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A FUNDAMENTAL SOLUTION OF N. ZEILON'S OPERATOR

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Abstract

In this paper, we resume earlier work of N. Zeilon and of J. Fehrman and derive an explicit representation by elliptic integrals of a fundamental solution of the partial differential operator $\partial_1^3 + \partial_2^3 + \partial_3^3$.

1. Introduction

The operator $\partial_1^3 + \partial_2^3 + \partial_3^3$ was considered – to my knowledge – for the first time in N. Zeilon's article of 1913 (see [16]), where he generalizes I. Fredholm's method of construction of fundamental solutions (see [5]) from homogeneous *elliptic* equations to arbitrary homogeneous equations in three variables with *real-valued* symbol (cf. [16, II, pp. 14–22]). In particular, he applies his theory to the operator $\partial_1^3 + \partial_2^3 + \partial_3^3$ (see [16, pp. 56–70]), though he concedes that this is "... une équation du troisième ordre, sans application à la Physique, il est vrai ..." (cf. [16, p. 3]). Probably, he was led to consider this operator as an example, since, a little earlier, I. Fredholm had calculated a fundamental solution of $\partial_1^4 + \partial_2^4 + \partial_3^4$ (cf [6]). Fredholm's result is (up to the constant factor) the following:

$$G(x) = -\frac{1}{8\pi}\sum_{j=1}^{3}|x_j|\int_{\zeta/(2x_j^2)}^{\infty}\frac{\mathrm{d}u}{\sqrt{4u^3-u}}$$
$$= -\frac{1}{8\pi}\sum_{j=1}^{3}x_jF\left(\arcsin\left(\frac{\sqrt{2}\,x_j}{\sqrt{\zeta+x_j^2}}\right),\frac{1}{\sqrt{2}}\right),$$

where ζ is the largest of the three real roots of the cubic

$$\zeta^3 - (x_1^4 + x_2^4 + x_3^4)\zeta - 2x_1^2x_2^2x_3^2 = 0$$

and F denotes the elliptic integral of the first kind (cf [8,3.131.8 and 8.111]). We mention that G is the only fundamental solution of $\partial_1^4 + \partial_2^4 + \partial_3^4$ which is

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homogeneous and even. Unfortunately, N. Zeilon did not obtain a representation for a fundamental solution of $\partial_1^3 + \partial_2^3 + \partial_3^3$ which is as explicit as Fredholm's formula in the case of $\partial_1^4 + \partial_2^4 + \partial_3^4$.

In 1975, J. Fehrman introduced the class of *hybrid* operators, which have fundamental solutions that are real-analytic outside proper cones. As an example, he shows that $\partial_1^3 + \partial_2^3 + \partial_3^3$ is hybrid with respect to the direction $N = (1, 1, 1)$ (see [3, p. 223]) and, therefore, it possesses a fundamental solution which is real analytic outside the wave front surface with respect to N, i.e. outside

$$\{x \in \mathsf{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_i^{3/2} = x_j^{3/2} + x_k^{3/2} \text{ for a permutation } i, j, k \text{ of } 1, 2, 3\},$$

see [3, Th. 4, p. 231]. He also proves that this fundamental solution of $\partial_1^3 + \partial_2^3 + \partial_3^3$ has (except at the origin) sharp fronts everywhere from within

$$L := \{x \in \mathsf{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0, x_i^{3/2} < x_j^{3/2} + x_k^{3/2} \text{ for all permutations } i, j, k \text{ of } 1, 2, 3\}, \tag{1}$$

see [3, p. 235]. However, he does not give an explicit formula for a fundamental solution exhibiting this behaviour.

Recently, R. Meise and his co-workers showed that, for the polynomial $P(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3$, the set R^3 is *P-convex with bounds* (i.e., $P(-\mathrm{i}\partial)$ admits a right-inverse on $\mathscr{E}(\mathsf{R}^3)$), although $P(-\mathrm{i}\partial)$ is not an evolution operator with respect to any direction (i.e., there does not exist a fundamental solution of $P(-\mathrm{i}\partial)$ with support in a half-space), and hence no bounded convex open set in R^3 is P-convex (cf. [13, Ex. 1, p. 463], [4, Ex. 3.7, p. 160]). It is still an open problem to decide whether there exist fundamental solutions of $P(-\mathrm{i}\partial)$ having conical lacunae different from L and $-L$.

In this paper, I shall give an explicit formula for a fundamental solution E of $\partial_1^3 + \partial_2^3 + \partial_3^3$ in terms of elliptic integrals. The result is the following:

THEOREM. *The limit*

$$T := \lim_{\epsilon \searrow 0} \frac{Y(|\xi_1^3 + \xi_2^3 + \xi_3^3| - \epsilon)}{\xi_1^3 + \xi_2^3 + \xi_3^3}$$

defines a distribution in $\mathscr{S}'(\mathsf{R}^3)$. *If* $E := \left(\frac{\mathrm{i}}{2\pi}\right)^3 \mathscr{F} T$ *and* L *is as in* (1), *then*

(a) *E is a fundamental solution of* $\partial_1^3 + \partial_2^3 + \partial_3^3$;
(b) *E is homogeneous of degree* 0;
(c) *E is odd and invariant under permutations of the co-ordinates*;
(d) $\operatorname{sing\,supp} E = \operatorname{sing\,supp}_{\mathrm{A}} E = \partial L \cup -\partial L$;
(e) *E is continuous in* $\mathsf{R}^3 \setminus \{0\}$;

(f) *E is constant in L and in* $-L$, *and*

$$E|_{\pm L} = \mp \frac{B(\frac{1}{3}, \frac{1}{3})}{8\sqrt{3}\,\pi} \approx \mp 0.12175;$$

(g) *for* $x \in \mathsf{R}^3 \setminus (\overline{L} \cup -\overline{L})$, *we have*

$$\begin{aligned} E(x) &= \frac{\sqrt[3]{2}}{8\sqrt{3}\,\pi} \operatorname{sign}(3\sqrt[3]{2}\,x_1x_2x_3 - x_1^3 - x_2^3 - x_3^3) \int_{-1}^{\zeta} \frac{\mathrm{d}u}{\sqrt{u^3+1}} \\ &= \frac{\sqrt[3]{2}}{8 \cdot 3^{3/4}\,\pi} \operatorname{sign}(3\sqrt[3]{2}\,x_1x_2x_3 - x_1^3 - x_2^3 - x_3^3) \\ &\quad \times F\left(\arccos\left(\frac{\sqrt{3}-1-\zeta}{\sqrt{3}+1+\zeta}\right), \frac{\sqrt{3}+1}{2\sqrt{2}}\right), \end{aligned}$$

where either ζ *is the only simple real root or, if* x *lies on one of the co-ordinate axes,* ζ *is the triple root 0, respectively, of the cubic equation*

$$\begin{aligned} (2) \qquad &(x_1^6 + x_2^6 + x_3^6 - 2x_1^3x_2^3 - 2x_1^3x_3^3 - 2x_2^3x_3^3)\zeta^3 - 9\sqrt[3]{4}\,x_1^2x_2^2x_3^2\zeta^2 \\ &- 3\sqrt[3]{16}\,x_1x_2x_3(x_1^3 + x_2^3 + x_3^3)\zeta - 4(x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3) = 0 \end{aligned}$$

and

$$F(\varphi, k) = \int_0^{\varphi} \frac{\mathrm{d}\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}, \qquad \varphi \in \mathsf{R},\ 0 \le k < 1;$$

(h) $\forall x \in \mathsf{R}^3 \setminus (\overline{L} \cup -\overline{L}) : E|_L < E(x) < -E|_L$ *and* $[E(x) = 0 \Longleftrightarrow x_1^3 + x_2^3 + x_3^3 = 3\sqrt[3]{2}\,x_1x_2x_3]$.

REMARK. Before proceeding, let us comment on the cubic $Q(\zeta, x)$ in (2) and on why the integral for E given in (g) is well-defined and represents – as it is required by (d) – an analytic function in $\Omega := \mathsf{R}^3 \setminus (\overline{L} \cup -\overline{L})$. First note that the leading coefficient

$$A(x) := x_1^6 + x_2^6 + x_3^6 - 2x_1^3x_2^3 - 2x_1^3x_3^3 - 2x_2^3x_3^3$$

of $Q(\zeta, x)$ is positive in Ω and vanishes on its boundary. Further, the discriminant of Q with respect to ζ is $-2^4 3^3 A(x)(x_1^3 - x_2^3)^2(x_1^3 - x_3^3)^2(x_2^3 - x_3^3)^2$. This is negative unless two co-ordinates are equal, and thus $Q(\zeta, x)$ has just one real root ζ except for the planes $x_1 = x_2$, $x_1 = x_3$, and $x_2 = x_3$. By formula (12) below, triple roots do not occur in Ω but along the three co-ordinate axes. Since

$$Q(-1, x) = -B(x)^2 \text{ with } B(x) := x_1^3 + x_2^3 + x_3^3 - 3\sqrt[3]{2}\,x_1x_2x_3,$$

and since $\zeta \geq -1$ on the co-ordinate planes (e.g., if $x_1 = 0$, $\zeta = \sqrt[3]{\frac{4x_2^3x_3^3}{(x_2^3-x_3^3)^2}}$), we conclude that $\zeta \geq -1$ holds throughout in Ω, and that ζ and hence also the integral representing E in (g) are real-analytic in Ω except possibly on the co-ordinate axes and on the surface $\Sigma := \{x \in \Omega : B(x) = 0\}$. In the course of the proof, we shall show that (g) holds true in some region of Ω. Using the precise description of $\operatorname{sing\,supp}_{\mathrm{A}} E$ in (d) and the odd parity of E, this already implies, by analytic continuation, that the representation in (g) remains valid in all points of Ω. (Notice that $\Omega \setminus \Sigma$ has just two connected components. In Fig. 1 at the end, Σ is represented by the curve passing through $(-1, 0)$ and $(0, -1)$.) As a matter of fact, $\operatorname{sign}\big(B(x)\big) \int_{-1}^{\zeta} \frac{\mathrm{d}u}{\sqrt{u^3+1}}$ can also directly be proven to be analytic along Σ: Since $Q(\zeta, x) = -B(x)^2 + (\zeta+1)R(\zeta, x)$ for some polynomial R with $R(-1, x) = \partial_\zeta Q(-1, x) = 3A(x) - 3\sqrt[3]{16}\, x_1x_2x_3B(x)$, we have

$$\zeta(x) + 1 = \frac{B(x)^2}{3A(x)} + O\big(B(x)^3\big)$$

near Σ. Furthermore, the integral

$$\int_{-1}^{\zeta} \frac{\mathrm{d}u}{\sqrt{u^3+1}} = 2\int_0^{\sqrt{\zeta+1}} \frac{\mathrm{d}t}{\sqrt{t^4 - 3t^2 + 3}}$$

equals $\sqrt{\zeta+1}$ times a real-analytic function of ζ, and hence $\operatorname{sign}\big(B(x)\big) \int_{-1}^{\zeta} \frac{\mathrm{d}u}{\sqrt{u^3+1}}$ is $B(x)$ times a real-analytic function of x near Σ.

Let us establish some notations. We consider R^n as a Euclidean space with the inner product $x \cdot y := x_1y_1 + \ldots + x_ny_n$ and write $|x| := \sqrt{x \cdot x}$. To display the variable referred to, notation as R^n_x is used. S_{n-1} denotes the unit sphere $\{\omega \in \mathsf{R}^n : |\omega| = 1\}$ in R^n and $\mathrm{d}\sigma(\omega)$ the Euclidean measure on S_{n-1}. The beta-function, also called Euler's integral of the first kind, is abbreviated by B, i.e., $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. We write $⨍$ for the Cauchy principal value.

When we make use of the theory of distributions, we adopt the notations from [10], [12], [14]. In particular, the Heaviside function is abbreviated by Y, and $\langle \varphi, T \rangle$ stands for the value of the distribution T on the test function φ. We use the Fourier transform $\mathscr{F}$ in the form

$$(\mathscr{F}\varphi)(x) = \int \exp(-\mathrm{i}x \cdot \xi)\varphi(\xi)\,\mathrm{d}\xi \qquad (\varphi \in \mathscr{S}(\mathsf{R}^n))$$

What concerns homogeneous distributions, we refer to [15].

2. Borovikov's formula, wave front sets, and lacunae

2.1. Let us consider first an arbitrary *real-valued, homogeneous* polynomial $P(\xi)$ *of principal type* in n variables.

Then $\nabla P(\xi) \neq 0$ for $\xi \in \mathsf{R}^n \setminus \{0\}$ (cf. [11, Def. 10.4.11, p. 38]). If m denotes the degree of homogeneity of $P(\xi)$, then $\xi \cdot \nabla P(\xi) = mP(\xi)$, and hence $P(\xi)$ fulfills

(α) $\{\omega \in \mathsf{S}_{n-1} : P(\omega) = 0\}$ is a $\mathscr{C}^\infty$ submanifold of S_{n-1};

(β) $\Phi := \mathrm{vp}\,\frac{1}{P(\omega)} \in \mathscr{D}'(\mathsf{S}_{n-1})$ is well-defined by

$$\left\langle \varphi, \mathrm{vp}\ \frac{1}{P(\omega)} \right\rangle := \lim_{\epsilon \searrow 0} \int_{|P(\omega)|>\epsilon} \frac{\varphi(\omega)}{P(\omega)}\, \mathrm{d}\sigma(\omega) \qquad (\varphi \in \mathscr{D}(\mathsf{S}_{n-1}));$$

(γ) $T := \mathrm{Pf}_{\lambda=-m}\big[\Phi\big(\frac{\xi}{|\xi|}\big)\,|\xi|^\lambda\big] \in \mathscr{S}'(\mathsf{R}^n_\xi)$ fulfills $P(\xi)T = 1$;

(δ) $E := \frac{\mathrm{i}^m}{(2\pi)^n}\,\mathscr{F}T$ is a fundamental solution of $P(\partial)$.

Th. 8.4.18 in [10, p. 294] allows to precisely determine the analytic wave front set of E. In fact, if T is as in (γ) above, then

$$\begin{aligned}\mathrm{WF}_{\mathrm{A}} T \cap [(\mathsf{R}^n \setminus \{0\}) \times (\mathsf{R}^n \setminus \{0\})] \\ = \{(\xi, t\nabla P(\xi)) : \xi \in \mathsf{R}^n \setminus \{0\},\ P(\xi) = 0,\ t \in \mathsf{R} \setminus \{0\}\}\end{aligned}$$

and hence

$$\mathrm{WF}_{\mathrm{A}} E = \{0\} \times (\mathsf{R}^n \setminus \{0\}) \cup \{(t\nabla P(\xi), \xi) : \xi \in \mathsf{R}^n \setminus \{0\},\ P(\xi) = 0,\ t \in \mathsf{R} \setminus \{0\}\}.$$

Therefore, the analytic singular support of E is given by

$$\operatorname{sing\,supp}_{\mathrm{A}} E = \{t\nabla P(\xi) : \xi \in \mathsf{R}^n,\ P(\xi) = 0,\ t \in \mathsf{R}\}$$

(cf. also [1, p. 251; Engl.: p. 69]). Of course, the singular support coincides with the analytic singular support on the basis of the same reasoning.

Since T is homogeneous in $\mathsf{R}^n \setminus \{0\}$, E can be represented by an $(n-1)$-dimensional integral. The shape of it depends on whether $m \geq n$ or $m < n$, and on whether n is even or odd. The corresponding formulae (cf. [7, Ch. I, 6.2, (2)–(6), p. 129]) are often called Herglotz-Petrovsky formulae. In the case of P being of principal type and $\{\omega \in \mathsf{S}_{n-1} : P(\omega) = 0\}$ being non-empty, they go back to Borovikov (see [1]).

2.2. Let us specialize now on the case of $m = n = 3$.

Then $\langle 1, \Phi \rangle = 0$ since $\Phi = \mathrm{vp}\big(\frac{1}{P(\omega)}\big)$ is odd, and, therefore, the meromorphic distribution-valued function $\lambda \longmapsto \Phi\big(\frac{\xi}{|\xi|}\big) \cdot |\xi|^\lambda$ is analytic in $\lambda = -3$. Hence T and E, which were defined in (γ) and (δ) above, are homogeneous of the degrees -3 and 0, respectively (cf. [15, Satz 2, p. 410]). Obviously, T and E

are of odd parity, and they are invariant under permutations of the co-ordinates. Making use of the estimate

$$\exists C > 0 : \forall \epsilon > 0 : \forall \rho > 0 : \forall \varphi \in \mathscr{S}(\mathsf{R}^3) :$$
$$\left| \int_{|P(\omega)|>\epsilon} \frac{\varphi(\rho\omega)}{P(\omega)} \, \mathrm{d}\sigma(\omega) \right| \leq C\rho \max\{|\nabla\varphi(\xi)| : |\xi| = \rho\}$$

and of Lebesgue's dominated convergence theorem we infer, for $\varphi \in \mathscr{S}(\mathsf{R}^3)$,

$$\begin{aligned}
\langle \varphi, T \rangle &= \int_0^\infty \langle \varphi(\rho\omega), \Phi \rangle \, \frac{\mathrm{d}\rho}{\rho} \\
&= \int_0^\infty \left(\lim_{\epsilon \searrow 0} \int_{|P(\omega)|>\epsilon\rho^{-3}} \frac{\varphi(\rho\omega)}{P(\omega)} \, \mathrm{d}\sigma(\omega) \right) \frac{\mathrm{d}\rho}{\rho} \\
&= \lim_{\epsilon \searrow 0} \int_{|P(\xi)|>\epsilon} \frac{\varphi(\xi)}{P(\xi)} \, \mathrm{d}\xi.
\end{aligned}$$

Thus T can be represented by the following limit, which converges in $\mathscr{S}'(\mathsf{R}^3_\xi)$:

$$T = \lim_{\epsilon \searrow 0} \frac{Y(|P(\xi)| - \epsilon)}{P(\xi)}$$

Borovikov's formula yields, in the case of $m = n = 3$, the following representation of $\langle \varphi, E \rangle$ for $\varphi \in \mathscr{S}(\mathsf{R}^3_x)$ (cf. [1, (5r), p. 204; Engl.: 95d), p. 16], [7, Ch. I, 6.2, (5), p. 129] or [15, Satz 3, p. 410]):

$$\begin{aligned}
\langle \varphi, E \rangle &= -\frac{1}{16\pi^2} \left\langle \int \varphi(x) \mathrm{sign}(\omega \cdot x) \, \mathrm{d}x, \Phi(\omega) \right\rangle \\
&= -\frac{1}{16\pi^2} \lim_{\epsilon \searrow 0} \int_{|P(\omega)|>\epsilon} \left(\int \varphi(x) \mathrm{sign}(\omega \cdot x) \, \mathrm{d}x \right) \frac{\mathrm{d}\sigma(\omega)}{P(\omega)}.
\end{aligned}$$

The estimate

$$\exists C > 0 : \forall \epsilon > 0 : \forall x \in \mathsf{R}^3 \setminus \{0\} : \left| \int\limits_{|P(\omega)|>\epsilon} \frac{\mathrm{sign}(\omega \cdot x)}{P(\omega)} \, \mathrm{d}\sigma(\omega) \right|$$
$$\leq C \max\left\{ 1 + \ln\left(\frac{|x| \, |\nabla P(\xi)|}{|x \times \nabla P(\xi)|} \right) : \xi \in \mathsf{R}^3 \setminus \{0\}, P(\xi) = 0 \right\}$$

(where it is understood that $\ln \infty = \infty$) implies that E is given by a locally integrable function, namely

$$E(x) = -\frac{1}{16\pi^2} \lim_{\epsilon \searrow 0} \int\limits_{|P(\omega)|>\epsilon} \frac{\operatorname{sign}(\omega \cdot x)}{P(\omega)} \, \mathrm{d}\sigma(\omega),$$

and, moreover, that the modulus of E is inferior to a constant multiple of the function

$$1 + \left| \ln \operatorname{dist}\left(\frac{x}{|x|}, \operatorname{sing\,supp} E\right)\right|.$$

(In the last formula, we put, as usual, $\operatorname{dist}(u, M) = \inf\{|u - v| : v \in M\}$.)

By the odd parity of the functions $P(\omega)$ and $\omega \longmapsto \operatorname{sign}(\omega \cdot x)$, the integral for $E(x)$ above can also be written as one over the two-dimensional projective space P_2. If

$$\mathsf{P}_2 = \mathsf{S}_2 \text{ modulo } \{\pm 1\} = \{[\omega] : \omega \in \mathsf{S}_2\}$$

is parametrized, as usually, by $u = \frac{\omega_1}{\omega_3}$, $v = \frac{\omega_2}{\omega_3}$, then

$$\mathrm{d}\sigma([\omega]) = \frac{\mathrm{d}u \, \mathrm{d}v}{(1 + u^2 + v^2)^{3/2}} = |\omega_3|^3 \, \mathrm{d}u \, \mathrm{d}v$$

and hence, using the equation $\operatorname{sign} = 2Y - 1$ and the substitution $\lambda = -ux_1 - vx_2$, we obtain (almost everywhere with respect to x)

$$E(x) = -\frac{1}{8\pi^2} \lim_{\epsilon \searrow 0} \int\limits_{|P(u,v,1)|>\epsilon} \frac{\operatorname{sign}(ux_1 + vx_2 + x_3)}{P(u, v, 1)} \, \mathrm{d}u \, \mathrm{d}v \tag{3}$$

$$= -E(0, 0, 1) - \frac{1}{4\pi^2} \lim_{\epsilon \searrow 0} \int\limits_{|P(u,v,1)|>\epsilon} \frac{Y(ux_1 + vx_2 + x_3)}{P(u, v, 1)} \, \mathrm{d}u \, \mathrm{d}v$$

$$= -E(0, 0, 1) - \frac{1}{4\pi^2 |x_2|} \int_{-\infty}^{x_3} \mathrm{d}\lambda \oint_{\mathbb{C}} \frac{\mathrm{d}u}{P\big(u, -(\lambda + ux_1)/x_2, 1\big)} \tag{4}$$

(comp. [16, p. 15]). Here we assumed $x_2 \neq 0$ and $(0, 0, 1) \notin \operatorname{sing\,supp} E$.

From the fact that, for all pairwise different $a, b, c \in \mathsf{R}$,

$$\oint_{\mathbb{C}} \frac{\mathrm{d}u}{(u - a)(u - b)(u - c)} = 0,$$

we conclude $\partial_3 E(x) = 0$ if the polynomial $u \longmapsto P(u, -(x_3 + ux_1)/x_2, 1)$ has three real zeros. The region of x where this is the case is bounded by such points x for which the projective plane $\{[\omega] \in \mathsf{P}_2 : \omega \cdot x = 0\}$ touches the projective variety $\{[\omega] \in \mathsf{P}_2 : P(\omega) = 0\}$. This happens iff $x = \pm \nabla P(\xi)$ for some $\xi \in \mathsf{R}^3$, i.e., iff $x \in \operatorname{sing\,supp} E$. Therefore, E is constant in those components of $\mathsf{R}^3_x \setminus \operatorname{sing\,supp} E$ in which $\{[\omega] \in \mathsf{P}_2 : \omega \cdot x = P(\omega) = 0\}$ consists of three points.

2.3. Finally, we specialize on the polynomial $P(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3$.

Then E is constant inside $\pm L$ (L having been defined in (1)), and $\operatorname{sing\,supp} E = \partial L \cup -\partial L$.

Inserting the substitution $v = \sqrt[3]{u^3+1}\, w$ into formula (3) yields

$$(5) \quad E(x) = -\frac{1}{8\pi^2} \int \frac{\operatorname{sign}(u^3+1)}{|u^3+1|^{2/3}}\, \mathrm{d}u \oint \frac{\operatorname{sign}(ux_1 + \sqrt[3]{u^3+1}\, wx_2 + x_3)}{w^3+1}\, \mathrm{d}w.$$

The application of the estimate

$$\exists C > 0 : \forall a, b \in \mathsf{R} : \left| \oint \frac{\operatorname{sign}(aw+b)}{w^3+1}\, \mathrm{d}w \right| \leq C\big(1 + \ln(|a|+1) + \big|\ln|b-a|\big|\big)$$

in eq. (5) shows that E is continuous in $\mathsf{R}^3 \setminus \{0\}$.

Let us calculate some values of E. Formula (5) and [9, 151.5a and 151.13] yield

$$E(0,0,1) = -\frac{1}{8\pi^2} \int \frac{\operatorname{sign}(u^3+1)}{|u^3+1|^{2/3}}\, \mathrm{d}u \oint \frac{\mathrm{d}w}{w^3+1} = -\frac{1}{24\sqrt{3}\,\pi} \int \frac{\operatorname{sign}(t+1)}{|t|^{2/3}|t+1|^{2/3}}\, \mathrm{d}t$$

$$= -\frac{1}{24\sqrt{3}\,\pi} \int_{-1}^{0} \frac{\mathrm{d}t}{t^{2/3}(t+1)^{2/3}} = -\frac{B(\frac{1}{3}, \frac{1}{3})}{24\sqrt{3}\,\pi} \approx -0.04058;$$

similarly, formula (4) furnishes

$$E|_L = E(0,1,1) = -E(0,0,1) - \frac{1}{4\pi^2} \int_{-\infty}^{1} \mathrm{d}\lambda \oint \frac{\mathrm{d}u}{u^3 - \lambda^3 + 1}$$

$$= -E(0,0,1) - \frac{1}{4\sqrt{3}\,\pi} \int_{-\infty}^{1} \frac{\mathrm{d}\lambda}{(1-\lambda^3)^{2/3}} = -\frac{B(\frac{1}{3}, \frac{1}{3})}{8\sqrt{3}\,\pi} \approx -0.12175.$$

3. Representation of E by elliptic integrals

3.1. We start from formula (4) where we set $P(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3$ and suppose that $x_2 \neq 0$, $x_1 \neq x_2$, $x_3 < 0$, and that the polynomial $u \longmapsto P(u, -(\lambda + ux_1)/x_2, 1)$ has only one real zero for $-\infty < \lambda \leq x_3$.

An easy calculation yields

$$P(u, -(\lambda + ux_1)/x_2, 1) = u^3\Big(1 - \frac{x_1^3}{x_2^3}\Big) - 3u^2\frac{\lambda x_1^2}{x_2^3} - 3u\frac{\lambda^2 x_1}{x_2^3} - \frac{\lambda^3}{x_2^3} + 1$$
$$= \Big(1 - \frac{x_1^3}{x_2^3}\Big)(s^3 + gs + h),$$

where $s := u - \lambda x_1^2/(x_2^3 - x_1^3)$ and

$$g := -3\frac{x_1 x_2^3 \lambda^2}{(x_2^3 - x_1^3)^2}, \qquad h := \frac{x_2^3}{x_2^3 - x_1^3} - \frac{x_2^3(x_1^3 + x_2^3)\lambda^3}{(x_2^3 - x_1^3)^3}. \tag{6}$$

Hence setting $s = \lambda t$ and $\mu = \lambda^{-3}$, and

$$p := -3\frac{x_1 x_2^3}{(x_2^3 - x_1^3)^2}, \qquad q := \mu\frac{x_2^3}{x_2^3 - x_1^3} - \frac{x_2^3(x_1^3 + x_2^3)}{(x_2^3 - x_1^3)^3}$$

we obtain

$$\begin{aligned} E(x) &= -E(0,0,1) - \frac{x_2^2 \operatorname{sign} x_2}{4\pi^2(x_2^3 - x_1^3)}\int_{-\infty}^{x_3} \mathrm{d}\lambda \oint_{C} \frac{\mathrm{d}s}{s^3 + gs + h} \\ &= -E(0,0,1) + \frac{x_2^2 \operatorname{sign} x_2}{12\pi^2(x_2^3 - x_1^3)}\int_{x_3^{-3}}^{0} \frac{\mathrm{d}\mu}{|\mu|^{2/3}} \oint_{C} \frac{\mathrm{d}t}{t^3 + pt + q}. \end{aligned} \tag{7}$$

Now we can apply the following lemma.

Lemma. *Let* $p, c, d, \mu_1, \mu_2 \in \mathsf{R}$ *with* $c \neq 0$, $\mu_1 < \mu_2$, *and*

$$\forall \mu \in [\mu_1, \mu_2] : D(\mu) := \frac{(c\mu + d)^2}{4} + \frac{p^3}{27} > 0.$$

If, furthermore, $f \in L^1([\mu_1, \mu_2])$, $S(\tau) := \frac{-\tau^2 - d\tau + p^3/27}{c\tau}$, *and* a_j, b_j *with* $a_j < b_j$ *are the roots of* $S(\tau) = \mu_j$, *i.e.,*

$$\left.\begin{matrix} b_j \\ a_j \end{matrix}\right\} = -\frac{c\mu_j + d}{2} \pm \sqrt{D(\mu_j)}, \qquad j = 1, 2,$$

then

$$\int_{\mu_1}^{\mu_2} f(\mu)\,\mathrm{d}\mu \oint_{C} \frac{\mathrm{d}t}{t^3 + pt + c\mu + d} = \frac{\pi}{c\sqrt{3}}\Big[\int_{b_1}^{b_2} - \int_{a_1}^{a_2}\Big] f(S(\tau))\,\frac{\mathrm{d}\tau}{|\tau|^{2/3}}.$$

Proof. Let $q := c\mu + d$. Due to the condition $D(\mu) > 0$, the quadratical resolvent $R(\mu, \tau) := \tau^2 + q\tau - p^3/27$ of the cubic $Q(t) := t^3 + pt + q$ has two real roots $\tau_{1,2} = \alpha, \beta$ depending on μ. Assume $\alpha < \beta$ and take $\sqrt[3]{\alpha}, \sqrt[3]{\beta} \in \mathsf{R}$. Then $Q(t)$ has one real root, namely $t_1 = \sqrt[3]{\alpha} + \sqrt[3]{\beta}$, and two further complex roots

$$t_2 = e^{2\pi i/3}\sqrt[3]{\alpha} + e^{-2\pi i/3}\sqrt[3]{\beta}, \qquad t_3 = \overline{t_2}, \qquad \operatorname{Im} t_3 > 0,$$

and, therefore,

$$\oint \frac{dt}{t^3 + pt + q} = \pi i\Big[\operatorname*{Res}_{t=t_3} - \operatorname*{Res}_{t=t_2}\Big]\frac{1}{t^3 + pt + q} = -2\pi \operatorname{Im}\frac{1}{(t_3 - t_1)(t_3 - t_2)}$$
$$= \frac{2\pi}{\sqrt{3}(\sqrt[3]{\beta} - \sqrt[3]{\alpha})}\operatorname{Re}\frac{1}{t_3 - t_1} = \frac{\pi}{\sqrt{3}}\frac{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}{\alpha - \beta} = \frac{\pi}{\sqrt{3}}\left(\frac{\sqrt[3]{\alpha}}{\partial_\tau R(\mu, \alpha)} - \frac{\sqrt[3]{\beta}}{\partial_\tau R(\mu, \beta)}\right).$$

Hence, with the substitutions $\tau = \alpha(\mu)$, $\tau = \beta(\mu)$, and taking into account that this implies $R(\mu, \tau) = 0$, $\mu = S(\tau)$, and thus

$$\partial_\mu R(\mu, \tau) + \frac{d\tau}{d\mu}\,\partial_\tau R(\mu, \tau) = 0, \qquad \frac{d\mu}{\partial_\tau R(\mu, \tau)} = -\frac{d\tau}{c\tau},$$

we conclude that

$$\int_{\mu_1}^{\mu_2} f(\mu)\,d\mu \oint \frac{dt}{t^3 + pt + c\mu + d}$$
$$= \frac{\pi}{\sqrt{3}}\int_{\mu_1}^{\mu_2} f(\mu)\left(\frac{\sqrt[3]{\alpha(\mu)}}{\partial_\tau R(\mu, \alpha(\mu))} - \frac{\sqrt[3]{\beta(\mu)}}{\partial_\tau R(\mu, \beta(\mu))}\right)d\mu$$
$$= \frac{\pi}{c\sqrt{3}}\left[\int_{b_1}^{b_2} - \int_{a_1}^{a_2}\right] f(S(\tau))\,\frac{d\tau}{|\tau|^{2/3}}.$$

We apply the assertion of the Lemma to eq. (7). Here

$$f(\mu) = |\mu|^{-2/3},\ p = -3\frac{x_1 x_2^3}{(x_2^3 - x_1^3)^2},\ c = \frac{x_2^3}{x_2^3 - x_1^3},$$
$$d = -\frac{x_2^3(x_1^3 + x_2^3)}{(x_2^3 - x_1^3)^3},\ \mu_1 = x_3^{-3}, \mu_2 = 0$$

and hence

$$E(x) = -E(0,0,1) + \frac{|x_2|}{12\sqrt{3}\,\pi|x_2^3 - x_1^3|^{2/3}}\left[\int_{b_1}^{b_2} - \int_{a_1}^{a_2}\right]\frac{d\tau}{|\tau^2 + d\tau - p^3/27|^{2/3}},$$

where $a_1 < b_1$ and $a_2 < b_2$ are the roots of

$$\tau^2 + d\tau - \frac{p^3}{27} = -\frac{c\tau}{x_3^3} \text{ and of } \tau^2 + d\tau - \frac{p^3}{27} = 0,$$

respectively. The subsequent substitutions $\mu = \tau + d/2$ and $\nu = 2(x_2^3 - x_1^3)^2 x_2^{-3}\mu$ yield

$$E(x) = -E(0,0,1) + \frac{|x_2|}{12\sqrt{3}\,\pi|x_2^3 - x_1^3|^{2/3}}\Big[\int_{b'_1}^{b'_2} - \int_{a'_1}^{a'_2}\Big]\frac{\mathrm{d}\mu}{|\mu^2 - x_2^6/(4(x_2^3 - x_1^3)^4)|^{2/3}}$$

$$= -E(0,0,1) + \frac{\sqrt[3]{2}}{12\sqrt{3}\,\pi}\Big[\int_{a''_2}^{b''_2} - \int_{a''_1}^{b''_1}\Big]\frac{\mathrm{d}\nu}{|\nu^2 - 1|^{2/3}}.$$

Here $a''_2 < b''_2$ are the roots of $\nu^2 = 1$, i.e., $a''_2 = -1$, $b''_2 = 1$, and $a''_1 < b''_1$ are the roots of

$$\nu^2 - 1 = \frac{2(x_1^3 - x_2^3)}{x_3^3}\,\nu - \frac{2(x_1^3 + x_2^3)}{x_3^3},$$

i.e.,

$$\text{(8)} \qquad \left.\begin{matrix} b''_1 \\ a''_1 \end{matrix}\right\} = \frac{x_1^3 - x_2^3 \mp \sqrt{A(x)}}{x_3^3},$$

$$A(x) := x_1^6 + x_2^6 + x_3^6 - 2x_1^3x_2^3 - 2x_1^3x_3^3 - 2x_2^3x_3^3.$$

Since, by [9, 421.3],

$$-E(0,0,1) + \frac{\sqrt[3]{2}}{12\sqrt{3}\,\pi}\int_{-1}^{1}\frac{\mathrm{d}\nu}{|\nu^2 - 1|^{2/3}} = -E|_L,$$

we derive, due to the odd parity of E and the principle of analytic continuation, the following formula for $E(x)$, which is valid for $x \in \Omega := \mathsf{R}^3 \setminus (\overline{L} \cup -\overline{L})$ with $x_3 \neq 0$:

$$E(x) = E|_L\ \mathrm{sign}(x_3) + \frac{\sqrt[3]{2}}{12\sqrt{3}\,\pi}\int_{(x_1^3 - x_2^3 - \sqrt{A(x)})/x_3^3}^{(x_1^3 - x_2^3 + \sqrt{A(x)})/x_3^3}\frac{\mathrm{d}\nu}{|\nu^2 - 1|^{2/3}}$$

3.2. In order to give a representation of E which is symmetric in the coordinates, we make use of the addition theorem for elliptic functions. Suppose that $x_1, x_2 > 0$, $x_3 \gg x_1 + x_2$. The substitution $\nu = \sqrt{1 - t^3}$ yields

$$E(x) = E|_L + \frac{\sqrt[3]{2}}{8\sqrt{3}\,\pi}\Big[\int_y^1 + \int_z^1\Big]\frac{\mathrm{d}t}{\sqrt{1 - t^3}},$$

wherein

$$\text{(9)} \qquad \left.\begin{matrix} y \\ z \end{matrix}\right\} = \sqrt[3]{1 - \big(\sqrt{A(x)} \pm \big(x_1^3 - x_2^3\big)\big)^2/x_3^6}\,.$$

The addition theorem (cf. [8, 8.166.2], [2, 9.7, p. 281]) states in our situation that

$$\Big[\int_y^1 + \int_z^1\Big] \frac{\mathrm{d}t}{\sqrt{1-t^3}} = \int_w^1 \frac{\mathrm{d}t}{\sqrt{1-t^3}},$$

if $y \neq z$ are near 1 and

$$w = 1 - \frac{3(y-z)^2}{2 - yz(y+z) + (y-z)^2 - 2\sqrt{1-y^3}\sqrt{1-z^3}} = 1 - \frac{3}{\zeta + 1}$$

with

$$\zeta := \frac{2(1 - \sqrt{1-y^3}\sqrt{1-z^3}) - yz(y+z)}{(y-z)^2}. \tag{10}$$

Note that

$$\frac{\sqrt[3]{2}}{8\sqrt{3}\,\pi} \int_{-\infty}^1 \frac{\mathrm{d}t}{\sqrt{1-t^3}} = -E|_L$$

and hence the final substitution $t = 1 - 3/(u+1)$ furnishes

$$E(x) = -\frac{\sqrt[3]{2}}{8\sqrt{3}\,\pi} \int_{-1}^{\zeta} \frac{\mathrm{d}u}{\sqrt{u^3+1}},$$

which is valid for positive x_j with $x_3 \gg x_1 + x_2$.

To prove the formula in (g) of the Theorem in Section 1, it only remains to show that ζ satisfies the cubic equation (2) given there. In fact, if this is the case, then

$$\zeta = -1 \Longleftrightarrow 3\sqrt[3]{2}\, x_1 x_2 x_3 - x_1^3 - x_2^3 - x_3^3 = 0$$

and thus, by analytic continuation and the parity of E, we conclude that

$$E(x) = \frac{\sqrt[3]{2}}{8\sqrt{3}\,\pi}\ \mathrm{sign}(3\sqrt[3]{2}\, x_1 x_2 x_3 - x_1^3 - x_2^3 - x_3^3) \int_{-1}^{\zeta} \frac{\mathrm{d}u}{\sqrt{u^3+1}}$$

for all $x \in \Omega$ (comp. the Remark in Section 1).

Though y, z are given explicitly as functions of x in (9), there is no easy way to derive therefrom the cubic equation (2) for ζ, which is given by (10). We just outline the procedure.

Denote by s_1 and s_2, s_3 the real and the two complex conjugate roots, respectively, of the equation $s^3 + gs + h = 0$, where g, h are as in (6) and $\lambda = x_3$. Then $s_1 = \sqrt[3]{\alpha} + \sqrt[3]{\beta}$, if α, β are the roots of $\tau^2 + h\tau - g^3/27$. A simple calculation shows that α, β coincide with $x_2^3 x_3^6 y^3/(4(x_2^3 - x_1^3)^3)$ and $x_2^3 x_3^6 z^3/(4(x_2^3 - x_1^3)^3)$. Therefore,

$$y+z=\frac{\sqrt[3]{4}(x_2^3-x_1^3)}{x_2x_3^2}\,s_1$$

and similarly

$$\text{(11)}\qquad y-z=\pm\frac{\sqrt[3]{4}\,\mathrm{i}(x_2^3-x_1^3)}{\sqrt{3}\,x_2x_3^2}\,(s_2-s_3),$$

$$\zeta=\frac{3\sqrt[3]{4}\,x_2^2}{x_2^3-x_1^3}\cdot\frac{x_1(x_2^3-x_1^3)s_1-x_3(x_2^3+x_1^3)}{(s_2-s_3)^2}.$$

Now the coefficients of the cubic equation (2) for ζ can be computed as symmetric functions of its roots, which in turn are obtained from (11) by permuting s_1, s_2, s_3.

3.3. We finally depict E by drawing some contour lines of the function $(x_1,x_2)\longmapsto E(x_1,x_2,1)$. For that purpose, we first solve eq. (2) for ζ. This yields, with $A(x)$ as in (8),

$$\text{(12)}\qquad \zeta=\frac{\sqrt[3]{4}}{A(x)}\left(3x_1^2x_2^2x_3^2+\sqrt[3]{\alpha_1}+\sqrt[3]{\alpha_2}\right)$$

$$\text{with } \alpha_{1,2}=\tfrac{1}{2}\big[54x_1^6x_2^6x_3^6+9x_1^3x_2^3x_3^3(x_1^3+x_2^3+x_3^3)A(x)$$
$$+(x_1^3x_2^3+x_1^3x_3^3+x_2^3x_3^3)A(x)^2\pm(x_1^3-x_2^3)(x_1^3-x_3^3)(x_2^3-x_3^3)A(x)^{3/2}\big].$$

The value of ζ corresponding to the level surfaces $E(x)=\pm cE|_L$, $c\in[0,1]$, can be found by solving the equation

$$F\left(\arccos\left(\frac{\sqrt{3}-1-\zeta}{\sqrt{3}+1+\zeta}\right),\frac{\sqrt{3}+1}{2\sqrt{2}}\right)=\frac{c\sqrt[4]{3}B(\frac{1}{3},\frac{1}{3})}{\sqrt[3]{2}}.$$

Hence

$$\zeta=\frac{\sqrt{3}-1-u(\sqrt{3}+1)}{u+1},\ \text{where } u=\mathrm{cn}\left(\frac{c\sqrt[4]{3}B(\frac{1}{3},\frac{1}{3})}{\sqrt[3]{2}}\right)$$

and cn denotes one of Jacobi's elliptic functions.

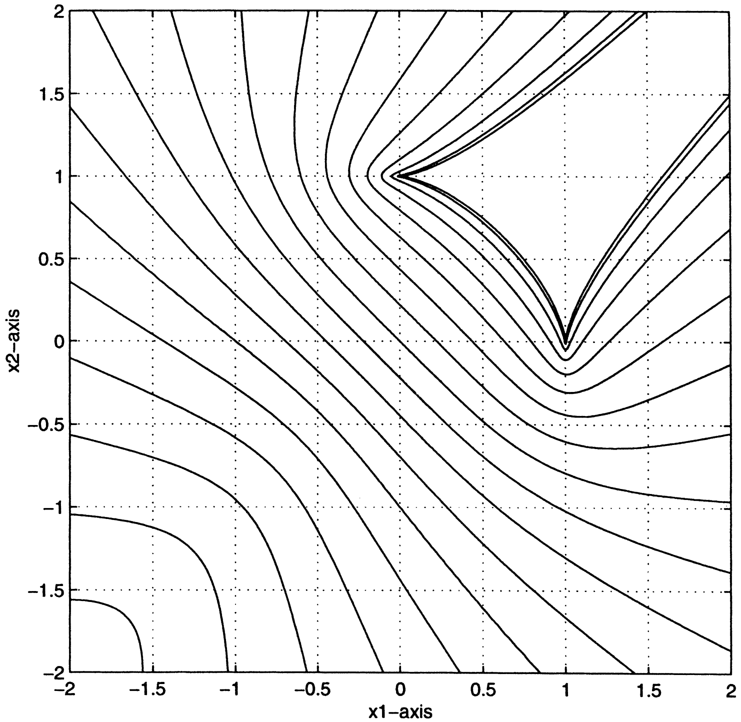

Figure 1: Contour lines of $E(x_1, x_2, 1)$ at height increments of $-\frac{1}{12}E|_L$

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NOTE ADDED IN PROOF. The Theorem has recently been generalized to the operators of the form $\partial_1^3 + \partial_2^3 + \partial_3^3 + 3a\partial_1\partial_2\partial_3$, $a \in \mathsf{R}$, cf. P. Wagner, *Fundamental solutions of real homogeneous cubic operators of principal type in three dimensions*, Acta. Math. 182 (1999), 283–300.

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