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ON ADDITIVE K-THEORY WITH THE LODAY - QUILLEN *-PRODUCT

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Abstract

The *-product defined by Loday and Quillen [17] on the additive K-theory (the cyclic homology with shifted degrees) $K_*^+(A)$ for a commutative ring A is naturally extended to a product (*-product) on the additive K-theory $K_*^+(\Omega)$ for a differential graded algebra (Ω, d) over a commutative ring. We prove that Connes' B-maps from the additive K-theory $K_*^+(\Omega)$ to the negative cyclic homology $\mathrm{HC}_*^-(\Omega)$ and to the Hochschild homology $\mathrm{HH}_*(\Omega)$ are morphisms of algebras under the *-product on $K_*^+(\Omega)$. Applications to topology of Connes' B-maps are also described.

§0. Introduction

Let A be an algebra over a commutative ring. Let $\mathrm{HC}_n^-(A)$ and $\mathrm{HH}_n(A)$ denote the negative cyclic homology and the Hochschild homology of A, respectively. In the algebraic K-theory, C. Hood and J. D. S. Jones [11] have constructed the Chern character $ch_n : K_n(A) \to \mathrm{HC}_n^-(A)$ which is a lift of the Dennis trace map $\mathrm{Dtr} : K_n(A) \to \mathrm{HH}_n(A)$ by modifying basic construction due to Connes [5] and Karoubi [12]. When the algebra A is commutative, the usual pairing of $K_*(A)$ and the product on $\mathrm{HC}_*^-(A)$ defined by Hood and Jones in [11] make the character ch_* into a morphism of algebras. In consequence, we can have the following commutative diagram in the category of graded algebras:

$$
\begin{array}{ccc}
K_*(A) & \xrightarrow{\ \mathrm{ch}_*\ } & \mathrm{HC}_*^-(A) \\
 & \searrow^{\mathrm{Dtr}} & \downarrow h \\
 & & \mathrm{HH}_*(A)
\end{array}
\tag{0.1}
$$

Here h is the map induced from the natural projection to the Hochschild complex from the cyclic bar complex. The Chern character $\mathrm{ch} : K_0(A) \to \mathrm{HC}_0^-(A) = \mathrm{HC}_0^{\mathrm{per}}(A)$ is connected with the ordinary Chern character $K(X) \to H^{\mathrm{even}}_{\mathrm{de\,Rham}}(X; \mathsf{C})$ when A is the ring consisting of smooth

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functions from a compact manifold X to the complex number C (see, for example, [19, 6.2.9. Example]). Therefore, one may expect that the Chern character $\mathrm{ch}_n : K_n(A) \to \mathrm{HC}_n^-(A)$ or the Dennis trace map $\mathrm{Dtr} : K_n(A) \to \mathrm{HH}_n(A)$ becomes a map with value in the de Rham (singular) cohomology of some manifold (space) by replacing the algebra A with an appropriate object concerning with the space.

Hochschild and (negative) cyclic homologies can be extended to functors defined on the category of commutative differential graded algebras (DGAs) over a commutative ring (see [8], [11], [4]). In particular, if we choose the de Rham complex $(\Omega(X), d)$ of a simply connected manifold X as the DGA, the Hochschild and the negative cyclic homology of $\Omega(X)$ can be regarded as the real cohomology and the real T-equivariant cohomology of the space of free loops on X respectively (see [8]), where T denotes the circle group. However, in algebraic K-theory, we can not expect such an extension. What is "K-theory" which addmits an extension to a functor on the category of DGAs and in which there is a commutative diagram corresponding to (0.1)? We can consider the additive K-theory $K_*^+(A)$ (see [6]) as "K-theory", which is isomorphic to the positive cyclic homology group $\mathrm{HC}_{*-1}(A)$. Let ϕ be the isomorphism form $K_*^+(A)$ to $\mathrm{HC}_{*-1}(A)$ defined by Loday and Quillen in [17] and independently Tsygan in [21]. Tillmann's commutative diagram [20, Theorem 1] connects the dual of the Dennis trace map with the Connes B-map by the dual of the isomorphism $\phi : K_*^+(A) \to \mathrm{HC}_{*-1}(A)$ when A is a Banach algebra. Therefore it is natural to choose the Connes B-map B_{HH}: $\mathrm{HC}_{*-1}(A) \to \mathrm{HH}_*(A)$ as a map in the additive K-theory corresponding to the Dennis trace map in algebraic K-theory. The Connes' B-map $B_{\mathrm{HH}} : K_*^+(A) \cong \mathrm{HC}_{*-1}(A) \to \mathrm{HH}_*(A)$ has a natural lift B, which is also called Conne's B-map, to the negative cyclic homology $\mathrm{HC}_*^-(A)$. Moreover functors HC_*, HC_*^-, HH_* and the connecting maps can be extend on the category of DGAs by using the cyclic bar complex in [7] and [8]. In the consequence, we can obtain the following commutative diagram corresponding to (0.1) in the category of graded modules:

$$
\begin{array}{ccc}
K_*^+(\Omega) \cong \mathrm{HC}_{*-1}(\Omega) & \xrightarrow{\quad B \quad} & \mathrm{HC}_*^-(\Omega) \\
& \searrow^{B_{\mathrm{HH}}} & \downarrow h \\
& & \mathrm{HH}_*(\Omega),
\end{array}
\tag{0.2}
$$

where Ω is a DGA. We propose a natural question that whether the diagram (0.2) is commutative in the category of graded algebras, as well as the diagram (0.1), under an appropriate product on $K_*^+(\Omega)$. To answer this question, we extend the *-product defined by Loday and Quillen [17] to a product on the additive K-theory (the cyclic homology with shifted degrees) of a

DGA, which is an explicit version of that of Hood and Jones [11, Theorem 2.6]. Since the product is defined at chain level, we can see that

THEOREM 0.1. *The diagram* (0.2) *is commutative in the category of graded algebras when the product of* $K_*^+(\Omega)$ *is given by the* *-*product.*

Let M be a simply connected manifold and LM the space of all smooth maps from circle group T to M. By using the Connes' B-map B_{HH}, we consider the vanishing problem of string class of a loop group bundle $L\mathrm{Spin}(n) \to LQ \to LM$. In the consequence, a generalization of the main theorem in [14] is obtained when the given manifold M is formal (see Theorem 2.1).

We also show that the algebra structure of $\mathrm{HC}_*^-(\Omega)$ can be described with the *-product on $K_*^+(\Omega)$ via Connes' B-map $B : K_*^+(\Omega) \to \mathrm{HC}_*^-(\Omega)$ when the DGA (Ω, d) over a field $\mathbf{k}$ of characteristic zero is formal. This fact allows us to deduce the following theorem.

THEOREM 0.2. *Let* X *be a formal simply connected manifold. Then*

$$H_{\mathsf{T}}^*(LX; \mathsf{R}) \cong \{H^*(LX; \mathsf{R})/\mathrm{Im}(B_{\mathrm{HH}} \circ I)\}^{*+1} \oplus \mathsf{R}[u]$$

as an algebra, where $I : H^*(LX; \mathsf{R}) = \mathrm{HH}_{-*}(\Omega(X)) \to K_{-*}^+(\Omega(X))$ *is the map in Connes' exact sequence* (1,1) *mentioned in* §1 *for the de Rham complex* $\Omega(X)$ *with negative degrees and* $\mathsf{R}[u]$ *is the polynomial algebra over* u *with degree* 2. *The multiplication of the algebra on the right hand side is given as follows;* $w * u^i = 0$ *and* $w * w' = w \cdot BIw'$, *where* $\cdot$ *is the cup product on* $H^*(LX; \mathsf{R})$. *In particular,*

(i) *if* $H^*(X; \mathsf{R}) \cong \mathsf{R}[x]/(x^{s+1})$ *and* $s > 1$, *then*

$$H_{\mathsf{T}}^*(LX; \mathsf{R}) \cong \oplus_{k \geq 0, 1 \leq j \leq s} \mathsf{R}\{\beta(j,k)\} \oplus \mathsf{R}[u]$$

as an algebra, where $\deg \beta(j,k) = j \deg x + k((s+1)\deg x - 2) - 1$, $\beta(j,k) * \beta(j',k') = 0$ *and* $\beta(j,k) * u = 0$ *for any* j, k, j', k', *and*

(ii) *if* $H^*(X; \mathsf{R}) \cong \Lambda(y)$, *then*

$$H_{\mathsf{T}}^*(LX; \mathsf{R}) \cong \oplus_{k \geq 0} \mathsf{R}\{\beta(k)\} \oplus \mathsf{R}[u]$$

as an algebra, where $\deg \beta(k) = (k+1)(\deg y - 1)$, $\beta(k) * \beta(j) = \beta(k+j+1)$ *and* $\beta(k) * u = 0$ *for any* j, k.

As for the algebra structure of $H_{\mathsf{T}}^*(LX; \mathsf{R})$, the above results cover [13, Theorem 2.4].

This paper is set out as follows. In Section 1, we define the additive K-Theory $K_*^+(\Omega)$ of a DGA (Ω, d) over a commutative ring and a product (*-product) on $K_*^+(\Omega)$. Some properties of the *-product will also be studied.

In Section 2, we will describe the applications of Connes' B-maps B and B_{HH} which are mentioned above.

§1. The *-product on K_*^+

Let (Ω, d) be a commutative differential graded algebra (DGA) over a commutative ring $\mathbf{k}$, $\Omega = \bigoplus_{i \le 0} \Omega_i$, with unit 1 in Ω_0, endowed with a differential d of degree -1 satisfying $d(1) = 0$. We assume that differential graded algebras are non-positively graded algebras with the above properties unless otherwise stated. We recall the cyclic bar complex defined in [7] and [8]. The complex $(\mathbf{C}(\Omega)[u^{-1}], b + uB)$ is defined as follows:

$$\mathbf{C}(\Omega) = \sum_{k=0}^{\infty} \Omega \otimes \bar{\Omega}^{\otimes k},$$

$$\begin{aligned} b(\omega_0, .., \omega_k) = & -\sum_{i=0}^{k} (-1)^{\varepsilon_{i-1}} (\omega_0, .., \omega_{i-1}, d\omega_i, \omega_{i+1}, .., \omega_k) \\ & -\sum_{i=0}^{k-1} (-1)^{\varepsilon_i} (\omega_0, .., \omega_{i-1}, \omega_i \omega_{i+1}, \omega_{i+2}, .., \omega_k) \\ & +(-1)^{(deg\omega_k - 1)\varepsilon_{k-1}} (\omega_k \omega_0, .., \omega_{k-1}), \ b(u^{-1}) = 0 \end{aligned}$$

and

$$B(\omega_0, .., \omega_k) = \sum_{i=0}^{k} (-1)(\varepsilon_{i-1} + 1)(\varepsilon_k - \varepsilon_{i-1})(1, \omega_i, .., \omega_k, \omega_0, .., \omega_{i-1}), \ B(u^{-1}) = 0,$$

where $\bar{\Omega} = \Omega/\mathbf{k}$, $\deg(\omega_0, .., \omega_k) = \deg \omega_0 + \cdots + \deg \omega_k + k$, for $(\omega_0, .., \omega_k)$ in $\mathbf{C}(\Omega)$, $\varepsilon_i = \deg \omega_0 + \cdots + \deg \omega_i - i$ and $\deg u = -2$. Note that the formulas $bB + Bb = 0$ and $b^2 = B^2 = 0$ hold, see [7]. The negative cyclic homology $\mathrm{HC}_*^-(\Omega)$, the periodic cyclic homology $\mathrm{HC}_*^{\mathrm{per}}(\Omega)$ and the Hochschild homology $\mathrm{HH}_*(\Omega)$ of a DGA (Ω, d) are defined as the homology of the complexes $(\mathbf{C}(\Omega)[[u]], b + uB)$, $(\mathbf{C}(\Omega)[[u, u^{-1}], b + uB)$ and $(\mathbf{C}(\Omega), b)$, respectively. Since a DGA in our case has negative degrees, the power series algebra $\mathbf{C}(\Omega)[[u]]$ agrees with the polynomial algebra $\mathbf{C}(\Omega)[u]$, similarly, $\mathbf{C}(\Omega)[[u, u^{-1}] = \mathbf{C}(\Omega)[u, u^{-1}]$.

We define the nth additive K-theory $K_n^+(\Omega, d)$ of a DGA (Ω, d) to be the $(n-1)$-th cyclic homology $\mathrm{HC}_{n-1}(\Omega, d)$ which is the $(n-1)$-th homology of the cyclic bar complex $(\mathbf{C}(\Omega)[u^{-1}], b + uB)$:

$$K_*^+(\Omega) = \mathrm{HC}_{*-1}(\Omega) = H_{*-1}(\mathbf{C}(\Omega)[u^{-1}], b + uB).$$

Unless we note the differential d of a DGA in particular, $K_n^+(\Omega, d)$ will be denoted by $K_n^+(\Omega)$. We define a product (*-product) on the complex $(\mathbf{C}(\Omega)[u^{-1}], b + uB)$ as follows:

$$\sum_{i=0}^{n} x_i u^{-i} * \sum_{j=0}^{m} y_j u^{-j} = \sum_{i=0}^{n} x_i \cdot B y_0 u^{-i},$$

where $\cdot$ is the shuffle product on $\mathbf{C}(\Omega)$.

Proposition 1.1. (i) *The *-product induces a degree* $+1$ *map of complexes* $\mathbf{C}(\Omega)[u^{-1}] \otimes \mathbf{C}(\Omega)[u^{-1}] \to \mathbf{C}(\Omega)[u^{-1}]$ *which is associative.*

(ii) *The *-product on the cyclic bar complex defines an associative graded commutative algebra structure on* $K_*^+(\Omega)$.

In [7], to give an A_∞-algebra structure to the graded $\mathbf{k}$-module $\mathbf{C}(\Omega)[[u]]$, E. Getzler and J. D. S. Jones have defined a sequence of operators $B_k\colon \mathbf{C}(\Omega)^{\otimes k} \to \mathbf{C}(\Omega)$ of degree k and have clarified relation of B_k, B_{k-1} and the shuffle products on $\mathbf{C}(\Omega)$. In particular, in order to prove Proposition 1.1, we need the following formula representing the relation of the operator B_2, Connes' B-operator $B : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$ and the shuffle products.

Lemma 1.2. ([7, Lemma 4.3]) *There exists an operator* $B_2\colon \mathbf{C}(\Omega)^{\otimes 2} \to \mathbf{C}(\Omega)$ of rank 2 *satisfying*

$$(1.2.1)\qquad (-1)^{|\alpha|+1} b B_2(\alpha, \beta) + B(\alpha \cdot \beta) = (-1)^{|b\alpha|+1} B_2(b\alpha, \beta) + (B\alpha) \cdot \beta + (-1)^{|\alpha|}\{\alpha \cdot B\beta + (-1)^{|\alpha|+1} B_2(\alpha, b\beta)\}.$$

The definitions of B_2 (see [7, page 280]) and B enable us to deduce that, for any elements z and z' in $\mathbf{C}(\Omega)$,

$$(1.2.2)\qquad B_2(z, Bz') = B_2(Bz, z') = 0.$$

Proof of Proposition 1.1. (i) From the formulas (1.2.1) and (1.2.2), by replacing the element β with $B\gamma$, it follows that $B(\alpha \cdot B\gamma) = B\alpha \cdot B\gamma$. For any elements $x = \sum x_i u^{-i}$, $y = \sum y_j u^{-j}$ and $z = \sum z_k u^{-k}$ in $\mathbf{C}(\Omega)[u^{-1}]$, we see that, on $\mathbf{C}(\Omega)[u^{-1}]$, $x * (y * z) = x * (\sum y_j \cdot Bz_0)u^{-j} = \sum x_i \cdot B(y_0 \cdot Bz_0)u^{-i} = (x * y) * z$. We will prove that *-product is a map of complexes. Since the differential b is a derivation under the shuffle product on $\mathbf{C}(\Omega)[u^{-1}]$, by the formula $b \circ B + B \circ b = 0$, we have

$$
\begin{aligned}
(b+uB)(x*y) &= (b+uB)\Big(\sum_{i\geq 0}(x_i\cdot By_0)u^{-i}\Big)\\
&= \sum_{i\geq 0}(bx_i)\cdot By_0u^{-i} + \sum_{i\geq 0}(-1)^{|x_i|}x_i\cdot bBy_0u^{-i} + \sum_{i\geq 0}B(x_i\cdot By_0)u^{-i+1}\\
&= \sum_{i\geq 0}(bx_i)\cdot By_0u^{-i} + \sum_{i\geq 0}(-1)^{|x_i|+1}x_i\cdot Bby_0u^{-i} + \sum_{i\geq 0}Bx_{i+1}\cdot By_0u^{-i}.
\end{aligned}
$$

On the other hand, by the formula $B\circ B=0$, we have

$$
\begin{aligned}
&(b+uB)x*y+(-1)^{|x|+1}x*(b+uB)y\\
&\qquad = \sum_{i\geq 0}(bx_i)\cdot By_0u^{-i} + \sum_{i\geq 0}Bx_{i+1}\cdot By_0u^{-i}\\
&\qquad\quad +(-1)^{|x|+1}x*\Big(\sum_{j\geq 0}by_ju^{-j} + \sum_{j\geq 0}By_{j+1}u^{-j}\Big)
\end{aligned}
$$

Thus we can conclude that $(b+uB)(x*y)=(b+uB)x*y+(-1)^{|x|+1}x*(b+uB)y$. Note that $(-1)^{|x|}=(-1)^{|x_i|}$ for any i.

(ii) To prove that the *-product defines a graded commutative algebra structure on $K_*^+(\Omega)$, it suffices to prove that, for any elements $x=\sum x_iu^{-i}$ and $y=\sum y_ju^{-j}$ in $\mathrm{Ker}(b+uB)$, there exists an element $\omega=\sum_{k\geq 0}\omega_{k-1}u^{-k}$ such that

$$
x_k\cdot By_0-(-1)^{(|x|+1)(|y|+1)}y_k\cdot Bx_0 = b\omega_{k-1}+B\omega_k
$$

for any $k\geq 0$. We will verify that

$$
\omega_k=(-1)^{(|y|+1)|x|}\Big(\sum_{i+j=k}y_i\cdot x_j-\sum_{i+j=k+1}(-1)^{|y_i|}B_2(y_i,x_j)\Big) \text{ for } k\geq 0 \text{ and}
$$

$$
\omega_{-1}=(-1)^{(|x|+1)(|y|+1)}B_2(y_0,x_0)
$$

are factors of the required element. Since equalities $by_i=-By_{i+1}$ and $bx_j=-B_{j+1}x_{j+1}$ hold, it follows that, if $k\geq 0$,

$$
\begin{aligned}
&(-1)^{(|y|+1)|x|}(b\omega_{k-1}+B\omega_k)\\
&=\sum_{i+j=k-1}\{-By_{i+1}\cdot x_j+(-1)^{|y_i|}y_i\cdot(-Bx_{j+1})\}-\sum_{i+j=k}(-1)^{|y_i|}bB_2(y_i,x_j)\\
&+\sum_{i+j=k}\{By_i\cdot x_j+(-1)^{|y_i|}y_i\cdot Bx_j+(-1)^{|y_i|}bB_2(y_i,x_j)\}\\
&=-\sum_{i+j=k,i\geq 1}By_i\cdot x_j-\sum_{i+j=k,j\geq 1}(-1)^{|y_i|}y_i\cdot Bx_j\\
&+\sum_{i+j=k}By_i\cdot x_j+\sum_{i+j=k}(-1)^{|y_i|}y_i\cdot Bx_j\\
&=By_0\cdot x_k+(-1)^{|y_k|}y_k\cdot Bx_0\\
&=(-1)^{(|y|+1)|x|}(x_k\cdot By_0-(-1)^{(|y|+1)(|x|+1)}y_k\cdot Bx_0)
\end{aligned}
$$

from the formulas (1.2.1) and (1.2.2). We can check that equality $b\omega_{-1}+B\omega_0=x_0\cdot By_0-(-1)^{(|x|+1)(|y|+1)}y_0\cdot Bx_0$ holds in a similar way.

We define Connes' B-maps $B_{\mathrm{HH}}:K_n^+(\Omega)\longrightarrow \mathrm{HH}_n(\Omega)$ and $B:K_n^+(\Omega)\longrightarrow \mathrm{HC}_n^-(\Omega)$ by $B_{\mathrm{HH}}(\sum_{i\geq 0}x_iu^{-i})=Bx_0$ and $B(\sum_{i\geq 0}x_iu^{-i})=Bx_0$. Note that the maps B_{HH} and B are connecting maps in Connes' exact sequences ([16, Theorem 2.2.1 and Proposition 5.1.5])

$$
(1.1)\qquad \cdots\to \mathrm{HH}_{n+1}(\Omega)\xrightarrow{I}K_{n+2}^+(\Omega)\xrightarrow{S}K_n^+(\Omega)\xrightarrow{B_{\mathrm{HH}}}\mathrm{HH}_n(\Omega)\to\cdots
$$

and

$$
(1.2)\qquad \cdots\to \mathrm{HC}_{n+1}^-(\Omega)\xrightarrow{\times u}\mathrm{HC}_{n-1}^{\mathrm{per}}(\Omega)\longrightarrow K_n^+(\Omega)\xrightarrow{B}\mathrm{HC}_n^-(\Omega)\to\cdots
$$

respectively.

Proof of Theorem 0.1. The product structure m_2 on $\mathbf{C}(\Omega)[u]$ defined by $m_2(\alpha_1,\alpha_2)=\alpha_1\cdot\alpha_2+(-1)^{|\alpha_1|+1}uB_2(\alpha_1,\alpha_2)$ induces the algebra structure of $\mathrm{HC}_*^-(\Omega)$. From (1.2.2), we see that the product m_2 agrees with the shuffle product if α_1 or α_2 belongs to the image of the operator $B:\mathbf{C}(\Omega)\to\mathbf{C}(\Omega)$. Therefore the formula $B(\alpha\cdot B\gamma)=B\alpha\cdot B\gamma$ implies that the map $B:K_*^+(\Omega)\to\mathrm{HC}_*^-(\Omega)$ is a morphism of algebras.

In study of the cyclic homology theory, it is often useful to consider the reduced theory. To prove some theorems below, we will use the reduced additive K-theory $\tilde{K}_*^+(\Omega)$ defined by $\tilde{K}_*^+(\Omega)=\mathrm{Coker}(\iota_*:K_*^+(\mathbf{k})\to K_*^+(\Omega))$, where $\iota:\mathbf{k}\to\Omega$ is the unit. The reduced additive K-theory $\tilde{K}_*^+(\Omega)$ is a direct summand of $K_*^+(\Omega)$ because the exact sequence $0\to\mathbf{C}(\mathbf{k})[u^{-1}]\to\mathbf{C}(\Omega)[u^{-1}]\to\mathbf{C}(\bar{\Omega})[u^{-1}]\to 0$ of cyclic chain complexes is a split sequence.

More precisely, $K_*^+(\Omega)$ is isomorphic to $\tilde{K}_*^+(\Omega) \oplus \mathbf{k}[u^{-1}]$ as a graded $\mathrm{HC}_*(\mathbf{k}) = \mathbf{k}[u^{-1}]$-module. When one notices the direct summand $\mathbf{k}[u^{-1}]$ of $K_*^+(\Omega)$, by definition of *-product, it follows that $\mathbf{k}[u^{-1}]$ is included in the annihilator ideal of $K_*^+(\Omega)$. Therefore we can also conclude that the algebra $(K_*^+(\Omega), *)$ does not have an unit.

Let us consider a relation of the *-product on $K_*^+(\Omega)$ to the suspension map $S : K_*^+(\Omega) \to K_{*-2}^+(\Omega)$ in Connes' exact sequence (1.1). Since the suspension map S is defined by $S\left(\sum_{i\geq 0} x_i u^{-i}\right) = \sum_{i\geq 0} x_{i+1} u^{-i}$, it follows that $Sx * y = S(x * y)$ on $\mathbf{C}(\Omega)[u^{-1}]$. From this fact and commutativity of the *-product, we have

Proposition 1.3. *For any elements ω and η in $K_*^+(\Omega)$,*

$$S\omega * \eta = S(\omega * \eta) = \omega * S\eta.$$

For the rest of this paper, unless otherwise mentioned, we will assume that any DGA (Ω, d) is a commutative algebra over a field $\mathbf{k}$ of characteristic zero, connected and simply connected, that is, $\Omega = \oplus_{i\leq 0}\Omega_i$, $\Omega_0 = \mathbf{k}$, $H_1(\Omega) = 0$ and $d(1) = 0$. A DGA (Ω, d) is said to be *formal* if there exists a DGA-morphism from the minimal model $\mathcal{M}$ of Ω to the DGA $(H^*(\Omega, d), 0)$ which induces a isomorphism between their homologies (see [10]).

For any DGA $(\Omega, 0)$ with the trivial differential, M. Vigué-Poirrier has given a decomposition of the negative cyclic homology $\mathrm{HC}_*^-(\Omega, 0)$: $\widetilde{\mathrm{HC}}_*^-(\Omega, 0) = \oplus_{q\geq 0} H(\mathcal{C}_*^q[u],\ b + uB)$, and has shown that the S-action on $\widetilde{\mathrm{HC}}_*^-(\Omega, 0)$ is trivial, see [22, Proposition 5], where $\mathcal{C}_n^q = \{(a_0, .., a_p) | \sum \deg a_i = -q, -q + p = -n\}$. This fact implies that the S-action on $\widetilde{\mathrm{HC}}_*^-(\Omega, d)$ for any formal DGA (Ω, d) is trivial ([22, Théorème A]). The proof of [22, Proposition 5] is based on Goodwillie's result [9, Corollary III.4.4], which is led from the following proposition.

Proposition 1.4. [9]. *Let (Ω, d) be a* DGA *over a commutative ring and D a derivation on Ω with degree $|D|$ satisfying that $D(ab) = (Da)b + (-1)^{|D||a|}a(Db)$ and $[D, d] = 0$. Then there exist chain maps $e_D : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$ of degree $|D| - 1$, $E_D : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$ of degree $|D| + 1$ and an operator $L_D : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$ of degree $|D|$ such that $[u^{-1}e_D + E_D, b + uB] = L_D$ in $\mathbf{C}(\Omega)[u, u^{-1}]$, where $[a, b] = ab - (-1)^{|a||b|}ba$ for any operators a and b.*

We can obtain a lemma by using Proposition 1.4 and the idea of the decomposition of cyclic homology due to Vigué-Poirrier [22].

Lemma 1.5. *Let $(\Omega, 0)$ be a* DGA *with the trivial differential. For any ele-*

ment ω *in* $\tilde{K}_*^+(\Omega,0) = \widetilde{\mathrm{HC}}_{*-1}(\Omega,0)$, *there exists an element* η_0 *in* $\mathbf{C}(\Omega) \cap \ker b$ *such that* $\omega = [\eta_0]$ *in* $\tilde{K}_*^+(\Omega,0)$.

Proof. According to Vigué-Poirrier [22], we define a derivation D on Ω by $D(a) = (\deg a)a$. Consider a decomposition $K_*^+(\Omega,0) = \sum_{q\geq 0} K_*^+(\Omega,0)^q$ defined by $K_*^+(\Omega,0)^q = H_{*-1}(\mathscr{C}_*^q[u^{-1}], b+uB)$. Since $\tilde{K}_*^+(\Omega)$ is isomorphic to $\sum_{q\geq 1} K_*^+(\Omega)^q$, in order to prove Lemma 1.5, it suffices to show that there exists an element η_0 with the property in Lemma 1.5 for any element ω in $K_*^+(\Omega)^q$ $(q \geq 1)$. Since the operation L_D on $\mathbf{C}(\Omega)$ is defined by $L_D(a_0,..,a_p) = \sum_{0\leq i\leq p}(a_0,..,Da_i,..,a_p)$, it follows that the operator L_D on $K_*^+(\Omega)^q$ is given by $L_D(\omega) = -q\omega$ in our case. On the other hand, for any element ω in $K_*^+(\Omega)^q$ which is represented by $\sum_{i\geq 0}\omega_i u^{-i}$ in $\mathbf{C}(\Omega)[u^{-1}]$, we have that $[u^{-1}e_D + E_D, b+uB]\omega = \ e_D B\omega_0 + uE_D B\omega_0 - \ (b+uB)(u^{-1}e_D + E_D)\omega$ in $\mathbf{C}(\Omega)[u,u^{-1}]$. By virtue of Proposition 1.4, we can conclude that $e_D B\omega_0 - \ (b+uB)(u^{-1}e_D + E_D)\omega = -q\omega$ in $\mathbf{C}(\Omega)[u^{-1}]$. Thus, we see that $-\frac{1}{q}e_D B\omega_0$ is the required element η_0.

We will consider the algebra structure of $K_*^+(\Omega)$ by using a minimal model of (Ω,d). Let $\varphi : (\mathscr{M}, d_{\mathscr{M}}) \longrightarrow (\Omega,d)$ be a minimal model of a DGA (Ω,d). Then φ induces an isomorphism of algebras $K(\varphi) : K_*^+(\mathscr{M}) \longrightarrow K_*^+(\Omega)$. Therefore if a DGA (Ω,d) is formal, then there exist isomorphisms $K_*^+(\Omega,d) \cong K_*^+(\mathscr{M},d_M) \cong K_*^+(H(\Omega),0)$. It follows immediately that the isomorphisms are compatible with the S-action. Since Lemma 1.5 asserts that any element of $\tilde{K}_*^+(\Omega,0)$ can be represented by an element with column degree 0, from the definition of S-action, we can get

Proposition 1.6. *If a* DGA (Ω,d) *is formal, then the suspension map* $S : \tilde{K}_*^+(\Omega) \longrightarrow \tilde{K}_{*-2}^+(\Omega)$ *is trivial.*

Let $(\mathscr{M}, d_{\mathscr{M}})$ be a free commutative differential graded algebra $(\Lambda V, d)$ over $\mathbf{k}$. We denote by $(\mathscr{E}(\mathscr{M}), \delta, \beta)$ the double complex defined in [4, Example 2] by D. Burghelea and M. Vigué-Poirrier. Namely, $\mathscr{E}(\mathscr{M}) = \Lambda(V \oplus \bar{V})$, β is the unique derivation of degree $+1$ defined by $\beta v = \bar{v}$ and δ is the unique derivation of degree -1 which satisfies $\delta|_V = d$ and $\delta\beta + \beta\delta = 0$. Here $\bar{V}$ is the vector space with $\bar{V}_{n+1} = V_n$. We here mention that the double complex induces the complex $(\mathscr{E}(\mathscr{M})[u^{-1}], \delta + u\beta)$ with a product defined by $\sum \omega_i u^{-i} * \sum \eta_j u^{-j} = \sum \omega_i \beta \eta_0 u^{-i}$. By [4, Theorem 2.4 (i)], we see that the map $\Theta : \mathbf{C}(\mathscr{M}) \to \mathscr{E}(\mathscr{M})$ defined by $\Theta(a_0,a_1,..,a_p) = 1/p!\ a_0\beta a_1 \cdots \beta a_p$ is a chain map between the double complexes $(\mathbf{C}(\mathscr{M}), b, B)$ and $(\mathscr{E}(\mathscr{M}), \delta, \beta)$. [4, Theorem 2.4 (iii)] shows that the induced map $K(\Theta)$ from $K_*^+(\mathscr{M})$ to $H_{*-1}(\mathscr{E}(\mathscr{M})[u^{-1}], \delta + u\beta)$ is an isomorphism of graded vector spaces. Moreover we have

Proposition 1.7. *The map* $K(\Theta) : K_*^+(\mathcal{M}) \to H_{*-1}(\mathcal{E}(\mathcal{M})[u^{-1}], \delta + u\beta)$ *is an isomorphism of algebras.*

The following lemma will be needed to prove that $K(\Theta)$ is a morphism of algebras.

Lemma 1.8. *Let* $(\mathcal{M}, d_{\mathcal{M}})$ *be a free commutative* DGA.

(i) *The chain map* $\Theta : \mathbf{C}(\mathcal{M}) \to \mathcal{E}(\mathcal{M})$ *is compatible with* $B : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$ *and* $\beta : \mathcal{E}(\mathcal{M}) \to \mathcal{E}(\mathcal{M}) : \beta\Theta = \Theta B$.

(ii) *Let* W *be a subspace of* $\mathbf{C}(\mathcal{M})$ *consisting of the elements whose first factor have even degree:* $W = \{\sum_i (a_{i_0}, .., a_{i_{k(i)}}) \in \mathbf{C}(\mathcal{M}) | \deg a_{i_0} \text{ is even } \}$. *Then* $\Theta(\omega \cdot \omega') = \Theta\omega \cdot \Theta\omega'$ *for any element* ω' *in* W *and any element* ω *in* $\mathbf{C}(\mathcal{M})$*, here* $\cdot$ *in the left hand side and right hand side are the shuffle product on* $\mathbf{C}(\mathcal{M})$ *and the natural product on* $\mathcal{E}(\mathcal{M})$ *respectively.*

Proof. It is straightforward to check that identities $\beta\Theta = \Theta B$ and $\Theta(\omega \cdot \omega') = \Theta\omega \cdot \Theta\omega'$ hold.

Proof of Proposition 1.7. From the definition of Connes' B-operator, it follows that Im B is a subspace of W in Lemma 1.8. By virtue of Lemma 1.8, we see that $\Theta(\omega \cdot B\omega') = \Theta\omega \cdot \beta\Theta\omega'$ for any element ω and ω' in $\mathbf{C}(\mathcal{M})$. Thus we can conclude that $K(\Theta)$ is a morphism of graded algebras.

By virtue of Proposition 1.7, we can determine $K_*^+(\Omega)$ explicitly as an algebra when the homology of (Ω, d) is generated with one generator.

Theorem 1.9. *For any formal* DGA (Ω, d),

$$K_*^+(\Omega) \cong \widetilde{\mathrm{HH}}_{*-1}(\Omega)/\mathrm{Im}(B_{\mathrm{HH}} \circ I : \widetilde{\mathrm{HH}}_{*-2}(\Omega) \to \widetilde{\mathrm{HH}}_{*-1}(\Omega)) \oplus \mathbf{k}\{1, u^{-1}, u^{-2}, ..\}$$

as an algebra, where $\deg u^{-k} = 2k+1$, $\omega * \omega' = \omega \cdot B\omega'$, $\omega * u^{-k} = 0$ *for any elements* ω *and* ω' *in* $\widetilde{\mathrm{HH}}_*(\Omega)/\mathrm{Im}(B_{\mathrm{HH}} \circ I)$ *and* $u^{-i} * u^{-j} = 0$. *In particular,*

(i) *when* $\deg x$ *is even,*

$$K_*^+(\mathbf{k}[x]/(x^{s+1})) \cong \oplus_{k \geq 0, 1 \leq j \leq s} \mathbf{k}\{\beta(j,k)\} \oplus \mathbf{k}\{1, u^{-1}, u^{-2}, ..\},$$

where $\deg \beta(j,k) = j \deg x + k((s+1)\deg x + 2) + 1$, $\omega * \omega' = 0$ *for any elements* ω *and* ω' *in* $K_*^+(\mathbf{k}[x]/(x^{s+1}))$, *and*

(ii) *when* $\deg y$ *is odd,*

$$K_*^+(\Lambda(y)) \cong \oplus_{k \geq 0} \mathbf{k}\{\beta(k)\} \oplus \mathbf{k}\{1, u^{-1}, u^{-2}, ..\}$$

where $\deg \beta(k) = \deg y + k(\deg y + 1) + 1$, $\beta(k) * \beta(j) = \beta(k+j+1)$, $1 * \beta(k) = 0$ *and* $\beta(k) * u^{-l} = 0$.

Proof. By Proposition 1.6, the suspension map $S : \tilde{K}_*^+(\Omega) \to \tilde{K}_{*-2}^+(\Omega)$ is trivial. From this fact and Connes' exact sequence (1.1) obtained by using $\bar{\Omega}$ instead of a DGA Ω, it follows that the map $I : \widetilde{\mathrm{HH}}_{*-1}(\Omega) \to \tilde{K}_*^+(\Omega)$ is surjective and that the kernel of I is the image of $B_{\mathrm{HH}} \circ I : \widetilde{\mathrm{HH}}_{*-2}(\Omega) \to \widetilde{\mathrm{HH}}_{*-1}(\Omega)$. Thus we can conclude that $K_*^+(\Omega) \cong \tilde{K}_*^+(\Omega) \oplus \mathbf{k}\{1, u^{-1}, u^{-2}, ..\} \cong \widetilde{\mathrm{HH}}_{*-1}(\Omega)/\mathrm{Im}(B_{\mathrm{HH}} \circ I) \oplus \ \mathbf{k}\{1, u^{-1}, u^{-2}, ..\}$ as algebras. From Proposition 1.7 and the explicit formulas of the Hochschild homology of $\mathbf{k}[x]/(x^{s+1})$ and $\Lambda(y)$ in [15], we can get (i) and (ii).

Remark. In Theorem 1.9, the elements $\beta(j,k)$ and $\beta(k)$ correspond to the elements $x^j\omega^k$ and $y\bar{y}^k$ in [15, Proposition 1.1(ii)], respectively.

As mentioned before Proposition 1.3, the algebra $K_*^+(\Omega)$ does not have an unit. Since $\tilde{K}_0^+(\Omega)$ is non zero in general, the algebra $\tilde{K}_*^+(\Omega)$ may be have an unit. However, the results of Theorem 1.9 (i) and (ii) enable us to conjecture that the reduced additive K-theory $\tilde{K}_*^+$ does not have an unit for any DGA either. The first assertion in the following proposition is an answer to the conjecture.

Proposition 1.10. (i) *Let* (Ω, d) *be a* DGA. *Assume that* $\tilde{K}_*^+(\Omega) \neq 0$. *Then the algebra* $\tilde{K}_*^+(\Omega)$ *does not have an unit.*

(ii) *If* deg $QH(\Omega, d) \geq n$, *then there exist n elements* $x_1, x_2, .. , x_n$ *in* $K_*^+(\Omega)$ *such that* $x_1 * x_2 * \cdots * x_n \neq 0$, *where* $QH(\Omega, d)$ *denotes the space of indecomposable elements in the graded algebra* $H(\Omega, d)$.

Proof. From the usual argument on a minimal model of Ω, we can assume that Ω is free.

(i) Suppose that there exists an element e in $\tilde{K}_*^+(\Omega)$ such that $e * x = x$ for any x in $\tilde{K}_*^+(\Omega)$. Let us consider the Hodge decomposition of Hochschild homology ([3], [4, Theorem 3.1]): $\widetilde{\mathrm{HH}}_*(\Omega) = \oplus_{i\geq 0}\widetilde{\mathrm{HH}}_*^{(i)}(\Omega)$. Since $B_{\mathrm{HH}} : \tilde{K}_*^+(\Omega) \to \widetilde{\mathrm{HH}}_*(\Omega)$ is a morphism of algebras by Theorem 0.1, it follows that $B_{\mathrm{HH}}(e) \cdot B_{\mathrm{HH}}(x) = B_{\mathrm{HH}}(x)$. We see that $B_{\mathrm{HH}}(e)$ belongs to $\mathrm{HH}_*^{(0)}(\Omega)$ because deg $B_{\mathrm{HH}}(e) = 0$. The definition of the Hodge decomposition and Lemma 1.8 (i) enables us to deduce that Im B_{HH} is included in $\oplus_{i\geq 1}\widetilde{\mathrm{HH}}_*^{(i)}(\Omega)$. Thus we have $B_{\mathrm{HH}}(e) = 0$. On the other hand, we see that $S^N e = 0$ for some sufficient large integer N. If $B_{\mathrm{HH}}(x) = 0$ for all $x \in \tilde{K}_*^+(\Omega)$, then the map $S : \tilde{K}_{*+2}^+(\Omega) \to \tilde{K}_*^+(\Omega)$ is epimorphism. Therefore, for any $x \in \tilde{K}_*^+(\Omega)$, there is an element $x' \in \tilde{K}_*^+(\Omega)$ such that $S^N x' = x$. It follows from Proposition 1.3 that $x = e * x = e * S^N x' = S^N e * x' = 0$ for any x, which a contradiction. Thus $B_{\mathrm{HH}}(x) \neq 0$ for some $x \in \tilde{K}_*^+(\Omega)$. However, $B_{\mathrm{HH}}(x) = B_{\mathrm{HH}}(e) \cdot B_{\mathrm{HH}}(x) = 0$. The result now follows.

(ii) We can choose n elements of Ω corresponding to x_i in $K_*^+(\Omega)$ which are part of generators of Ω. We represent the elements with the same notation $x_1, ..., x_n$, respectively. Under the isomorphism $H(\Theta)$ in [4, Theorem 2.4

(ii)], $B_{\mathrm{HH}}(x_1 * \cdots * x_n) = B_{\mathrm{HH}}x_1 \cdot \cdots \cdot B_{\mathrm{HH}}x_n = \bar{x}_1 \cdots \bar{x}_n$ in $H_*(\mathrm{Tot}\, \mathscr{E}(\Omega), \delta)$. Since $\mathrm{Im}\, \delta$ consists of elements whose factors have an element in Ω, it follows that $\bar{x}_1 \cdots \bar{x}_n \neq 0$ in $\mathrm{HH}_*(\Omega) \cong H_*(\mathrm{Tot}\, \mathscr{E}(\Omega), \delta)$. By virtue of Proposition 1.7, we can see that $x_1 * \cdots * x_n \neq 0$ in $K_*^+(\Omega)$.

From Proposition 1.10 (ii), Theorem 1.9 (i) and (ii), we can conclude that $K_*^+(\Omega)$ has trivial algebra structure if and only if the homology of (Ω, d) is generated by one element with even degree.

§2. Applications of Connes' B-maps B_{HH} and B

Let M be a simply connected manifold and LM the space of C^∞-free loops on M. When an $\mathrm{SO}(n)$-bundle $P \to M$ over M has a spin structure $Q \to M$, the string class $\mu(Q)$, which belongs to $H^3(LM; \mathsf{Z})$, is defined as an obstruction to lift the structure group $L\mathrm{Spin}(n)$ of $LQ \to LM$ to $\widehat{L\mathrm{Spin}}(n)$, for details see [18]. Here $\widehat{L\mathrm{Spin}}(n)$ is the universal central extension of $L\mathrm{Spin}(n)$ by the circle. One of important properties for the string class $\mu(Q)$ is the fact that the class $\mu(Q)$ is the image of $\frac{1}{2} p_1$ by the map $\int_{S^1} \circ \mathrm{ev}^* : H^*(M; \mathsf{Z}) \to H^*(LM \times S^1; \mathsf{Z}) \to H^{*-1}(LM; \mathsf{Z})$, where p_1 is the first Pontrjagin class of the bundle $P \to M$, $\mathrm{ev} : LM \times S^1 \to M$ is the evaluation map and $\int_{S^1}$ is the integration along S^1. Let G be a linear Lie group and $\xi : Q \to M$ a G-bundle over M. Let $\mathrm{Ch}^{p+1}(\xi)$ be the Chern character of the bundle ξ. The higher string classes $\tilde{C}^p(L\xi)$ $(p \geq 1)$ (see [2]) in $H^{2p+1}(LM; \mathsf{C})$ defined for the LG-bundle $L\xi : LQ \to LM$ has a similar property to the ordinary string class $\mu(Q)$. Indeed, the pth string class $\tilde{C}^p(L\xi)$ is the image of $-(2\pi\sqrt{-1})^{p+1} p! Ch^{p+1}(\xi)$ by the map $\int_{S^1} \circ \mathrm{ev}^*$. As mentioned in the introduction, in the study of the problem of whether the map $\int_{S^1} \circ \mathrm{ev}^*$ is injective, the Connes' B-map $B_{\mathrm{HH}} : K_*^+(\Omega(M)) \to \mathrm{HH}_*(\Omega(M)) \cong H^*(LM; \mathsf{R})$ plays an important role. We will have the following theorem which is a generalization of [14, Theorem 2]. We may call a simply connected manifold *formal* if its de Rham complex is formal (see [10]).

THEOREM 2.1. *Let M be a simply connected manifold and formal.*

(i) *For any $SO(n)$-bundle $P \to M$ with a spin structure $Q \to M$, if $H^3(M; \mathsf{Z})$ is torsion free, then the string class $\mu(Q)$ vanishes if and only if $\frac{1}{2} p_1$ vanishes.*

(ii) *Let G be a linear Lie algebra and $\xi : Q \to M$ a G-bundle. The string class $\tilde{C}^p(L\xi)$ vanishes if and only if the Chern character* $\mathrm{Ch}^{p+1}(\xi)$ *of the bundle ξ vanishes.*

By virtue of [14, Proposition 2.1], we can regard the map $\int_{S^1} \circ \mathrm{ev}^* : H^*(M; \mathsf{R}) \to H^{*-1}(LX; \mathsf{R})$ as the map $\alpha : H^*(M; \mathsf{R}) \to \mathrm{HH}_{-*}(\Omega(M), d)$ defined by $\alpha(x) = (1, x)$ under the identification by the iterated integral map $\sigma : \mathrm{HH}_{-*}(\Omega(M)) \to H^*(LM; \mathsf{R})$ ([8]), where $\Omega_{-i}(M)$ is the ith de Rham complex $\Omega^i_{\mathrm{de\,Rham}}(M)$ and the differential $d : \Omega_{-i}(M) \to \Omega_{-i-1}(M)$ is the exterior differential on the de Rham complex $\Omega^*_{\mathrm{de\,Rham}}(M)$. Thus, to prove Theorem 2.1, it suffices to show that the map α is injective when M is formal. Note that, for any DGA (Ω, d), we can define the map $\alpha : H_*(\Omega) \to \mathrm{HH}_*(\Omega)$ by $\alpha(x) = (1, x)$. The definition of the map α allows us to deduce that α factors through Connes' B-map B_{HH} as follows: $\alpha = B_{\mathrm{HH}} \circ I \circ i$, where $i : H_*(\Omega) \to \mathrm{HH}_*(\Omega)$ and $I : \mathrm{HH}_*(\Omega) \to K^+_*(\Omega)$ are the homomorphisms induced by the natural inclusions $\Omega \to \mathbf{C}(\Omega)$ and $\mathbf{C}(\Omega) \to \mathbf{C}(\Omega)[u{-}1]$ respectively. For any DGA (Ω, d), we have

Lemma 2.2. *The map* $H_{-*}(\Omega) \xrightarrow{i} \mathrm{HH}_{-*}(\Omega) \xrightarrow{I} K^+_{-*+1}(\Omega)$ *is injective.*

Proof. It suffices to prove that Lemma 2.2 holds when Ω is free. In this case, we can identify $K^+_*(\Omega)$ with the homology of the complex $(\mathscr{E}(\Omega)[u^{-1}], \delta + u\beta)$ by Proposition 1.7. Since $\mathrm{Im}(\delta + u\beta) \cap \Omega$ is contained in $\mathrm{Im}\, d$ which is a subspace of Ω, it follows that if $Ii(x)$ is zero in $K^+_*(\Omega)$, then so is x in $H_*(\Omega)$.

Proof of Theorem 2.1. The reduced additive K-theory $\tilde{K}^+_*(\Omega)$ includes $\mathrm{Im}(I \circ i : H_{*-1}(\Omega) \to K^+_*(\Omega))$ for $* < 1$. By Proposition 1.6, Connes' B-map $B_{\mathrm{HH}} : \tilde{K}^+_*(\Omega) \to \widetilde{\mathrm{HH}}_*(\Omega)$ is injective. Therefore we can have Theorem 2.1 by virtue of Lemma 2.2.

In general case, we can show that $Ii(\mathrm{Ker}\,\alpha)(= \mathrm{Im}(I \circ i) \cap \mathrm{Ker}\, B_{\mathrm{HH}})$ is contained in the space of annihilators of $K^+_*(\Omega)$.

Proposition 2.3. For any DGA (Ω, d), $K^+_*(\Omega) * \{\mathrm{Im}(I \circ i) \cap \mathrm{Ker}\, B_{\mathrm{HH}}\} = 0$.

Proof. For any element ω in $\mathrm{Im}(I \circ i) \cap \mathrm{Ker}\, B_{\mathrm{HH}}$, we can write $\beta\omega = \delta\eta$ for some element η in $\mathscr{E}(\Omega)$. For any element ω' in $\mathrm{Ker}(u\beta + \delta)$ which is the subspace of $\mathscr{E}(\Omega)[u^{-1}]$,

$$\begin{aligned}(u\beta + \delta)(\omega' \cdot \eta) &= (-1)^{deg\omega'} \omega' \cdot (u\beta + \delta)\eta \\ &= (-1)^{deg\omega'} \omega' \cdot (0 + \beta\omega) \\ &= (-1)^{deg\omega'} \omega' * \omega\end{aligned}$$

Note that $\beta\eta = 0$ in $\mathscr{E}(\Omega)[u^{-1}]$. Thus we see that $\omega' * \omega = 0$ in $K^+_*(\Omega)$.

We will describe some applications of Connes' B-map $B : K^+_*(\Omega) \to \mathrm{HC}^-_*(\Omega)$.

Proposition 2.4. *The following diagram is commutative:*

$$\begin{array}{ccc} K_*^+(\Omega) & \xrightarrow{B} & \mathrm{HC}_*^-(\Omega) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ K_{*-2}^+(\Omega) & \xrightarrow[B]{} & \mathrm{HC}_{*-2}^-(\Omega) \end{array}$$

Proof. For any element $\omega = \sum \omega_i u^{-i}$ in $\mathrm{Ker}(b + uB)$, by the definition of the S-action, we have that $BS\omega = B\omega_1$. On the other hand, $SB\omega = B\omega_0 u$. Since $b\omega_0 + B\omega_1 = 0$, it follows that $Bw_0u - B\omega_1 = (b + uB)\omega_0$. Thus we have $SB\omega = BS\omega$ in $\mathrm{HC}_*^-(\Omega)$.

If the S-action on $\widetilde{\mathrm{HC}}_*^-(\Omega)$ is trivial, then we can represent the algebra structure of the negative cyclic homology $\mathrm{HC}_*^-(\Omega)$ with the *-product on $K_*^+(\Omega)$.

Theorem 2.5. (i) *The map* $B : \tilde{K}_*^+(\Omega) \longrightarrow \widetilde{\mathrm{HC}}_*^-(\Omega)$ *induced by Connes' B-map is an isomorphism of algebras.*

(ii) *The S-action on* $\tilde{K}_*^+(\Omega)$ *is trivial if and only if so is the S-action on* $\widetilde{\mathrm{HC}}_*^-(\Omega)$.

(iii) *If the S-action on* $\widetilde{\mathrm{HC}}_*^-(\Omega)$ *is trivial, then* $\mathrm{HC}_*^-(\Omega) \cong \mathbf{k}[u] \oplus \tilde{K}_*^+(\Omega) \cong \mathbf{k}[u] \oplus \widetilde{\mathrm{HH}}_{*-1}(\Omega)/\mathrm{Im}(B_{\mathrm{HH}} \circ I)$ as algebras. By the assertions (i) *and* (ii)*, we see that* $\mathbf{k}[u] \oplus \widetilde{\mathrm{HC}}_*^-(\Omega) \cong \mathbf{k}[u] \oplus \tilde{K}_*^+(\Omega)$ *as an algebra.*

Proof. (i) The result [9, Theorem III.5.1] enables us to conclude that $\mathrm{HC}^{\mathrm{per}}(\Omega) \cong \mathbf{k}[u, u^{-1}]$. From Connes' exact sequence (1.2) for $\bar{\Omega}$, we can get (i). From (i) and Proposition 2.4, we have (ii). Since the S-action on $\mathrm{HC}_*^-(\Omega)$ is trivial, it follows that $\mathrm{HC}_*^-(\Omega) \cong \mathbf{k}[u] \oplus \widetilde{\mathrm{HC}}_*^-(\Omega)$ as an algebra. From the proof of Theorem 1.9, we deduce the results of (iii).

We can now prove Theorem 0.2.

Proof of Theorem 0.2. If $H^*(X; \mathsf{R})$ is isomorphic to the algebra $\mathsf{R}[x]/(x^{s+1})$ or $\Lambda(y)$, then X is a formal. By virtue of Theorem 2.5 (iii), we see that $H_T^*(LX; \mathsf{R}) \cong \mathrm{HC}_{-*}^-(\Omega(X)) \cong \mathrm{HC}_{-*}^-(H^*(X; \mathsf{R})) \cong \mathsf{R}[u] \oplus \tilde{K}_{-*}^+(H^*(X; \mathsf{R}))$. Therefore, Theorem 1.9 yields Theorem 0.2. In particular, we deduce (i) and (ii) by virtue of Theorem 1.9 (i) and (ii).

Let M be a simply connected manifold (simplicial complex) and $\Omega^*(M)$ its de Rham algebra of differential forms (simplicial differential forms) with coefficient in $\mathbf{k} = \mathsf{R}, \mathsf{C}$ ($\mathbf{k} = \mathsf{R}, \mathsf{C}, \mathsf{Q}$). Then the isomorphism $B : \tilde{K}_*^+(\Omega) \to \widetilde{\mathrm{HC}}_*^-(\Omega)$ in Theorem 2.5 (i) agrees with the isomorphism $b_M : \widetilde{\mathrm{HC}}_{-*-1}(\Omega(M)) \to \tilde{H}_{\mathsf{T}}^*(LM; \mathbf{k})$ in [3, Theorem B]. Therefore, if we regard $\tilde{K}_*^+(\Omega)$ as a graded algebra with the *-product, the isomorphism b_M becomes a morphism of algebras.

Let (Ω, d) and (Ω', d') be DGAs over a field $\mathbf{k}$ of characteristic zero. If one wants to know about the $\mathbf{k}$-module structure of the negative cyclic homology $\mathrm{HC}_*^-(\Omega \otimes \Omega')$, the use of the Künneth theorem [11, Theorem 3.1 (a)] for negative cyclic homology theory may be effective, because the exact sequence

$$0 \to \mathrm{HC}_*^-(\Omega) \otimes_{\mathbf{k}[u]} \mathrm{HC}_*^-(\Omega') \to \mathrm{HC}_*^-(\Omega \otimes \Omega') \to$$
$$\mathrm{Tor}_{\mathbf{k}[u]}(\mathrm{HC}_*^-(\Omega), \mathrm{HC}_*^-(\Omega'))_{*-1} \to 0$$

is split. However, it is not easy to determine the algebra structure of $\mathrm{HC}_*^-(\Omega \otimes \Omega')$ from the exact sequence even if Ω and Ω' are formal. Theorem 2.5 (ii) enables us to represent the graded algebra structure of $\mathrm{HC}_*^-(\Omega \otimes \Omega')$ with the Hochschild homologies $\mathrm{HH}_*(\Omega)$, $\mathrm{HH}_*(\Omega')$ and the *-product when Ω and Ω' are formal. In term of spaces, we also assert that the T-equivariant cohomology of the space of loops on the product space $M \times M'$ can be represented with the cohomologies of the spaces of loops on M and M', Connes' B-map B_{HH} and *-product.

Corollary 2.6. *Let M and M' be formal simply connected manifolds. Then*

$$H_{\mathsf{T}}^*(L(M \times M'); \mathsf{R}) \cong$$
$$\{(H^*(LM; \mathsf{R}) \otimes H^*(LM'; \mathsf{R})/\mathrm{Im}(B \circ I \otimes 1 \pm 1 \otimes B \circ I))\}^{*+1} \oplus \mathsf{R}[u]$$

as an algebra, where $\deg u = 2$. *Here the multiplication $*$ of the algebra on the right hand side is given as follows: $\omega \otimes \omega' * u = 0$, $\omega \otimes \omega' * \eta \otimes \eta' = \omega \otimes \omega' \cdot (BI\eta \otimes \eta' \; +(-1)^{|\eta|}\eta \otimes BI\eta')$ for any $\omega \otimes \omega'$ and $\eta \otimes \eta'$ in $H^*(LM; \mathsf{R}) \otimes H^*(LM'; \mathsf{R})/\mathrm{Im}\ (B \circ I \otimes 1 \pm 1 \otimes B \circ I)$, where $\cdot$ is the cup product on $H^*(LM; \mathsf{R}) \otimes H^*(LM'; \mathsf{R})$.*

Proof. Let $(\mathscr{M}, d)$ and $(\mathscr{M}', d)$ be minimal models of de Rham complexes $(\Omega(M), d)$ and $(\Omega(M'), d)$ respectively. We know that $\mathrm{HH}_{-*}(\mathscr{M}) \cong H^*(LM; \mathsf{R})$ and $\mathrm{HC}_{-*}^-(\mathscr{M}) \cong H_{\mathsf{T}}^*(LM; \mathsf{R})$ as algebras ([8]). By virtue of [22, Proposition 5], the S-action on $\mathrm{HC}_{-*}^-(\mathscr{M})$ is trivial. Therefore, it follows from Theorem 2.5 (ii) that, as algebras,

$$\begin{aligned} H^*_{\mathsf{T}}(L(M \times M'); \mathsf{R}) &\cong \mathrm{HC}^-_{-*}(\mathscr{M} \otimes \mathscr{M}') \\ &\cong \mathrm{HH}_{-*-1}(\mathscr{M} \otimes \mathscr{M})/\mathrm{Im}(B_{\mathrm{HH}} \circ I) \oplus \mathsf{R}[u] \\ &\cong H_{-*-1}(\mathscr{E}(\mathscr{M} \otimes \mathscr{M}'))/\mathrm{Im}(B_{\mathrm{HH}} \circ I) \oplus \mathsf{R}[u] \\ &\cong \{H_*(\mathscr{E}(\mathscr{M}) \otimes H_*(\mathscr{E}(\mathscr{M}'))/ \\ &\quad \mathrm{Im}(\beta \circ I \otimes 1 \pm 1 \otimes \beta \circ I)\}_{-*-1} \oplus \mathsf{R}[u] \\ &\cong \{H^*(LM; \mathsf{R}) \otimes H^*(LM'; \mathsf{R})/ \\ &\quad \mathrm{Im}(B \circ I \otimes 1 \pm 1 \otimes B \circ I)\}^{*+1} \oplus \mathsf{R}[u]. \end{aligned}$$

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