$

Since $\mathbf{Z}$ is free, one sees that the short exact sequence of C^*-algebras gives the zero element in $\mathrm{Ext}(M_k, I)$.

In this paper, we will consider the case that $\delta(A) \neq 0$. In fact, we will consider an even larger class of C^*-algebras, namely, unital C^*-algebras of continuous trace. We show that, if A is a unital C^*-algebra of continuous trace, B is a unital C^*-algebra, ϕ, $\psi : A \to B$ are two homomorphisms, if also $[\phi] = [\psi]$ in $\mathrm{KL}(A, B)$, then ϕ and ψ are stably approximately unitarily equivalent (see 2.11). In the case that B is purely infinite, we show that, with the assumption that $[\phi] = [\psi]$ in $\mathrm{KL}(A, B)$, there is a sequence of unitaries $\{u_n\} \subset B$ such that

$$\phi(f) = \lim_{n \to \infty} u_n^* \psi(f) u_n$$

for all $f \in A$ (3.4).

These results have interesting application in study of classification of C^*-algebras. These applications will appear in a subsequent paper.

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1. Preparation

1.1. Let A be a C^*-algebra of type I and let $x \in A_+$. For each irreducible representation $\pi \in \hat{A}$, define $\hat{x} : \hat{A} \to [0, \infty]$ by $\hat{x}(\pi) = Tr(\pi(x))$, where Tr is the canonical trace. The positive element x is said to have continuous trace if $\hat{x} \in C^b(\hat{A})$. Recall that A is said to have continuous trace if the set of elements with continuous trace is dense in A_+. A C^*-algebra A is said to be homogeneous of rank n, if $\pi(A) \cong M_n$ for every irreducible representation π of A. The following proposition says that a separable unital C^*-algebra of continuous trace is a finite direct sum of homogeneous C^*-algebras of finite rank.

1.2. Proposition. *Let A be a separable unital C^*-algebra of continuous trace. Then $A = \oplus_{i=1}^n B_i$, where each $B_i = (B_i(t), \Gamma_i)$ is a unital locally trivial continuous field of C^*-algebras over a compact Hausdorff space $\hat{X}_i$ and $B_i(t) = M_{n(i)}$.*

Proof. The set of positive elements with continuous trace is the positive part of a dense hereditary ideal. Let I_0 be the dense ideal and let $P(A)$ be the Pedersen ideal. Then $P(A) \subset I_0$. Since A has an identity, it follows from 5.6.3 in [Pd] that $P(A) = A$. Thus $1_A \in I_0$. In particular, 1_A has continuous trace and all irreducible representations have dimension less than some positive integer. Thus $\hat{A} = \oplus_{i=1}^n V_i$, where V_i are clopen subsets of $\hat{A}$ and dimension of each irreducible representation $\xi \in V_i$ is the same. It is then immediate that $A = \oplus_{i=1}^n B_i$, where every irreducible representation of B_i has the same dimension $k(i)$. Furthermore, since A is unital, each B_i is unital. It follows from 3.2 in [Fe] that B_i is a separable locally trivial continuous field of $k(i) \times k(i)$ matrix algebras over compact Hausdorff space $\hat{B}_i$.

We refer the reader to Chapter 10 of [Dix] for other information about continuous fields of C^*-algebras. It follows from 3.2 in [Fe] that a homogeneous C^*-algebra of rank n is a locally trivial continuous field $(A(t), \Gamma)$ of C^*-algebras over the compact Hausdorff space $\hat{A}$ and $A(t) \cong M_n$.

1.3. Definition. Let $A = (A(t), \Gamma)$ be a continuous field of C^*-algebras over X and $B = (B(t), \Theta)$ be a continuous field of C^*-algebras over Y with $Y \subset X$. A homomorphism $h : A \to B$ is said to be *spatial*, if, for each $t \in Y$, there is a homomorphism $h_t : A(t) \to B(t)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \pi_t \downarrow & & \downarrow \pi_t \\ A(t) & \xrightarrow{h_t} & B(t) \end{array}$$

where π_t denotes both maps from A to $A(t)$ and from B to $B(t)$.

1.4. Defintion. Let A be as in 1.3 with each $A(t) \cong M_{n(t)}$ and B is a C^*-algebra. For each $t \in X$, we denote by $\pi_t : A \to M_{n(t)}$ the irreducible representation corresponding to the point t. To save the notation, we will often write $f(t)$ for $\pi_t(f)$. Suppose that $t_1, t_2, ..., t_n \in X$ are fixed points in X and $p_1, p_2, ..., p_n \in B$ are mutually orthogonal projections in B. Define a homomorphism $h : A \to B$ by $h(f) = \sum_{i=1}^n \gamma_i(f(t_i))p_i$ for $f \in A$, where $\gamma_i : M_{n(t_i)} \to p_iBp_i$ is a homomorphism. Such a homomorphism will be called a point-evaluation. If A is a homogeneous C^*-algebra of rank n, then the existence of a unital point-evaluation $h : A \to B$ implies that there is a unital monomorphism $\phi : M_n \to B$. Note that a homomorphism $\phi : A \to B$ has finite dimensional range if and only if ϕ is a point-evaluation.

1.5. Lemma (cf. Lemma D in [BDR]). *Let X be a connected finite CW complex of dimension d. Let $\eta \in K^0(X)$, and suppose that the rank of η is at least $d/2$,* (i.e., $\eta = [F_1] - [F_2]$, *where* $\dim(F_1) - \dim(F_2) \geq d/2$). *Then there is a vector bundle E over X, unique up to isomorphism, such that $\eta = [E]$.*

Proof. We first show the uniqueness. Suppose that $[E] = [F]$ with $\dim(E), \dim(F) \geq d/2$. Then there is k such that $E \oplus (X \times \mathsf{C}^k) \cong F \oplus (X \times \mathsf{C}^k)$. The cancellation theorem for the vector bundle (see Theorem 9.1.5 [Hu]) implies that $E \cong F$.

To find such E, write $\eta = [E_1] - [E_2]$, with E_1, E_2 being vector bundles over X. Then $\dim(E_1) - \dim(E_2) \geq d/2 \geq (d-1)/2$. It follows that $E_1 \cong E \oplus E_2$ for some vector bundle E. (see 1.5 (3) of [Ph2]). So $\eta = [E]$.

1.6. Lemma. *Let X be a connected finite* CW*-complex and let E be a vector bundle over X. Then there exists a nonzero vector bundle F over X such that $E \otimes F$ is trivial.*

Proof. Let $n = \dim(E)$, and let $\eta = n - [E] \in K^0(X)$. By Corollary 3.1.6 of [At], there is an integer $r > 0$ such that $\eta^r = 0$ (see p.120 of [At] for the definition of $K_1(X)$ there). Define

$$\sigma = n^{r-1} + n^{r-2}\eta + \cdots + n\eta^{r-2} + \eta^{r-1} \in K^0(X).$$

Using the fact that $\eta^r = 0$, it is easy to check that $(n - \eta)\sigma = n^r$.

Choose an integer k such that $kn^{r-1} \geq \frac{\dim(X)}{2}$. Then $\dim(k\sigma) = kn^{r-1}$, so by

1.5 there exists a vector bundle F with $[F] = k\sigma$ in $K^0(X)$. Also, since $[E] = n - \eta$, we have

$$[E \otimes F] = (n - \eta) \cdot k\sigma = kn = [X \times \mathsf{C}^{kn}]$$

and $knr \geq \frac{\dim(X)}{2}$. So, by 1.5, we obtain $E \otimes F \cong X \times \mathsf{C}^{kn^r}$, as desired.

1.7. Lemma. *Let X be a finite* CW *complex, and let A be a locally trivial fiber bundle over X with fiber M_n and the structure group* $\mathrm{Aut}(M_n)$. *Then there exists an integer $r > 0$ and a unital homomorphism $\Phi_1 : A \to X \times M_r$ and $\Phi_2 : X \times M_r \to M_r \otimes A$ such that $\Phi_2 \circ \Phi_1(a) = 1 \otimes a$ for all $a \in A$.*

Proof. By considering each summand separately, without loss of generality, we may assume that X is connected. We find a locally trivial bundle B with fiber M_k for some k such that $B \otimes A$ (fiberwise tensor product) is trivial. Once this is done, we take isomorphisms $\beta : B \otimes A \to X \times M_r$ (where $r = kn$) and $\bar{\beta} : A \otimes B \to X \times M_r$, and define

$$\Phi_1(a) = \beta(1 \otimes a) \qquad \text{and} \quad \Phi_2(x) = (\bar{\beta} \otimes \mathrm{id}_A)(1 \otimes \beta^{-1}(x))$$

where $a \in A$, $x \in X \times M_r$ and $1 \otimes \beta^{-1}(x) \in A \otimes (B \otimes A)$ $(= (A \otimes B) \otimes A)$. Clearly Φ_1 and Φ_2 satisfy the required conditions.

To find B, we proceed as follows. Let A^{op} be the opposite bundle to A : the multiplication in the algebra is reversed (i.e., $x \cdot y = yx$). Further, regard A as an ordinary complex vector bundle by forgetting the structure, and (using 1.7) find a nonzero vector bundle E such that $E \otimes A$ is trivial. Then set $B = L(E) \otimes A^{\mathrm{op}}$, where $L(E)$ is the bundle whose fiber $L(E)_x$ is just $L(E_x)$, where $L(Y)$ is the set of all of linear maps on vector space Y. To complete the proof, we first observe that there is an isomorphism $A^{\mathrm{op}} \otimes A \cong L(A)$, where on the right A is regarded as an ordinary complex vector bundle. The representation is defined by $(a \otimes b)(\xi) = b\xi a$ (the multiplication on the right side of the equation is the multiplication in A). So

$$(L(E) \otimes A^{,\mathrm{op}}) \otimes A \cong L(E) \otimes L(A) \cong L(E \otimes A) \cong X \otimes M_n,$$

Since $E \otimes A \cong X \times \mathsf{C}^r$ for some integer r.

The proof of 1.7 (and 1.6, 1.5) were supplied by N. Chris Phillips. We would like to express our gratitude for his proof.

1.8. Lemma. *Let X be a finite* CW *complex and A be a unital homogeneous C^*-algebra with finite rank n and with spectrum $\hat{A} = X$. Then there are an integer $r > 1$, unital spatial homomorphisms $\Phi_1 : A \to M_r(C(X))$, $\Phi_2 : M_r(C(X)) \to M_r(A)$ and $\Phi_3 : A \to M_{r-1}(A)$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & M_r(A) \\ {\scriptstyle \Phi_1} \searrow & & \nearrow {\scriptstyle \Phi_2} \\ & M_r(C(X)) & \end{array}$$

where $\phi = \mathrm{diag}(\mathrm{id}_A, \Phi_3)$, $\Phi_3 = \mathrm{diag}(\mathrm{id}_A, \cdots, \mathrm{id}_A)$ *and* $\mathrm{id}_A : A \to A$ *is the identity.*

Proof. It follows from [Fe] that A is a locally trivial bundle with fiber M_n. So 1.8 follows immediately from 1.7.

1.9. Corollary. *Let X and A be as in* 1.8. *Then, for any $\varepsilon > 0$ and any finite subset $\mathscr{F} \subset A$, there are an integer $k > 1$, a unital homomorphism $\psi : A \to M_k(A)$ and a unital homomorphism $\phi_0 : A \to M_{k+1}(A)$ with finite dimensional range such that*

$$\|f \oplus \psi(f) - \phi_0(f)\| < \varepsilon$$

for all $f \in \mathscr{F}$. Moreover ψ can be chosen so that $\mathrm{diag}(\mathrm{id}_A, \psi)$ *is homotopy to a point-evaluation $h : A \to M_{k+1}(A)$.*

Proof. By 1.8, there are an integer $r > 1$, homomorphisms $\Phi_1 : A \to M_r(C(X))$, $\Phi_2 : M_r(C(X)) \to M_r(A)$ and $\Phi_3 : A \to M_{r-1}(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & M_r(A) \\ {\scriptstyle \Phi_1} \searrow & & \nearrow {\scriptstyle \Phi_2} \\ & M_r(C(X)) & \end{array}$$

where $\phi = \mathrm{diag}(\mathrm{id}_A, \Phi_3)$.

By 1.2 in [D1] (see also [EG]), for any $\varepsilon > 0$ and any finite subset $\mathscr{G}$, there is an integer $m > 1$, a unital homomorphism $\sigma : M_r(C(X)) \to M_{(m-1)r}(C(X))$ and a unital homomorphism $\tau : M_r(C(X)) \to M_{mr}(C(X))$ with finite dimensional range such that

$$\|g \oplus \sigma(g) - \tau(g)\| < \varepsilon$$

for all $g \in \mathscr{G}$. Choose

$$\mathscr{G} = \{\Phi_1(f) : f \in \mathscr{F}\}.$$

Thus

$$\|\mathrm{diag}(\phi(f), (\Phi_2 \otimes \mathrm{id}_{m-1}) \circ \sigma \circ \Phi_1(f)) - (\Phi_2 \otimes \mathrm{id}_m) \circ \tau \circ \Phi_1(f)\| < \varepsilon$$

for all $f \in \mathscr{F}$. Set $k = mr - 1$, $\psi = \mathrm{diag}(\Phi_3, (\Phi_2 \otimes \mathrm{id}_{m-1}) \circ \sigma \circ \Phi_1)$ and $\phi_0 = (\Phi_2 \otimes \mathrm{id}_m) \circ \tau \circ \Phi_1$. One checks that so chosen k, ψ and ϕ_0 satisfy the first part of the requirements.

To conclude the last part of the lemma, we note that from the proof of 1.2

in [D1] one can choose σ so that $\mathrm{diag}(\mathrm{id}_{M_r(C(X))}, \sigma)$ is homotopy to a point-evaluation.

1.10. Definition (see [D1]). *A C^*-algebra A is said to have property* (H)*, if, for any finite subset $\mathscr{F} \subset A$ and any $\varepsilon > 0$, there exist $k \in \mathbb{N}$, a homomorphism $h : A \to M_k(A)$ and a homomorphism $h_0 : A \to M_{k+1}(A)$ with finite dimensional range such that*

$$\|\mathrm{diag}(a, h(a)) - h_0(a)\| < \varepsilon$$

for all $a \in \mathscr{F}$.

By 1.9, every unital C^*-algebra of continuous trace has property (H).

1.11. Let X be a connected finite CW complex and let $A = ((A(t), \Gamma)$ be a locally trivial continuous field of C^*-algebras over X, where each $A(t) \cong M_n$. Suppose that $\xi_0 \in X$. Let

$$I = \{x \in \Gamma : x(\xi_0) = 0\}.$$

Then I is an ideal of A.

1.12. Lemma. *Let I be as above and let $j_m : M_m(I) \to I \otimes \mathscr{K}$ by identifying I with $I \otimes e_{11}$. Then there is a positive integer k and there is a homomorphism $i^* : I \to I \otimes M_k$ such that $j \circ \mathrm{diag}(\mathrm{id}_I, i^*) : I \to I \otimes \mathscr{K}$ is null-homotopic.*

Proof. Let r, Φ_1, Φ_2, and Φ_3 be as in 1.8. By [EG], there is an integer $s\,(= 2\dim(X) + 1)$ and a homomorphism $h : M_r(C(X)) \to M_{rs}(C(X))$ such that $\mathrm{diag}(\Phi_2, (\Phi_2 \otimes 1_s) \circ h) : M_r(C_0(X \setminus \xi_0)) \to M_{r(s+1)}(I)$ is null-homotopic. Set $\Phi_4 = [(\Phi_2 \otimes 1_s) \circ h] \circ \Phi_1$. Then $\mathrm{diag}(\mathrm{id}_I, \Phi_3, \Phi_4) : I \to M_{r(s+1)}(I)$ is null-homotopy.

1.13. Definition. Let A and B be two C^*-algebras. Following [DL], we denote by $[[A, B]]$ the set of homotopy classes of asymptotic morphisms from A into B.

From above and a result in [DL], we have the following.

1.14. Lemma. *Let I be as in* 1.11. *Then, for any separable stable C^*-algebra,*

$$[[I, B]] \cong KK(I, B).$$

In particular, if $\phi_1, \phi_2 : I \to B$ are two asymptotic contractive completely positive linear morphisms with $[\phi_1] = [\phi_2]$ in $KK(I, B)$ then there is an asymptotic contractive completely positive linear morphism $\Phi : I \to C([0,1]) \otimes B$ such that $\pi_0 \circ \Phi = \phi_1$ and $\pi_1 \circ \Phi = \phi_2$, where $\pi_t : C([0,1]) \otimes B \to B$ is the evaluation at $t \in [0,1]$.

Proof. This follows from 1.12 and Theorem 4.3 in [DL] immediately.

2. Stably approximately unitarily equivalent homomorphisms

The purpose of this section is to prove Theorem 2.11. We will show that two homomorphisms ϕ, $\psi: A \to B$ are stably approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $\mathrm{KL}(A, B)$, where A is a unital C^*-algebra of continuous trace. The proof of the "only if" part follows from an argument of Rørdam ([Ro2]). A proof of the version that we need here is given by Dadarlat (see the proof "(b) $\Rightarrow$ (a)" in p. 126 of [D1]). So the task here is to prove the "if" part of Theorem 2.11.

The group $\mathrm{KL}(A, B)$ first appeared in [Ro2]. A special version of it has been used in [Br]. We will avoid using the Universal Coefficient Theorem.

2.1. Definition. Let A be a separable nuclear C^*-algebra and B be a σ-unital C^*-algebra. Identify $\mathrm{KK}(A, B)$ with $\mathrm{KK}^1(A, SB)$, where SB is the suspension of B, i.e., $SB = C_0(\mathbb{R}) \otimes B$. Then identify $\mathrm{KK}^1(A, SB)$ with $\mathrm{Ext}(A, SB)$. Let $\mathrm{PK}(A, B)$ be those extensions $0 \to SB \otimes \mathcal{K} \to E \to A \to 0$ in $\mathrm{Ext}(A, SB)$ such that its six-term exact sequence break into the following two short exact sequences

$$0 \to K_0(SB) \to K_0(E) \to K_0(A) \to 0 \text{ and } 0 \to K_1(SB) \to K_1(E) \to K_1(A) \to 0$$

which are *pure* extensions (i.e., a torsion element in $K_i(A)$ lifts to a torsion element in $K_i(E)$ with the same order). Note that $K_i(SB) = K_{i+1}(B)$ and $\mathrm{PK}(A, B)$ is a subgroup. We define $\mathrm{KL}(A, B) = \mathrm{KK}^1(A, SB)/\mathrm{PK}(A, B)$.

It is worth to point out that in the case that $K_i(A)$ are torsion free $\mathrm{KL}(A, B) = \mathrm{Hom}(K_*(A), K_*(B))$. So Theorem 2.11 states, in this special case, ϕ and ψ are stably approximately unitarily equivalent, if (and only if) ϕ and ψ induce the same maps from $K_i(A)$ into $K_i(B)$, $i = 0, 1$.

2.2. Lemma (Lemma 1.4 in [D1]). *Let A be a C^*-algebra with property* (H). *Let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There are $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ such that if B is any unital C^*-algebra and $\phi_0, \phi_1, \ldots, \phi_n$ are finitely many δ-$\mathcal{G}$-multiplicative contractive completely positive linear morphisms, then there exist $k \in \mathbb{N}$, a homomorphism $h: A \to M_k(B)$ with finite dimensional range and a unitary $u \in U(M_{k+1}(B))$ such that*

$$\|u^* \mathrm{diag}(\phi_0(f), h(f))u - \mathrm{diag}(\phi_n(f), h(f))\|$$
$$< \varepsilon + \max_{f \in \mathcal{F}} \max_{0 \le j \le n-1} \|\phi_{j+1}(f) - \phi_j(f)\|$$

for all $f \in \mathcal{F}$.

The following Lemma is a version of 1.5 in [D1]. This type of argument first appeared in [Ph1] in a special case and developed to this form in [EGLP] (see 3.14 in [EGLP]).

2.3. LEMMA. *Let X be a finite connected* CW *complex with the base point ξ, let $A = (A(t), \Gamma)$ be a unital locally trivial continuous field of C^*-algebras over X with $A(t) \cong M_n$ and let*

$$I = \{a \in \Gamma : a(\xi) = 0\}.$$

Let B be a unital C^*-algebra and let $\{\phi_t\}$, $\{\psi_t\} : I \to B \otimes M_\infty$ be two asymptotic contractive completely positive linear morphisms such that images of ϕ_t and ψ_t are contained in $B \otimes M_{\alpha(t)}$ for some map $\alpha : [1, \infty) \to \mathsf{N}$. Suppose that $[[\phi_t]] = [[\psi_t]]$ in $\mathrm{KK}(I, B)$. Then, for any finite subset $\mathcal{F} \subset I$ and any $\varepsilon > 0$, there are $t_0 \geq 1$ and maps $\beta, k : [1, \infty) \to \mathsf{N}$ such that, for any $t \geq t_0$, there exist a unitary $u \in U(B \otimes M_{(\beta(t)+1)k(t)})$ and a point-evaluation $h : I \to B \otimes M_{\beta(t)k(t)}$ such that

$$\|u^* \mathrm{diag}(\phi_t(f), h(f))u - \mathrm{diag}(\psi_t(f), h(f))\| < \varepsilon$$

for all $f \in \mathcal{F}$.

PROOF. By Lemma 1.14, we find an asymptotic morphism $\{\Phi_t\} : I \to C([0,1]) \otimes (B \otimes \mathcal{K})$ such that $\pi_0 \circ \Phi_t = \phi_t$ and $\pi_1 \circ \Phi_t = \psi_t$, where $\pi_t : C([0,1]) \otimes (B \otimes \mathcal{K}) \to B \otimes \mathcal{K}$ is the evaluation at point t. Let $\mathcal{G}$ be a finite subset of I and δ be a positive number. There is $t_0 \geq 1$ such that Φ_t is $\delta/2$-$\mathcal{G}$-multiplicative for $t \geq t_0$. Fix $t \geq t_0$. We can find a finitely many points $0 = s_0 < s_1 < s_2 < \cdots < s_m = 1$ such that

$$\max_{0 \leq j \leq m-1} \|\pi_{s_j} \circ \Phi_t(f) - \pi_{s_{j+1}} \circ \Phi_t(f)\| < \varepsilon/4$$

for all $f \in \mathcal{G}$.

Let $\{e_{ij}\}$ be the matrix unit for $\mathcal{K}$ and $e_k = \sum_{i=1}^k 1_B \otimes e_{ii}$. Then $\{e_k\}$ forms an approximate identity for $B \otimes \mathcal{K}$. There is, (for that fixed t), a sufficiently large k such that $k \geq \alpha(t)$ and

$$\|\pi_{s_j} \circ \Phi_t(f) - e_k(\pi_{s_j} \circ \Phi_t(f))e_k\| < \min(\varepsilon/4, \delta/2)$$

for all $f \in \mathcal{G}$, $j = 1, 2, ..., m-1$. Set $L_j(f) = e_k(\pi_{s_j} \circ \Phi_t(f))e_k$ for all $f \in I$, $j = 1, 2, ..., m_1$ and $L_0 = \phi_t$ and $L_1 = \psi_t$. Then L_j are δ-$\mathcal{G}$-multiplicative contractive completely positive linear morphisms and

$$\max_{0 \leq j \leq m_1} \|L_j(f) - L_{j-1}(f)\| < \varepsilon/2$$

for all $f \in \mathcal{G}$. It follows from Lemma 1.9 and Lemma 2.2 that there is a point-evaluation $h : I \to B \otimes M_{\beta(t) \times k}$ and a unitary $u \in U(B \otimes M_{(\beta(t)+1)k})$ such that

$$\|u^* \mathrm{diag}(\phi_t(f), h(f))u - \mathrm{diag}(\psi_t(f), h(f))\| < \varepsilon$$

for all $f \in \mathscr{F}$, provided that $\mathscr{G}$ is sufficiently large and δ is sufficiently small (as required by Lemma 2.2).

2.4. Let X be a connected finite CW complex and $A = (A(t), \Gamma)$ be a unital locally trivial continuous field of C^*-algebras over X, where $A(t) \cong M_n$. Fix a point $\xi \in X$. Let

$$I = \{a \in \Gamma : a(\xi) = 0\}.$$

We have the following short exact sequence:

$$0 \to I \to A \to M_n \to 0.$$

The key difference from the case that $A = PM_n(C(X))P$, where P is a projection in $M_n(C(X))$, and the general case is that the above extension may give a nonzero element in $\mathrm{Ext}(M_n I)$.

We will use π_ξ for the quotient map from A onto M_n which is also the evaluation at point ξ. We choose ξ so that ξ has a closed neighborhood D which is homeomorphic to a finite dimensional disk. Since A is locally trivial, we also assume that $A|_D = M_n(C(D))$.

2.5. Lemma. *Let X and I be as in* 2.4. *For any $\varepsilon > 0$, $\eta > 0$ and finite subset $\mathscr{F}$ in the unit ball of A, there exist $\delta > 0$ and a finite subset $\mathscr{G}$ in the unit ball of I satisfying the following:*

Suppose that B is a unital C^*-algebra, $\phi : A \to B$ is a unital point-evaluation and $\psi : A \to B$ is a unital homomorphism. Suppose also that

$$\|\psi(g) - \phi(g)\| < \delta$$

for all $g \in \mathscr{G}$. Then there are a point-evaluations $h_1 : A \to pBp$ for some projection $p \in B$ and an η-$\mathscr{F}$-multiplicative contractive completely positive linear morphism $L : M_n(C(D)) \to (1-p)B(1-p)$ such that

$$\|\psi(f) - h_1 \oplus L \circ s(f)\| < \varepsilon$$

for all $f \in \mathscr{F}$, where D is a closed neighborhood of ξ which is homeomorphic to a finite dimensional closed dick and $s : A \to M_n(C(D))$ is the spatial surjection.

Proof. Let $\mathscr{G}_1 \supset \mathscr{F}$ be a finite subset of the unit ball of A which contains 1_A. Let δ be a positive number to be determined. Choose a neighborhood $O(\xi)$ of ξ such that the closure D of $O(\xi)$ is homeomorphic to a finite dimensional disk and $A|_D = M_n(C(D))$. Let D_1 be a compact subset of $O(\xi)$ such that its interior contains ξ. Let $f_1 \in C_0(X \setminus \{\xi\})$ such that $0 \le f_1 \le 1$, $f_1(t) = 1$ if $t \in X \setminus D$ and $f_1(t) = 0$ if $t \in D_1$. Note that (see 10.5.6 in [Dix]), for any $f \in A$, $f_1(t) \cdot f \in I$. We let $\mathscr{G}_1$ contains $f_1(t) \cdot 1_A$. There exists a compact subset $F \subset X \setminus \{\xi\}$ which contains $X \setminus D_1$. Choose a strictly positive function $b \in C_0(X \setminus \{\xi\})$ such that

$$b(t) = 1$$

for all $t \in F$. It follows from 10.5.6 in [Dix] that $b(t)1_A \in I$. Let $\mathcal{G}$ be a finite subset of the unit ball which contains $\mathcal{G}_1 \cup \{g \cdot b(t) \cdot 1_A : g \in \mathcal{G}_1\}$.

Now let ψ and ϕ be as described in the lemma (with $\mathcal{G}$ as above and with δ to be determined later). We may write, for all $a \in A$,

$$\phi(a) = \sum_{i=1}^{m} \gamma_i(a(t_i))p_i$$

where $t_1, t_2, \ldots, t_m \in X$, $p_1, p_2, \ldots, p_m$ are mutually orthogonal projections with $\sum_{i=1}^{n} p_i = 1_B$ and $\gamma_i : M_n \to p_i B p_i$. We have

$$\|\psi(g \cdot b(t) \cdot 1_A) - \phi(g \cdot b(t) \cdot 1_A)\| < \delta \quad \text{and} \quad \|\psi(b(t) \cdot 1_A) - \phi(b(t) \cdot 1_A)\| < \delta$$

for all $g \in \mathcal{G}$. Thus

$$\|[\psi(g) - \phi(g)]\phi(b(t) \cdot 1_B)\| < 2\delta$$

for all $g \in \mathcal{G}$. Set

$$P = \sum_{t_i \in F} p_i$$

Thus

$$\|[\psi(g) - \phi(g)]P\| \leq \|[\psi(g) - \phi(g)] \sum_{i=1}^{m} b(t_i)p_i\| \|P\| < 2\delta$$

for all $g \in \mathcal{G}$, since $b(t) = 1$ for $t \in F$. Note that $\phi(g)P = P\phi(g)$ for all $g \in I$. So

$$\|P\psi(g) - \psi(g)P\| < 4\delta$$

for all $g \in \mathcal{G}$. We also have

$$\|\psi(f_1(t) \cdot 1_A) - \phi(f_1(t) \cdot 1_A)\| < \delta.$$

Note that $P\phi(f_1(t) \cdot 1_A) = \phi(f_1(t) \cdot 1_A)$. So

$$\|P\psi(f_1(t) \cdot 1_A) - \psi(f_1(t) \cdot 1_A)\| < 2\delta.$$

Therefore, if $g_0 = (1 - f_1) \cdot 1_A$,

$$\|(1 - P)\psi(g_0) - (1 - P)\| < 4\delta. \tag{1}$$

By [CE], there is a contractive completely positive linear map $\sigma : M_n(C(D)) \to A$ such that $s \circ \sigma = \mathrm{id}_{M_n(C(D))}$, where $s : A \to A|_D = M_n(C(D))$ is the quotient map. Define $L : M_n(C(D)) \to (1 - P)B(1 - P)$ by $L(f) = (1 - P)\psi(\sigma(f))(1 - P)$ for all $f \in M_n(C(D))$. So L is a contractive completely positive linear map. Note that $\sigma \circ s(f)(t) - f(t) = 0$ for all $t \in D$. We also have

$$\begin{aligned}
&\|L(s(f)) - (1-P)\psi(f)(1-P)\| \\
&\quad = \|(1-P)\psi(\sigma \circ s(f))(1-P) - (1-P)\psi(f)(1-P)\| \\
&\quad \le \|(1-P)\psi(\sigma \circ s(f))(1-P) - (1-P)\psi(g_0 \cdot \sigma \circ s(f))(1-P)\| \\
&\quad + \|(1-P)\psi(g_0(\sigma \circ s(f) - f))(1-P)\| + \|(1-P)\psi(g_0 f - f)(1-P)\|
\end{aligned}$$

for all $f \in \mathcal{G}$. By (1) above, the first term is less than 4δ. The second term is zero since $g_0(t) = 0$ for any $t \notin D$. The third term is less than 4δ, again by (1) above. Thus we have

$$\|L(s(f)) - (1-P)\psi(f)(1-P)\| < 8\delta$$

for all $f \in \mathcal{G}$.

Note also

$$\|(1-P)\psi(fg)(1-P) - (1-P)\psi(f)(1-P)\psi(g)(1-P)\| < 4\delta$$

for all $f, g \in \mathcal{G}$. Combining the above two inequalities, we conclude that L is η-$\mathcal{F}$-multiplicative if δ is small enough. Set $h_1(f) = \sum_{t_i \in F} \gamma_i(f(t_i))p_i$ for $f \in A$. Finally,

$$\begin{aligned}
\|\psi(f) - h_1(f) \oplus L \circ s(f)\| &\le \|\psi(f)P - h_1(f)\| + \|\psi(f)(1-P) - L \circ s(f)\| \\
&\le 2\delta + 4\delta + \|(1-P)\psi(f)(1-P) - L \circ s(f)\| < 14\delta
\end{aligned}$$

for all $f \in \mathcal{F}$. We can require that $\delta < \varepsilon/14$.

2.6. Lemma. *Let X be a connected finite* CW *complex, let $A = (A(t), \Gamma)$ be a unital locally trivial continuous field of C^*-algebras over X, where $A(t) = M_n$ and let B be a unital C^*-algebra. Suppose that h_1, $h_2 : A \to B$ are two unital point-evaluations. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist a (unital) point-evaluation $h_0 : A \to M_N(B)$ and a unitary $U \in M_{N+1}(B)$ (for some integer $N > 0$) such that*

$$\|\mathrm{ad}(U) \circ \mathrm{diag}(h_1(f), h_0(f)) - \mathrm{diag}(h_2(f), h_0(f))\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. Write $h_1(f) = \sum_{i=1}^m \gamma_i(f(t_i))p_i$ for all $f \in A$, where $t_i \in X$, $p_1, p_2, \dots, p_m$ are mutually orthogonal projections with $\sum_{i=1}^n p_i = 1_B$ and $\gamma_i : M_n \to p_iBp_i$. Since X is connected and compact and A is locally trivial, it is routine but easy to show that h_1 is homotopy to h_{00}, where $h_{00}(f) = f(t_1)1_B$. Also h_2 is homotopy to h_{00}. Thus, by applying 2.2, the lemma follows.

2.7. LEMMA. *Let X be a compact metric space, let $A = (A(t), \Gamma)$ be a unital locally trivial continuous field of C^*-algebras over X, where $A(t) \cong M_n$. There are finite* CW *compleces X_n and unital locally trivial continuous field of C^*-algebras over X_n $A_n = (B(t), \Gamma_n)$ (each fiber $B(t) \cong M_n$) such that $A = \lim_{n\to\infty}(A_n, \sigma_n)$.*

PROOF. We write $X = \lim_{\leftarrow}(X_n, \alpha_{n,m})$, where each X_n is a finite simplicial complex and $\alpha_{n,m} : X_m \to X_n$ $(n < m)$ are continuous maps. By Lemma 1 of section 2 in [M], we may assume that $\alpha_{\infty,n} : X \to X_n$ is surjective. Define a map $\beta_n : \prod_{x\in X_n} B(x) \to \prod_{t\in X} A(t)$ by $\beta_n(f)(t) = f(\alpha_{\infty,n}(t))$, where $B(x), A(t) \cong M_n$ for all $x \in X_n$ and $t \in X$. Since $\alpha_{\infty,n}$ is surjective, β_n is injective. Set $\Gamma_n = \{f \in \prod_{x\in X_n} B(t) : \beta_n(f) \in \Gamma\}$. Since $1_A \in \Gamma$, Γ_n is not empty. It the follows that Γ_n is an algebra. Since both X_n and X are compact Huasdorff spaces and $\alpha_{\infty,n}$ is a continuous surjective map (so open sets maps to open sets), one easily verifies that

(i) $x \to \|f(x)\|$ is continuous, if $\beta_n(f) \in \Gamma_n$;

(ii) if $f \in \prod_{x\in X_n} B(x)$ and if, for every $x \in X_n$ and every $\varepsilon > 0$, there exists $g \in \Gamma_n$ such that $\|f(x) - g(t)\| < \varepsilon$ throughout a neighborhood, then $f \in \Gamma_n$.

Let $A_n = (B(x), \Gamma_n)$. Then A_n is a continuous field of C^*-algebras (homogeneous of rank n). Again, using the facts that both X_n and X are compact and $\alpha_{\infty n}$ is surjective, one checks that A_n is locally trivial, since A is. Furthermore, if $A|_D$ is trivial for some neighborhood D of some $t_0 \in X$, then $A_n|_{\alpha_{\infty,n}(D)}$ is trivial.

The map $\beta_n : A_n \to A$ is an injective homomorphism. Let $\sigma_{m,n} : A_m \to A_n$ by $\sigma_{m,n}(f)(t) = f(\alpha_{n,m})(t)$ for $f \in A_m$ and $t \in X_n$. It is clear that $\beta_n \circ \sigma_{m,n} = \beta_m$. Therefore we obtain an inductive limit $\lim_n(A_n, \sigma_{m,n})$ and an injective homomorphism $h : \lim_{n\to\infty} A_n \to A$. To show that h is surjective, we use the fact that A is locally trivial and X is compact. We now identify $\lim_n(A_n, \sigma_{n,m})$ with a C^*-subalgebra of A. Let $\{U_i\}_{i=1}^k$ be a finite open cover of X such that $A|_{U_i}$ is trivial. From the above, $A_n|_{\alpha_{\infty,n}(U_i)}$ is trivial. Therefore $A|_{U_i} = \lim A_n|_{\alpha_{\infty,n}(U_i)}$. Let $g_1, g_2, ..., g_k$ be a partition of unity (subordinate to $\{U_i\}_{i=1}^k$) consisting of compactly supported functions. Given $f \in A$. Let $f_i = f(g_i \cdot 1_A)$. Thus there is $c_i \in \lim A_n|_{\alpha_{\infty,n}(U_i)}$ such that $c_i(t) = f_i(t)$ for $t \in U_i$. There is a $k_i \in C(X)$ with $0 \le k_i \le 1$, $k_i(t) = 1$ for $t \in \mathrm{supp}(g_i)$ and $k_i(t) = 0$ for $t \in X \setminus U_i$. Let $c_i' \in \lim(A_n, \sigma_{n,m})$ such that $c_i'|_{U_i} = c_i$. Then $b_i = c_i'(k_i \cdot 1) \in \lim(A_n, \sigma_{n,m})$, since $k_i \cdot 1 \in \lim(A_n, \sigma_{n,m})$. Then $b_i = f_i$, $i = 1, 2, ..., k$. Since $f = \sum_{i=1}^k f_i$, this implies that $f \in \lim(A_n, \sigma_{n,m})$. This shows that $A = \lim(A_n, \sigma_{n,m})$.

2.8. LEMMA. *Let X be a finite* CW *complex and $A = (A(t), \Gamma)$ be a unital locally trivial continuous field of C^*-algebras with $A(t) \cong M_n$. Then $K_i(A)$ is finitely generated, $i = 0, 1$.*

PROOF. Since X is a finite CW complex and A is locally trivial, we obtain a finite open cover $\{U_i\}_1^k$ of X such that $A|_{\bar{U}_i}$ is trivial. Let

$$I_1 = \{f \in A : f(t) = 0 \ \text{if} \ t \in X \setminus U_1\}$$

and

$$I_i = \{f \in A : f(t) = 0, \quad \text{if} \ t \in X \setminus \cup_{j=1}^i U_j\},$$

$i = 2, 3, ..., k$. Note that $K_i(I_1)$ and $K_i(I_2/I_1)$ are finitely generated and we have the following six-term exact sequence:

$$\begin{array}{ccccc} K_0(I_1) & \rightarrow & K_0(I_2) & \rightarrow & K_0(I_2/I_1) \\ \uparrow & & & & \downarrow \\ K_1(I_2/I_1) & \rightarrow & K_1(I_2) & \rightarrow & K_1(I_1) \end{array}$$

So $K_i(I_2)$ is finitely generated, $i = 0, 1$. Note that $K_i(I_{j+1}/I_j)$ is finitely generated, $i, j = 1, 2, ..., k$. Then we employ an inductive argument (on k).

2.9. LEMMA. *Let X be a compact metric space, let $A = (A(t), \Gamma)$ be a unital locally trivial continuous field of C^*-algebras over X, where $A(t) = M_n$ and let B be a unital C^*-algebra. Suppose that $\phi : A \to B$ is a unital homomorphism and $\psi : A \to B$ is a unital point-evaluation such that*

$$[\phi] = [\psi] \ \ \textit{in} \ \mathrm{KL}(A, B)$$

Then, for any $\varepsilon > 0$ and any finite subset $\mathscr{F} \subset A$, there are an integer $k > 0$, a point-evaluation $h : A \to M_k(B)$ and a unitary $u \in M_{k+1}(B)$ such that

$$\|u^* \mathrm{diag}(\phi(f), h(f))u - \mathrm{diag}(\psi(f), h(f))\| < \varepsilon$$

for all $f \in \mathscr{F}$.

PROOF. We will prove the case that X is a connected finite CW complex. The case that X is a general finite CW complex can be reduced to the connected case by considering each component separately. The general case follows from 2.7 that $A_n = \lim_{n\to\infty}(A_n, \sigma_n)$, where each A_n is a locally trivial continuous field over a finite simplicial complex X_n with homogeneous of rank n. Set $\phi_n = \phi \circ \sigma_n$, $\psi_n = \psi \circ \sigma_n : A_n \to B$. Since ψ is a point-evaluation, so is ψ_n. The condition that $[\phi] = [\psi]$ in $\mathrm{KL}(A, B)$ implies that $[\phi_n] - [\psi_n] = 0$ in $\mathrm{KL}(A_n, B)$ for all large n. Since $K_i(A_n)$ is finitely generated (by 2.8), any pure extension is trivial. So $\mathrm{KK}(A_n, B) = \mathrm{KL}(A_n, B)$. We also note that for any finite subset $\mathscr{F}$ and ε, with an arbitrary small error, we may assume that $\mathscr{F} \subset A_n$ for some large n.

Therefore now we reduce to the general case to the case that X is a connect CW complex and $[\phi] = [\psi]$ in $\mathrm{KK}(A, B)$.

It follows from Lemma 2.6 that it is sufficient to show the following: for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there are an integer $k > 0$, two unital point-evaluations $h : A \to M_k(B)$, $h'_0 : A \to M_{k+1}(B)$ and a unitary $u \in M_{k+1}(B)$ such that

$$\|u^*\mathrm{diag}(\phi(f), h(f))u - h'_0(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

We will then show the above.

Fix a point $\xi \in X$ and let I be as in 2.4. For any $\eta_1 > 0$ and finite subset $\mathcal{G}' \subset I$, by applying Lemma 2.3, we obtain a point-evaluation $h_0 : I \to B \otimes M_l$ and a unitary $V \in B \otimes M_{l+1}$ ($l \geq n$) such that

$$\| V^*\mathrm{diag}(\phi(f)), h_0(f))V - \mathrm{diag}(\psi(f), h_0(f))\| < \eta_1$$

for all $f \in \mathcal{G}'$. Suppose that $h_0(f) = \sum_{k=1}^{s} \sigma_k(f(\xi_k))p_k$ for $f \in I$, where $\xi_k \in X \setminus \{\xi\}$, $p_1, ..., p_s$ are mutually orthogonal projections in $B \otimes M_l$ and $\sigma_k : M_n \to p_k(B \otimes M_l)p_k$ are unital injective homomorphisms. We define $h_0(f) = \sum_{k=1}^{s} \sigma_k(f(\xi_k))p_k$ for all $f \in A$. This gives a homomorphism from A into $B \otimes M_l$ and we will keep the same notation h_0. Let

$$P = \mathrm{diag}(1_B, h_0(1_A)), \quad \text{and} \quad Q = (1_{B\otimes M_{l+1}}) - P.$$

Note that

$$Q \oplus 1_{B\otimes M_{l+1}\otimes M_{n-1}} = \mathrm{diag}(P, P, ..., P) \oplus \mathrm{diag}(Q, Q, ..., Q),$$

where P repeats $n - 1$ times and Q repeats n times. So there is a unital injective homomorphism $\psi' : M_n \to C$, where

$$C = (Q \oplus 1_{B\otimes M_{(l+1)(n-1)}})(B \otimes M_{(l+1)n})(Q \oplus 1_{B\otimes M_{(l+1)(n-1)}}).$$

Define $h_1 : A \to C$ by $h_1(f) = \psi'(f(\xi)), f \in A$.

(*The purpose to introduce* h_1 *is to obtain a unital homomorphism to apply Lemma* 2.5. *So if* h_0 *is unital, then we do not need the projection* Q *nor* h_1. *Also, there may not be any unital homomorphism from* M_n *to* $(1 - \sum_{k=1}^{k} p_k)(B \otimes M_l)(1 - \sum_{k=1}^{k} p_k)$. That is why we have to work in $B \otimes M_{(l+1)n}$.)

In particular, $h_1|_I = 0$. Set $h_2 = W^*\mathrm{diag}(\phi, h_0, h_1)W$ and $h_3 = \mathrm{diag}(\psi, h_0, h_1)$, where $W = \mathrm{diag}(V, 1_{B\otimes M_{(l+1)(n-1)}})$. Note now that $h_2, h_3 : A \to B \otimes M_{(l+1)n}$ are *unital*, and $h_2|_I = V^*\mathrm{diag}(\phi|_I, h_0|_I)V$ and $h_3|_I = \mathrm{diag}(\psi|_I, h_0|_I)$.

We have the following

$$\|h_2(f) - h_3(f)\| < \eta_1$$

for all $f \in \mathcal{G}'$. To save the notation, let $K = (l + 1)n$.

For any $\delta_1 > 0, \delta_2 > 0$ and any finite subset $\mathcal{G} \subset A$, by applying Lemma 2.5 (to h_2 and h_3), if η_1 is small enough and $\mathcal{G}'$ is large enough, we obtain a point-evaluation $h_4 : A \to pM_K(B)p$ for some projection $p \in M_K(B)$ and a δ_1-$\mathscr{G}$-multiplicative contractive completely positive linear morphism $L : M_n(C(D)) \to (1-p)M_K(B)(1-p)$ such that

$$\|h_2(g) - h_4(g) \oplus L \circ s(g)\| < \delta_2$$

for all $g \in \mathscr{G}$, where D is a closed neighborhood of ξ which is homeomorphic to a finite dimensional closed disk and $s : A \to M_n(C(D))$ is the spatial surjection.

Now we apply 1.7 in [Ln6] (see also [EGLP]). By 1.7 in [Ln6], for any finite subset $\mathscr{F}_1 \in M_n(C(D))$, we obtain point-evaluations $h_0' : M_n(C(D)) \to M_N((1-p)M_K(B)(1-p))$, $h_0'' : M_n(C(D)) \to (1-p)M_K(B)(1-p)$ and a unitary $V_1 \in M_N((1-p)M_K(B)(1-p))$ such that

$$\|\mathrm{ad}(V_1) \circ \mathrm{diag}(L \circ s(f), h_0'(f)) - \mathrm{diag}(h_0''(f), h_0'(f))\| < \varepsilon$$

for all $f \in \mathscr{F}_1$, where N is a positive integer, if δ_1 is sufficiently small and $\mathscr{G}$ is sufficiently large.

Let $\sigma : M_n(C(D)) \to A$ be the completely positive linear map such that $s \circ \sigma = \mathrm{id}_{M_n(C(D))}$. Define $h_5(f) = h_0'' \circ s(f)$ for all $f \in A$. Suppose that $h_0''(g) = \sum_{i=1}^{m_1} \gamma_i(g(\xi_i))p_i$, where $\xi_i \in D$ and $p_1, p_2, \ldots, p_{m_1}$ are mutually orthogonal projections in $(1-p)M_K(B)(1-p)$. Note that $\sigma \circ s(f)(\xi_i) = f(\xi_i)$ for all $f \in A$ and $i = 1, 2, \ldots, m_1$. Thus $h_5(f) = \sum_{i=1}^{m_1} \gamma_i(f(\xi_i))p_i$. So h_5 is a point-evaluation. Similarly $h_6 = h_0'' \circ s : A \to M_N((1-p)M_K(B)(1-p))$ is also a point-evaluation. Thus, for any $\varepsilon > 0$, and finite subset $\mathscr{F}_1$, we have

$$\|V_2^* \mathrm{diag}(\phi(f), h_0(f), h_1(f), h_6(f))V_2 - \mathrm{diag}(h_4(f), h_5(f), h_6(f))\| < \varepsilon$$

for all $f \in \mathscr{F}_1$ and $V_2 = V \oplus V_1$. Therefore the lemma follows.

2.10. Notation. Let $h_1, h_2 : A \to B$ be two linear maps from C^*-algebra A into a unital C^*-algebra B. We will write

$$h_1 \overset{\varepsilon}{\sim} h_2$$

on $\mathscr{G}$, if there exists a partial isometry $u \in M_k(B)$ such that

$$\|u^* h_1(g)u - h_2(g)\| \leq \varepsilon$$

for all $g \in \mathscr{G}$. If A is unital and both h_1 and h_2 are unital, then

$$h_1 \overset{\varepsilon}{\sim} h_2$$

on $\mathscr{G}$ which contains the identity implies that there is a unitary $v \in B$ such that

$$\|v^* h_1(g)v - h_2(g)\| \leq 2\varepsilon$$

for all $g \in \mathcal{G}$, provided that $\varepsilon < 1$. In fact, since $h_i(1_A) = 1_B$, we have

$$\|v^*v - 1_B\| < 1 \quad \text{and} \quad \|1_B - vv^*\| < 1.$$

Because vv^* and vv^* are assumed to be projections, we see that v is a unitary.

If, for any $\varepsilon > 0$ and any finite subset $\mathcal{F}$, $h_1 \overset{\varepsilon}{\sim} h_2$ on $\mathcal{F}$, then we say h_1 and h_2 are *approximately unitarily equivalent*. Let A be a unital C^*-algebra of continuous trace. We say that h_1 and h_2 are *stably approximately unitarily equivalent* if, for any $\varepsilon > 0$ and any finite subset $\mathcal{F}$, there is a point-evaluation $h_0 : A \to M_n(B)$ for some integer $n > 0$ such that

$$\mathrm{diag}(h_1, h_0) \overset{\varepsilon}{\sim} \mathrm{diag}(h_2, h_0)$$

on $\mathcal{F}$.

2.11. THEOREM. *Let A be a unital C^*-algebra of continuous trace and let B be a unital C^*-algebra. Suppose that $\phi, \psi : A \to B$ are two unital homomorphisms such that*

$$[\phi] = [\psi] \quad in \ \mathrm{KL}(A, B)$$

Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there are an integer $k > 0$, a point-evaluation $h : A \to M_k(B)$ and a unitary $u \in M_{k+1}(B)$ such that

$$\|u^*\mathrm{diag}(\phi(f), h(f))u - \mathrm{diag}(\psi(f), h(f))\| < \varepsilon$$

for all $f \in \mathcal{F}$. The converse is also true, i.e., *if ϕ and ψ are stably approximately unitarily equivalent, then $[\phi] = [\psi]$ in* $\mathrm{KL}(A, B)$.

PROOF. By 1.2 and by considering each summand separately, we may assume that $A = (A(t), \Gamma)$, a unital locally trivial continuous field of C^*-algebras over a compact Hausdorff space X, where $A(t) \cong M_n$. As in 2.9, we may further assume that X is a connected CW complex. Note also, with this assumption, $\mathrm{KK}(A, B) = \mathrm{KL}(A, B)$. By Corollary 1.9, there exist an integer $r > 0$, a homomorphism $h_1 : A \to M_r(B)$ and a point-evaluation $h_0 : A \to M_{r+1}(B)$ such that $\mathrm{diag}(\phi, h_1)$ is homotopy to h_0. Since $[\psi] = [\phi]$ and $\mathrm{diag}(\phi, h_1)$ is homotopy to h_0,

$$[\mathrm{diag}(\psi, h_1)] = [h_0] \quad \text{in } \mathrm{KK}(A, B).$$

Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$. By applying Lemma 2.9, there are integer L, L', point-evaluations $h_{00} : A \to M_{rL}(B)$ and $h'_{00} : A \to M_{rL'}(B)$ that

$$\mathrm{diag}(\psi, h_1, h_{00}) \overset{\varepsilon/2}{\sim} \mathrm{diag}(h_0, h_{00}) \quad \text{and} \quad \mathrm{diag}(\phi, h_1, h'_{00}) \overset{\varepsilon/2}{\sim} \mathrm{diag}(h_0, h'_{00})$$

on the finite subset $\mathcal{F}$. We have

$$\mathrm{diag}(\psi, h_0, h_{00}, h'_{00}) \overset{\varepsilon/2}{\sim} \mathrm{diag}(\psi, \phi, h_1, h_{00}, h'_{00}) \overset{0}{\sim} \mathrm{diag}(\phi, \psi, h_1, h_{00}, h'_{00}) \overset{\varepsilon/2}{\sim} \mathrm{diag}(\phi, h_0, h_{00}, h'_{00})$$

on $\mathscr{F}$. Therefore

$$\mathrm{diag}(\psi, h_0, h_{00}, h'_{00}) \overset{\varepsilon}{\sim} \mathrm{diag}(\phi, h_0, h_{00}, h'_{00})$$

on $\mathscr{F}$.

3. Approximately unitarily equivalence

3.1. Let A be a unital C^*-algebra of continuous trace. Then we may write $A = \oplus_i^k A_i$, where each $A_i = (A_i(t), \Gamma_i)$ is a locally trivial continuous field over a compact metric space $\hat{A}_i$ and $A_i(t) = M_{n(i)}$. Let $X = \sqcup \hat{A}_i$. We may write $A = (A(t), \Gamma)$ as a continuous field of C^*-algebras over X, where $A(t) = A_i(t)$, if $t \in \hat{A}_i$ and $\Gamma = \oplus \Gamma_i$. For each point $t \in X$, there exists a close neighborhood F_t such that $A|_{F_t}$ is spatially isomorphic to $M_{n(t)}(C(F_i))$. We denote G_t the interior of F_t. Let $\{e_{ij}\}$ be a constant matrix unit for $A|_{F_t}$. Let $f_t \in C(X)$ such that $0 \le f_t(\xi) \le 1$, $f_t(\xi) > 0$ for $\xi \in G_t$ and $f_t(\xi) = 0$ for $\xi \in X \setminus G_t$. Set $h_t = f_t \cdot e_{11}$. We will view h_t as an element in A.

3.2. Lemma (cf [Ln1,1] and [EGLP, 4.1]). *Let X be a compact metric space, A be a unital C^*-algebra of continuous trace with $\hat{A} = X$. Then, for any $\varepsilon > 0$, any finite subset $\mathscr{F} \subset A$, any unital C^*-algebra B and any unital homomorphism $\phi : A \to B$, there exists $\delta > 0$ satisfying the following: if*

(1) *$\xi_1, \xi_2, ..., \xi_n \in X$ and $S_k \subset \{\xi \in X : dist(\xi, \xi_k) < \delta\}$ then $A|_{\bar{S}_k} = M_{n(k)}(C(\bar{S}_k))$.*

Furthermore, if

(2) *$S_k \cap S_i \neq \emptyset$, if $k \neq i$, and S_k is an open neighborhood of ξ_k,*

(3) *$h_k = h_{\xi_k}$ is a positive element as in 3.1,*

(4) *p_k is a projection in the hereditary C^*-subalgebra of B generated by $\phi(h_k)$,*

then there exist projections $d_k \in B$ with $d_k \ge p_k$ and d_k is equivalent to $n(k)$ many direct sum of p_k such that

$$\left\| \phi(f) - \left(L(f) + \sum_{i=1}^{n} \gamma_i(f(\xi_i)) d_i \right) \right\| < \varepsilon$$

and

$$\left\| \left(1 - \sum_{i=1}^{n} d_i\right) \phi(f) - \phi(f) \left(1 - \sum_{i=1}^{n} d_i\right) \right\| < \varepsilon$$

for all $f \in \mathscr{F}$, *where* $f(\xi_i) = \pi_{\xi_i}(f)$, $\gamma_i : M_{n(\xi_i)} \to p_i B p_i$ *is a monomorphism and* $L(f) = (1 - \sum_{i=1}^n d_i)\phi(f)(1 - \sum_{i=1}^n d_i)$.

PROOF. By 1.2, we may write $A = \oplus_{i=1}^m B_i$, where each B_i is a homogeneous C^*-algebra of finite rank. Clearly, without loss of generality, we may assume that $A = (A(t), \Gamma)$ is a unital locally trivial continuous field over a compact metric space X, where each $A(t) = M_N$. Since X is compact and A is locally trivial, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $A|_{D_t} = M_N(C(D_t))$, where

$$D_t \subset \{\xi \in X : \mathrm{dist}(t, \xi) \le \delta\}$$

is a neighborhood of t, and

$$\|f(\xi) - f(\xi')\| < \varepsilon/4$$

for all $f \in \mathscr{F}$, whenever $\mathrm{dist}(\xi, \xi') < \delta$. Here we veiw $A|_{D_t} = M_N(C(D_t))$, $k = 1, 2, ..., n$.

With such δ, (1) follows immediately.

Now assume (2), (3) and (4).

Let h_k be as described in 3.2 (with $F_k = \bar{S}_k$). We denote by $\{e_{ij}\}$ be a constant matrix unit for $M_N(C(\bar{S}_k))$ for all k. Let $g_k = f_k \cdot 1_A$ (see 3.1). Then $g_k \in A$. Let

$$q_k = \lim_{m \to \infty} (g_k)^{1/m}$$

in A^{**}. So q_k is an open projection in A^{**}. Similarly, we obtain an open projection q'_k corresponding to the open subset $X \setminus \bar{S}_k$. Set $\bar{q}_k = 1 - q'_k$. So $\bar{q}_k$ is a closed projection in A^{**} and $q_k \le \bar{q}_k$. Moreover,

$$q_k f = f q_k, \bar{q}_k f = f \bar{q}_k \quad \text{and} \quad \|q_k f - f(\xi_k) q_k\| < \varepsilon/2.$$

for all $f \in \mathscr{F}$. Here we identify $A|_{\bar{S}_k}$ with $M_N(C(\bar{S}_k))$. Denote by $\phi : A^{**} \to B^{**}$ the extension of ϕ. Note that $p_k \le \phi(q_k)$ and p_k are mutually orthogonal. Let $\phi^{(k)} : M_N(C(\bar{S}_k)) \to \phi(\bar{q}_k) B \phi(\bar{q}_k)$ be the homomorphism induced by ϕ. For any $a \in \overline{(h_k A h_k)}$, there is $c \in A$ such that $e_{j1} a e_{1j}(t) = c(t)$ for $t \in S_k$ and $c(t) = 0$ for $t \in X \setminus S_k$. Therefore

$$\phi^{(k)}(e_{j1}) \phi(a) \phi^{(k)}(e_{1j}) \in B_k,$$

where $a \in \overline{(h_k A h_k)}$ and B_k is the hereditary C^*-subalgebra of B generated by $\phi(g_k)$. Set $u_{ij}^{(k)} = \phi^{(k)}(e_{ij})$. The above implies that

$$u_{1j}^* b u_{1j} \in B_k$$

for all b in the hereditary C^*-subalgebra of B generated by $\phi(h_k)$. Denote by $p_k^j = u_{1j}^* p_k u_{1j}$ and $d_k = \sum_{j=1}^N p_k^j$. Note that $\{p_k^1, p_k^2, ..., p_k^N\}$ are mutually orthogonal and mutually equivalent projections in B_k (since $p_k \in$

$(\phi(h_k)B\phi(h_k))\bar{)}$ and $p_k \le d_k$. One computes that

$$\phi^{(k)}(f(\xi_k)\cdot 1_{M_N(C(S_k))})d_k = d_k\phi^{(k)}(f(\xi_k)\cdot 1_{M_N(C(S_k))})$$

for all $f \in M_N(C(S_k))$. Define $\gamma_k' : M_N(C(\bar{S}_k)) \to d_kBd_k$ by writing

$$\gamma_k'(f) = \phi^{(k)}(f(\xi_k)\cdot 1_{M_N(C(S_k))})d_k.$$

This induces a homomorphism $\gamma_k : M_n \to d_kBd_k$.

We estimate that

$$\|\phi(f)\phi(q_k) - \phi^{(k)}(f(\xi_k)\cdot 1_{M_N(C(S_k))})\| < \varepsilon/4$$

for all $f \in \mathscr{F}$. We have

$$\left\|\phi(f)\left(\sum_{k=1}^n d_k\right) - \sum_{k=1}^n \gamma_k(f(\xi_k))d_k\right\|$$

$$= \|\phi(f)\phi\left(\sum_{k=1}^n q_k\right)\left(\sum_{k=1}^n d_k\right) - \sum_{k=1}^n \gamma_k(f(\xi_k))d_k\|$$

$$= \left\|\sum_{k=1}^n \phi(q_k)[\phi(f)d_k - \gamma_k(f(\xi_k))d_k]\right\| < \varepsilon/4$$

for all $f \in \mathscr{F}$. Similarly,

$$\left\|\left(\sum_{k=1}^n d_k\phi(f) - \sum_{k=1}^n \gamma_k(f(\xi_k)\right)d_k\right\| < \varepsilon/4$$

for all $f \in \mathscr{F}$. Moreover,

$$\left\|\left(1-\sum_{k=1}^n d_k\right)\phi(f)\left(1-\sum_{k=1}^n d_k\right) - \phi(f)\left(1-\sum_{k=1}^n d_k\right)\right\|$$

$$= \left\|\left[\left(1-\sum_{k=1}^n d_k\right)\phi(f) - \sum_{k=1}^n \gamma_k(f(\xi_k))d_k\right]\left(1-\sum_{k=1}^n d_k\right)\right\| < \varepsilon/2$$

for all $f \in \mathscr{F}$. Similarly,

$$\left\|\left(1-\sum_{k=1}^n d_k\right)\phi(f) - \left(1-\sum_{k=1}^n d_k\right)\phi(f)\left(1-\sum_{k=1}^n d_k\right)\right\| < \varepsilon/2$$

for all $f \in \mathscr{F}$. Set

$$L(f) = \left(1 - \sum_{k=1}^{n} d_k\right)\phi(f)\left(1 - \sum_{k=1}^{n} d_k\right).$$

Then

$$\left\|\phi(f) - \left(L(f) + \sum_{k=1}^{n} \gamma_k(f(\xi_k))d_k\right)\right\| < \varepsilon$$

and

$$\left\|\left(1 - \sum_{k=1}^{n} d_k\right)\phi(f) - \phi(f)\left(1 - \sum_{k=1}^{n} d_k\right)\right\| < \varepsilon$$

for all $f \in \mathscr{F}$.

3.3. LEMMA. *Let B be a unital purely infinite simple C^*-algebra and let $\phi, \psi : M_n \to B$ be two monomorphisms. Suppose that $[\psi] = 0$ in $\mathrm{KK}(M_n, B)$. then*

$$\mathrm{diag}(\phi, \psi) \overset{0}{\sim} \phi.$$

PROOF. Let $\{e_{ij}\}$ be a matrix unit for M_n. Note that $[\psi] = 0$ implies that $[\phi(e_{11}) + \psi(e_{11})] = [\phi(e_{11})]$ in $K_0(B)$. Since B is purely infinite simple, $\phi(e_{11}) + \psi(e_{11})$ is equivalent to $\phi(e_{11})$. It follows easily that $\mathrm{diag}(\phi, \psi)$ is equivalent to ϕ.

3.4. THEOREM. *Let B be a unital purely infinite simple C^*-algebra and A be a unital C^*-algebra of continuous trace. Suppose that ϕ, $\psi : A \to B$ are two unital monomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $\mathrm{KL}(A, B)$.*

PROOF. To save the notation, without loss of generality, we may assume that $A = (A(t), \Gamma)$ is a unital locally trivial continuous field of C^*-algebras over a compact metric space X with each $A(t) \cong M_n$. Let $\varepsilon > 0$ be a positive number and $\mathscr{F}$ be a finite subset in the unit ball of A which contains the identity of A. By 2.11, there exists a unital point-evaluation $h_0 : A \to M_N(B)$ such that

$$\mathrm{diag}(\phi, h_0) \overset{\varepsilon/5}{\sim} \mathrm{diag}(\psi, h_0)$$

on $\mathscr{F}$. We write

$$h_0(f) = \sum_{i=1}^{k} \gamma_i(f(\xi_i))p_i$$

for $f \in A$, where $p_1, p_2, ..., p_k$ are mutually orthogonal projections in $M_N(B)$, $\xi_i \in X$ and $\gamma_i : M_n \to p_i M_N(B) p_i$, $i = 1, 2, ..., k$. Since B is purely infinite simple, by adding another point-evaluation, if necessary, we may assume that $[\gamma_i] = 0$ in $\mathrm{KK}(M_n, B)$. Furthermore, we may assume that $\xi_i \neq \xi_j$ if $i \neq j$.

By applying 3.2, we have

$$\|\phi(f) - [L(f) \oplus h_{00}(f)]\| < \varepsilon/5$$

for all $f \in \mathscr{F}$, where $h_{00}(f) = \sum_{i=1}^{k} \beta_i(f(\xi_i))d_i$, where $d_1, d_2, ..., d_k$ are mutually orthogonal projections in B, $\beta_i : M_n \to d_i B d_i$ are monomorphism, $i = 1, 2, ..., k$ and $L : A \to (1 - \sum_{i=1}^{k} d_i)B(1 - \sum_{i=1}^{k} d_i)$ is a positive linear map. Note $\beta_i(f(\xi_i)) = \beta_i \circ \pi_{\xi_i}(f)$ and $\gamma_i(f(\xi_i)) = \gamma_i \circ \pi_{\xi_i}(f)$ for $f \in A$. By applying 3.3, we have

$$\mathrm{diag}(L, h_{00}) \overset{0}{\sim} \mathrm{diag}(L, h_{00}, h_0)$$

(on A). Thus

$$\phi\varepsilon/5{\sim}\mathrm{diag}(L, h_{00}, h_0) \overset{\varepsilon/5}{\sim} \mathrm{diag}(\phi, h_0)$$

on $\mathscr{F}$. Therefore

$$\phi \overset{2\varepsilon/5}{\sim} \mathrm{diag}(\phi, h_0)$$

on $\mathscr{F}$. Exactly the same argument shows that

$$\psi \overset{2\varepsilon/5}{\sim} \mathrm{diag}(\psi, h_0)$$

on $\mathscr{F}$. Hence we conclude that

$$\phi \overset{\varepsilon}{\sim} \psi$$

on $\mathscr{F}$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OREGON
EUGENE, OREGON 97403-1222
USA

DEPARTMENT OF MATHEMATICS
EAST CHINA NORMAL UNIVERSITY
SHANGHAI
CHINA 2000 62