MATH. SCAND. 87 (2000), 27–63

MEET IRREDUCIBLE IDEALS IN DIRECT LIMIT ALGEBRAS

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We study the meet irreducible ideals (ideals I so that $I = J \cap K$ implies $I = J$ or $I = K$) in certain direct limit algebras. The direct limit algebras will generally be strongly maximal triangular subalgebras of AF C*-algebras, or briefly, strongly maximal TAF algebras. Of course, all ideals are closed and two-sided.

These ideals have a description in terms of the coordinates, or spectrum, that is a natural extension of one description of meet irreducible ideals in the upper triangular matrices. Additional information is available if the limit algebra is an analytic subalgebra of its C*-envelope or if the analytic algebra is trivially analytic with an injective 0-cocycle. In the latter case, we obtain a complete description of the meet irreducible ideals, modeled on the description in the algebra of upper triangular matrices. This applies, in particular, to all full nest algebras.

One reason for interest in the meet irreducible ideals of a strongly maximal TAF algebra is that each meet irreducible ideal is the kernel of a nest representation of the algebra (Theorem 2.4). A *nest representation* of an operator algebra A is a norm continuous representation of A acting on a Hilbert space with the property that the lattice of closed invariant subspaces for the representation is totally ordered. These representations were introduced in [L1] as analogues for a general operator algebra of the irreducible representations of a C*-algebra. The meet irreducible ideals seem analogous to the primitive ideals in a C*-algebra. Indeed, in a C*-algebra, the meet irreducible ideals are precisely the primitive ideals [L3, Theorem 2.1].

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* Partially supported by an NSF grant.
† Partially supported by an NSERC grant.
Received January 17, 1997; in revised form October 28, 1997.

This analogy can be extended by noting that the meet irreducible ideals form a topological space under the hull-kernel topology and every ideal is the intersection of the meet irreducible ideals which contain it. There is a one-to-one correspondence between closed sets in the meet irreducible ideal space and ideals in the strongly maximal TAF algebra; thus the topological space of meet irreducible ideals determines completely the lattice of ideals of the limit algebra, exactly as the primitive ideal space does for C*-algebras. Similar results have been obtained for other operator algebras, including the compacts in a nest algebra, the disc algebra, and various nonselfadjoint crossed products [L1, L2, L3].

An interesting subset of the meet irreducible ideals is the set of completely meet irreducible ideals, namely those ideals satisfying an analogous condition, only for arbitrary intersections instead of finite intersections. We describe these ideals and show that, for direct limit algebras generated by their order preserving normalizers, this subset is isomorphic, as a set, to the spectrum of the limit algebra (Theorem 5.4). Also, there is a distance formula for ideals in a strongly maximal TAF algebra (Theorem 6.2) that is analogous to Arveson's distance formula for nest algebras and to the distance formulae in [MS2].

0. Algebras & Coordinates

An analysis of ideals in direct limit algebras is greatly facilitated by the technique of coordinatization. After outlining the algebraic setting, we describe the essential ingredients for coordinatization in the context in which we need it; for more detail on coordinatizations and more general results the reader is referred to [R], [MS1], and [P4]. The book [P4] by Power is also a convenient reference for direct limit algebras.

If A is a strongly maximal triangular subalgebra of a unital AF C*-algebra, B, then $D = A \cap A^*$ is a canonical masa in B and $A + A^*$ is dense in B. (This is one definition of "strongly maximal triangular".) Since B is AF, it may be written as a direct limit of finite dimensional C*-algebras:

$$B_1 \to B_2 \to B_3 \to \cdots \to B.$$

In turn, A can be written as a direct limit

$$A_1 \to A_2 \to A_3 \to \cdots \to A,$$

where each A_n is a maximal triangular subalgebra of B_n; also, D is a direct limit

$$D_1 \to D_2 \to D_3 \to \cdots \to D,$$

where each $D_n = A_n \cap A_n^*$ is a masa in B_n. If I is a two-sided ideal of A then, essentially because I is a D-bimodule, it follows that I is the closed union of the $I \cap A_n$.

Furthermore, it is possible to select a system of matrix units for B so that each of A and D are generated by the matrix units which they contain. Of course, it follows that every ideal is also generated by the matrix units it contains. The system of matrix units can also be chosen so that each matrix unit in B_n is a sum of matrix units in B_{n+1}. By identifying each B_n with its natural image in B, we may consider all the embeddings which appear in the direct system to be inclusions.

A direct system whose limit is A will be referred to as a *presentation* for A. Given a presentation for A as above, we can construct another presentation by choosing a subsequence $A_{n_1}, A_{n_2}, \ldots$, with maps given by composing the maps from the original presentation. We call this new presentation a *contraction* of the original.

We now coordinatize the triple of algebras (D, A, B), where B is an AF C*-algebra and A is a strongly maximal triangular subalgebra of B whose diagonal is D. Also assume that a system of matrix units for B has been selected with the properties described above. We need to define a *spectral triple* (X, P, G) for (D, A, B). The first ingredient, X, is simple: it is just the maximal ideal space for D. So D is isomorphic to $C(X)$ and, since D is a direct limit of finite dimensional algebras, X is a compact, totally disconnected set. If, in addition, B is simple, then X is homeomorphic to the Cantor set (we exclude the finite dimensional case, where B is some M_n and X finite).

Since the C*-algebra, B, is AF, it is a groupoid algebra; G will be the groupoid associated with B. While we will use some of the language of groupoids and a couple of results about groupoids, the reader does not need extensive knowledge of groupoids in order to follow our arguments. Indeed, G is a special type of groupoid and we can describe it completely in a very naive fashion. Each matrix unit, e, from the system of matrix units for B acts on D by conjugation ($e^*De \subseteq D$); consequently, each matrix unit, e, induces a partial homeomorphism of X into itself (i.e., a homeomorphism between two clopen subsets of X). Let denote the graph of this homeomorphism.

As a set, G is simply the union of the graphs of all the partial homeomorphisms induced by matrix units. Thus, G is a subset of $X \times X$; it is not difficult to check that it is an equivalence relation. There is, however, an additional structure, a topology, on G. This topology is the smallest topology in which every is an open subset. It turns out that every is also closed, and hence compact. This description of G appears to be dependent on the choice of matrix unit system (and hence on the choice of presentation); in point of fact the same topological equivalence relation arises from any choice of presentation and any choice of matrix unit system. Indeed, in place of matrix units one may use the collection of all partial isometries in B which normalize D. (A partial isometry, v, *normalizes* D if $vDv^* \subseteq D$ and $v^*Dv \subseteq D$.)

A topological equivalence relation such as G is an r-discrete, principal, topological groupoid. We won't use all this terminology, but we do need to say what the groupoid operations are. Two elements (x, y) and (w, z) are composable if, and only if, $y = w$. In that case, the product is given by $(x, y) \circ (y, z) = (x, z)$. Inverses are given by $(x, y)^{-1} = (y, x)$.

The graph, ν, of the partial homeomorphism associated with a matrix unit (or with a normalizing partial isometry) has the following properties:

i) if $(x, y_1) \in \nu$ and $(x, y_2) \in \nu$, then $y_1 = y_2$,

ii) if $(x_1, y) \in \nu$ and $(x_2, y) \in \nu$, then $x_1 = x_2$.

A subset of G with these properties is called a G-set It is a property of the topology on G that any point has a neighborhood basis which consists of compact, open G-sets. All G-sets which appear in this paper can be taken to to be compact and open; assume that any G-set which appears is compact and open even if these adjectives are absent.

If ν_1 and ν_2 are G-sets, then so is the composition $\nu_1 \circ \nu_2$, which is defined to be the set $\{a \circ b : a \in \nu_1, b \in \nu_2$ and a and b are composable$\}$. In the case of graphs $\hat{e}$ and $\hat{f}$ of matrix units (or normalizing partial isometries), $\hat{e} \circ \hat{f}$ will be the graph of the product ef in B.

The space, X, can be identified with the diagonal of G via the homeomorphism $x \leftrightarrow (x, x)$. In particular, the diagonal of G is an open subset of G. (For readers familiar with groupoids, the diagonal is the space, G^0, of units of G. The fact that it is open means that G is r-discrete.) One should also note that in the present context, the two coordinate projection maps π_1 and π_2, when restricted to G, are open maps (from the groupoid topology on G to the topology on X); in fact, they are local homeomorphisms.

It remains to describe the middle component, P, of the spectral triple. The short way is to invoke the spectral theorem for bimodules [MS1]: P is the unique open subset of G which is the support set for the subalgebra A. The fact that A is generated by the matrix units which it contains permits a naive definition of P: it is simply the union of the graphs, $\hat{e}$, for the matrix units e in A. As such, it is a subrelation of G and it carries the relative topology induced by the topology on G. The apparent dependence of P on choice of matrix unit system (or presentation) is illusory and P is, in fact, an isometric isomorphism invariant for A [P2]. We shall call P the *spectrum* of A.

As is to be expected, properties of A are reflected in properties of P. The fact that A is an algebra means that $P \circ P \subseteq P$. The triangularity of A becomes the property that $P \cap P^{-1}$ is the diagonal of G. Finally, strong maximality for A is equivalent to $P \cup P^{-1} = G$. Note, in particular, that the topological relation, P, induces a total order on each equivalence class in G. We shall need a notation for equivalence classes: if $z \in X$, let $\mathrm{orb}_z = \{x \in X : (x,z) \in G\}$. We sometimes emphasize the induced order on each equivalence class by writing $x \leq y$ when $(x,y) \in P$.

Some of our results are valid in the context of triangular subalgebras of B which are analytic. The simplest definition of analytic subalgebras is in terms of real valued cocycles. A continuous function $c : G \longrightarrow \mathsf{R}$ is a *1-cocycle* provided that $c(x,y) + c(y,z) = c(x,z)$, for all $(x,y),(y,z) \in G$. We say that A is *analytic* if $P = c^{-1}[0,\infty)$. We say that A is *trivially analytic* when c has the special form $c(x,y) = b(y) - b(x)$ for a continuous function $b : X \longrightarrow \mathsf{R}$. (Such a function, b, is called a *0-cocycle* and c is the *coboundary* of b.) The material in Section 3 will be valid for trivially analytic algebras with the additional requirement that the 0-cocycle be an injective function. This family of algebras includes all full nest algebras. (While the 0-cocycle most naturally associated with a full nest algebra will not be injective, it can be replaced by an injective 0-cocycle whose coboundary yields the same analytic algebra.)

Just as the algebra A has a natural support set $P \subset G$, each two sided closed ideal $\mathscr{I}| \subseteq A$ has a support set, σ. The existence of σ is given by the spectral theorem for bimodules and a complete description of coordinatization for ideals is given in [MS1]. Also, just as before, a naive description of σ is available based on the fact that an ideal is generated by the matrix units which it contains [P1]. So, σ is the union of the graphs associated with matrix units of $\mathscr{I}$ and the topology is the relative topology from P. The definition of σ is, of course, independent of choice of matrix unit system or presentation.

The fact that $\mathscr{I}$ is an ideal is reflected in the following property for σ:

$$(w,x) \in P, (x,y) \in \sigma, (y,z) \in P \Longrightarrow (w,z) \in \sigma.$$

We say that an open subset of P which satisfies this property is an *ideal set*. Lemma 4.3 in [MS1] shows that each ideal set is the support set of a closed, two sided ideal in A.

We say that an ideal set, σ_1, is *meet irreducible* if, whenever $\sigma = \tau_1 \cap \tau_2$ with τ_1, τ_2 ideal sets, either $\sigma = \tau_1$ or $\sigma = \tau_2$. Since intersection of ideals corresponds to intersection of ideal sets, an ideal in A is meet irreducible if, and only if, the corresponding ideal set is meet irreducible.

1. MI-chains

In T_n, the algebra of $n \times n$ upper triangular matrices, each meet irreducible ideal is determined by a matrix unit. If e_{st} is a matrix unit in T_n, then the meet irreducible ideal $\mathscr{I}$ associated with e_{st} is the largest ideal in T_n which does not contain e_{st}. This ideal is generated as a linear subspace by the set of matrix units e_{nm} where either $n < s$ or $m > t$.

The meet irreducible ideals in T_n can also be described in terms of the coordinates, rather than matrix units. Let $X = \{1, \ldots, n\}$ and $P = \{(s, t) : s, t \in X \text{ and } s \leq t\}$. Then P is the support set for T_n. Let I be an interval contained in X; i.e., $I = \{i : s \leq i \leq t\}$ for some s, t. Then the meet irreducible ideal $\mathscr{I}$ associated with I is the set of all matrices supported on $P \backslash P \cap (I \times I)$.

In the TAF algebra context, the description of meet irreducible ideals in terms of coordinates needs almost no modification from the finite dimensional case. The description in terms of matrix units is considerably more complicated than the finite dimensional description. In [La], Lamoureux gave a procedure for constructing meet irreducible ideals from certain sequences of matrix units, provided that the embeddings satisfy a special condition. (This condition is met by standard embeddings, by refinement embeddings, and, more generally, by nest embeddings.) However, this procedure fails to give all meet irreducible ideals even in the simplest TAF algebras.

There is, in fact, a more general family of matrix unit sequences from which meet irreducible ideals can be constructed. This concept – MI-chains of matrix units – yields all the meet irreducible ideals (provided we consider all possible contractions of a given presentation); furthermore, it is valid for all TAF algebras.

Let A be a TAF algebra with presentation

$$A_1 \to A_2 \to A_3 \to \cdots \to A.$$

Notation. If $e \in A_n$, then $\mathrm{Id}_n(e)$ will denote the ideal generated by e in A_n. If $k > n$, then $e \in A_k$ also; therefore $\mathrm{Id}_k(e)$ is defined and $\mathrm{Id}_n(e) \subseteq \mathrm{Id}_k(e)$.

Definition 1.1. A sequence $(e_k)_{k \geq N}$ of matrix units from A will be called an *MI-chain* if the following two conditions are satisfied for all $k \geq N$:

(A) $e_k \in A_k$.

(B) $e_{k+1} \in \mathrm{Id}_{k+1}(e_k)$.

If (e_k) is an MI-chain for A, let $\mathscr{I}$ be the join of all ideals which do not contain any matrix unit e_k from the chain. In other words, $\mathscr{I}$ is the largest ideal in A which does not contain any e_k.

THEOREM 1.2. *Let A be a strongly maximal TAF algebra with some presentation. For each MI-chain $(e_k)_{k\geq N}$ from the presentation, the ideal $\mathscr{I}$ associated with (e_k) is meet irreducible. Conversely, every proper meet irreducible ideal in A is induced by some MI-chain, chosen from some contraction of this presentation.*

PROOF. Let (e_k) be an MI-chain of matrix units and let $\mathscr{I}$ be the corresponding ideal. Suppose that $\mathscr{J}$ and $\mathscr{K}$ are two ideals each of which properly contains $\mathscr{I}$. Since $\mathscr{I}$ is the largest ideal containing no matrix units from the MI-chain, there exist indices s and t such that $e_s \in \mathscr{J}$ and $e_t \in \mathscr{K}$. Condition (B) in the definition of MI-chain implies that $e_n \in \mathscr{J}$ for all $n > s$ and $e_m \in \mathscr{K}$ for all $m > t$. Thus, $\mathscr{J} \cap \mathscr{K}$ contains matrix units from the MI-chain, which implies that $\mathscr{J} \cap \mathscr{K}$ properly contains $\mathscr{I}$. This proves that $\mathscr{I}$ is meet irreducible.

For the converse, suppose that $\mathscr{I} \neq A$ is a meet irreducible ideal in A and that

$$A_1 \to A_2 \to A_3 \to \cdots \to A$$

is a presentation for A. Each A_k is a maximal triangular subalgebra of a finite dimensional C*-algebra. Let $\mathscr{I}_k = \mathscr{I} \cap A_k$, for each k. While $\mathscr{I}$ is the closed union of the $\mathscr{I}_k$, it is not necessarily the case that each $\mathscr{I}_k$ is meet irreducible as an ideal in A_k. Note that, by contracting the presentation if necessary, we may also assume that $\mathscr{I}_k$ is a proper ideal in A_k, for each k.

From the known structure of ideals in maximal triangular subalgebras of finite dimensional C*-algebras, it follows that for each k there is a minimal set E_k of matrix units in $A_k \backslash \mathscr{I}_k$ such that any ideal of A_k which is larger than $\mathscr{I}_k$ must contain one of the matrix units in E_k. Begin the construction of an MI-chain for $\mathscr{I}$ by letting e_1 be any matrix unit from E_1.

For each $e \in E_1$, let $\mathscr{J}_e$ denote the ideal in A generated by $\mathscr{I}$ and e. Since each such e is not in $\mathscr{I}_1$ but is in A_1, e does not belong to $\mathscr{I}$; thus $\mathscr{I}$ is a proper subset of each $\mathscr{J}_e$. Let $\mathscr{J} = \cap\{\mathscr{J}_e : e \in E_1\}$. This is a finite intersection and $\mathscr{I}$ is meet irreducible, so $\mathscr{J}$ properly contains $\mathscr{I}$. Consequently, for some $k \geq 2$, $\mathscr{J} \cap A_k$ properly contains $\mathscr{I}_k = \mathscr{I} \cap A_k$. By replacing the presentation by a contraction and relabeling, we may assume that $k = 2$.

Since $\mathscr{J} \cap A_2$ properly contains $\mathscr{I}_2$, there is a matrix unit $e_2 \in E_2$ such that $e_2 \in \mathscr{J} \cap A_2$. By the construction of $\mathscr{J}$, $e_2 \in \mathrm{Id}_2(e_1)$; thus condition (B) for MI-chains is satisfied by the pair e_1, e_2.

If we now iterate this construction, we obtain a presentation which is a contraction of the original presentation and a sequence of matrix units

$(e_k)_{k\geq 1}$ which is an MI-chain. Since $\mathscr{I}$ contains none of the e_k, $\mathscr{I}$ is a subset of the meet irreducible ideal associated with the MI-chain. But if $\mathscr{K}$ is an ideal larger than $\mathscr{I}$, then $\mathscr{K} \cap A_k$ properly contains $\mathscr{I}_k$ for some k and hence $\mathscr{K}$ contains some element of E_k. By the construction of the sequence (e_n), e_{k+1} is in the ideal generated by each element of E_k; hence $e_{k+1} \in \mathscr{K}$. Thus, $\mathscr{I}$ is the largest ideal which contains none of the e_k and so it is the meet irreducible ideal associated with the MI-chain.

It is natural to ask if there is a 1-1 correspondence between MI-chains and meet-irreducible ideals. Without other conditions, the answer is clearly no. For example, take an MI-chain for the zero ideal and change the first finitely many matrix units in the MI-chain. To fix this trivial kind of counterexample, the appropriate condition on the MI-chain is

(C) for a matrix unit f in A_k, if $f \in \mathrm{Id}_k(e_k)$ and $f \neq e_k$, then $e_{k+1} \notin \mathrm{Id}_{k+1}(f)$.

In fact, Theorem 1.2 always gives an MI-chain satisfying this condition. Using the notation of the proof, observe that if f is a matrix unit in A_1 which is not equal to e_1 but is in $\mathrm{Id}_1(e_1)$, then f belongs to $\mathscr{I}_1$, and hence to $\mathscr{I}$. Observe that $\mathrm{Id}_2(f)\mathscr{I}$, and so $e_2 \notin \mathrm{Id}_2(f)$. This verifies condition (C) for the pair e_1, e_2 and, by induction, the MI-chain (e_k) satisfies the condition.

However, restricting to MI-chains satisfying condition (C) still does not give a 1-1 correspondence, as the following example shows. Thus the correspondence between meet irreducible ideals and MI-chains is rather subtle.

Example 1.3. For $n \geq 1$, let $A_n = T_{2^n} \oplus T_{2^n}$ and let $\alpha_n\colon A_n \to A_{n+1}$ be given by the block matrix map

$$\begin{bmatrix} A & B \\ & C \end{bmatrix} \oplus \begin{bmatrix} D & E \\ & F \end{bmatrix} \longrightarrow \begin{bmatrix} A & & & B \\ & D & E & \\ & & F & \\ & & & C \end{bmatrix} \oplus \begin{bmatrix} D & & & E \\ & A & B & \\ & & C & \\ & & & F \end{bmatrix}.$$

Consider the algebra A which is the direct limit of the algebras A_n with respect to the maps α_n. For each n, let e_n be the upper-right matrix unit of B in each A_n, and let f_n be the upper-right matrix unit of E in each A_n. Observe that for each n, e_{n+1} is a summand of e_n, and so $e_{n+1} \in \mathrm{Id}_{n+1}(e_n)$. Similarly, $f_{n+1} \in \mathrm{Id}_{n+1}(f_n)$, and so both $(e_n)_n$ and $(f_n)_n$ are MI-chains. Moreover, since there is no matrix unit f in A_k with $f \in \mathrm{Id}_k(e_k)$ and $f \neq e_k$, then (e_n) satisfies condition (C), and similarly for (f_n). It is easy to see that both chains correspond to the zero ideal of A.

2. Meet Irreducible Ideals and Nest Representations

In this section we will construct meet irreducible ideals using coordinate methods. Fix notation as follows:

Notation. Let A be a strongly maximal TAF algebra whose enveloping C*-algebra is B and whose diagonal is D, a canonical masa in B. Also, (X, P, G) will denote the spectral triple for (D, A, B).

For subsets of G, the closure operator will always denote closure with respect to the groupoid topology on G, never the relative product topology on the larger set $X \times X$. Also, by an order interval in an equivalence class of G we mean the set of points $\{y \in X : (x, y), (y, z) \in P\}$, where $(x, z) \in P$, possibly excluding the endpoints x and z.

THEOREM 2.1. *With notation as above, let I be an order interval from an equivalence class from G and let $\sigma = P \backslash \overline{P \cap (I \times I)}$. Then σ is a meet irreducible ideal set.*

PROOF. We will first show that σ is an ideal set in P. To that end, assume that $(u, x) \in P$ and $(x, y) \in \sigma$. We will show that $(u, y) \in \sigma$.

Suppose, to the contrary, that $(u, y) \in \overline{P \cap (I \times I)}$. Then there are sequences u_n and y_n in I such that $(u_n, y_n) \in P$ and $(u_n, y_n) \longrightarrow (u, y)$ in P. Let T and S be compact, open G-sets containing (u, x) and (x, y) respectively. We may select T and S so that each is a subset of P. (These sets may be chosen to be the graphs of matrix units in A.) Then $T \circ S$ is a (compact, open) G-set containing (u, y). For large n, $(u_n, y_n) \in T \circ S$. Hence, for large n, there is $x_n \in X$ such that $(u_n, x_n) \in T$ and $(x_n, y_n) \in S$. The coordinate projection maps are local homeomorphisms; consequently $(u_n, x_n) \longrightarrow (u, x)$ and $(x_n, y_n) \longrightarrow (x, y)$ in P. For all large n, u_n, x_n, and y_n are in the same equivalence class, u_n and y_n are in I, and x_n is in between u_n and y_n; so, $x_n \in I$. Thus, $(x_n, y_n) \in P \cap (I \times I)$ and hence $(x, y) \in \overline{P \cap (I \times I)}$, contradicting the assumption that $(x, y) \in \sigma$.

This proves that $(u, x) \in P$, $(x, y) \in \sigma \Longrightarrow (u, y) \in \sigma$. The proof that $(x, y) \in \sigma$, $(y, v) \in P \Longrightarrow (x, v) \in \sigma$ is similar; the two implication together show that σ is an ideal set.

Next, we show that σ is meet irreducible. Suppose that τ_1 and τ_2 are ideal sets and that $\sigma = \tau_1 \cap \tau_2$. Assume that σ is a proper subset of both τ_1 and τ_2.

First observe that there is a point $(x, y) \in P \cap (I \times I)$ such that $(x, y) \in \tau_1 \backslash \sigma$. Indeed, assume the contrary. Since no point of $P \cap (I \times I)$ lies in σ, we have $P \cap (I \times I) \subseteq P \setminus \tau_1$. But $P \backslash \tau_1$ is closed, so $\overline{P \cap (I \times I)} \subseteq P \backslash \tau_1$. This implies $\tau_1 \subseteq \sigma$ (and therefore $\tau_1 = \sigma$), contradicting our assumptions.

Since (x, y) is in $\tau_1 \backslash \sigma$, we have $(x, y) \in P \backslash \tau_2$. (Otherwise, $(x, y) \in \tau_1 \cap \tau_2 = \sigma$, a contradiction.)

Let $(a,b) \in P \cap (I \times I)$ and let $u = \min\{a,x\}$ and $v = \max\{b,y\}$. (Here min and max are with respect to the order on I.) Then we have

$$(u,x) \in P,\ (x,y) \in \tau_1,\ (y,v) \in P \Longrightarrow (u,v) \in \tau_1.$$

Since $(u,v) \in P \cap (I \times I) \subseteq P \backslash \sigma$ we also have $(u,v) \notin \tau_2$. But since $(u,a) \in P$, $(b,v) \in P$ and τ_2 is an ideal set, this implies that $(a,b) \notin \tau_2$. As (a,b) is arbitrary in $P \cap (I \times I)$, we obtain $P \cap (I \times I) \subseteq P \backslash \tau_2$. Since the latter is a closed set, this yields $\overline{P \cap (I \times I)} \subseteq P \backslash \tau_2$, which implies $\sigma = \tau_2$, contrary to assumption. This shows that σ must equal one of τ_1 or τ_2 and hence is meet irreducible.

REMARK. The mapping from intervals contained in some equivalence class of G to meet irreducible ideal sets is not one-to-one, even in a context as simple as a refinement algebra. Some meet irreducible ideal sets can be written as the complement of $\overline{P \cap (I \times I)}$ for a unique interval I from a unique equivalence class. For others, there is at least one such interval I for each equivalence class from G. It is also possible that different intervals from the same equivalence class yield the same meet irreducible ideal set. (Here, the latitude lies in whether or not to include "end points".)

Theorem 2.1 has a converse, whose proof requires the following elementary fact.

FACT 2.2. *For each element e of T_n, let $\mathrm{Id}(e)$ denote the ideal in T_n generated by e. If e_{ii}, e_{jj} and e_{kk} are three diagonal matrix units with $i < j < k$, then $\mathrm{Id}(e_{ii}) \cap \mathrm{Id}(e_{kk}) \subseteq \mathrm{Id}(e_{jj})$.*

THEOREM 2.3. *With notation as above, let $\mathcal{I}$ be a meet irreducible ideal in A. Then there is an interval I contained in an equivalence class from G so that the support set of $\mathcal{I}$ is $P \backslash \overline{P \cap (I \times I)}$.*

PROOF. The first step is to determine the equivalence class which will contain I. By Theorem 1.2, there is a presentation

$$A_1 \to A_2 \to A_3 \to \cdots \to A$$

together with an MI-chain $(e_k)_{k \geq 1}$ for which $\mathcal{I}$ is the largest ideal which contains no matrix unit e_k from the MI-chain. We shall use the MI-chain to construct a decreasing sequence of projections $p_1 \geq p_2 \geq p_3 \geq \ldots$ with each $p_k \in D_k = A_k \cap A_k^*$. Each such decreasing sequence of projections corresponds in a natural way to a point of X and thereby determines an equivalence class in G.

Observe that $\mathrm{Id}_2(e_1)$ is equal to the linear span of matrix units of the form fsg, where $f, s, g \in A_2$ and s is a subordinate of e_1 in A_2. Since e_2 is a matrix unit and is in $\mathrm{Id}_2(e_1)$, it has this form. In particular, there is a matrix unit s_2

in A_2 which is a subordinate of e_1 such that $e_2 \in \mathrm{Id}_2(s_2)$. Let p_2 and q_2 be the range and domain projections of s_2; i.e., $p_2 = s_2 s_2^*$ and $q_2 = s_2^* s_2$. If we let $p_1 = e_1 e_1^*$ and $q_1 = e_1^* e_1$, then we have $p_1 \geq p_2$ and $q_1 \geq q_2$. Note also that $e_2 \in \mathrm{Id}_2(p_2)$ and $e_2 \in \mathrm{Id}_2(q_2)$, since both of these ideals contain $\mathrm{Id}_2(s_2)$.

By property (B) for MI-chains, $e_3 \in \mathrm{Id}_3(e_2)$; consequently $e_3 \in \mathrm{Id}_3(s_2)$. Therefore, there is a matrix unit s_3 in A_3 which is subordinate to s_2 (and hence to e_1) such that $e_3 \in \mathrm{Id}_3(s_3)$. Let $p_3 = s_3 s_3^*$ and $q_3 = s_3^* s_3$. We have $p_2 \geq p_3$, $q_2 \geq q_3$, $e_3 \in \mathrm{Id}_3(p_3)$ and $e_3 \in \mathrm{Id}_3(q_3)$.

It is now clear that an inductive argument will yield a sequence of matrix units s_n in A_n with range projections p_n and domain projections q_n such that:

1) $s_1 = e_1$,
2) s_{n+1} is a subordinate of s_n, for all n,
3) $e_n \in \mathrm{Id}_n(s_n)$, $e_n \in \mathrm{Id}_n(p_n)$ and $e_n \in \mathrm{Id}_n(q_n)$, for all n, and
4) $p_n \geq p_{n+1}$ and $q_n \geq q_{n+1}$, for all n.

Clearly, (p_n) and (q_n) give points p and q in X. Since $(p, q) \in \hat{e}_1$, p and q determine the same equivalence class in G. This is the equivalence class which will contain I.

If $x \in X$ then, for each k, there is a unique minimal projection x_k in D_k, the diagonal of A_k, such that $x \in \hat{x}_k$. Define I as follows:

$$I = \{x \in \mathrm{orb}_p : e_k \in \mathrm{Id}_k(x_k) \text{ for all large } k\}.$$

Note that both p and q are in I.

For later use we need an observation. Fix $k > 1$. Let f_k and g_k be matrix units in A_k for which $e_k = f_k s_k g_k$. For each $n \geq k$, let $\tilde{s}_n = f_k s_n g_k$ and let $\tilde{p}_n$ and $\tilde{q}_n$ be the range and domain projections of $\tilde{s}_n$. Then $\tilde{s}_n$, $\tilde{p}_n$, $\tilde{q}_n$, $n \geq k$ satisfy properties analogous to the properties 1)–4) above for s_n, p_n, q_n, $n \geq 1$. In particular, $e_n \in \mathrm{Id}_n(\tilde{p}_n)$ and $e_n \in \mathrm{Id}_n(\tilde{q}_n)$ for all $n \geq k$; the points $\tilde{p}$ and $\tilde{q}$ in X corresponding to $(\tilde{p}_n)$ and $(\tilde{q}_n)$ lie in I; and $(\tilde{p}, \tilde{q}) \in \hat{e}_k$. Thus, for any e_k we can construct a point $(\tilde{p}, \tilde{q})$ in $\hat{e}_k \cap (I \times I)$.

We must show that I is an interval in orb_p. Suppose $w < x < y$ where $w, y \in I$ and (w_k), (y_k) are the nested sequences of projections associated to w and y. Recall that we sometimes write $w \leq x$ when $(w, x) \in P$. There is an integer N such that, for any $k \geq N$, all of the following are true:

i) $e_k \in \mathrm{Id}_k(w_k)$,
ii) $e_k \in \mathrm{Id}_k(y_k)$,
iii) there is a matrix unit in A_k with initial projection x_k and range projection w_k, and
iv) there is a matrix unit in A_k with initial projection y_k and range projection x_k.

Now, A_k is a maximal triangular subalgebra of a finite dimensional C*-algebra and so is a direct sum of T_n's. Conditions iii) and iv) imply that w_k, x_k and y_k all lie in the same summand; furthermore within that summand x_k lies in between w_k and y_k in the diagonal ordering. Since the context is now that of a T_n, Fact 2.2 tells us that $\mathrm{Id}_k(w_k) \cap \mathrm{Id}_k(y_k) \subseteq \mathrm{Id}_k(x_k)$. In particular, $e_k \in \mathrm{Id}_k(x_k)$. Since this holds for any $k \geq N$, $x \in I$, this proves that I is an interval.

It remains to show that $\mathscr{I}$ has support set $P \backslash \overline{P \cap (I \times I)}$. Let $\mathscr{I}'$ be the ideal with support set $P \backslash \overline{P \cap (I \times I)}$.

Suppose e is a matrix unit which is in $\mathscr{I}$ but not in $\mathscr{I}'$. Then $\hat{e} \cap (I \times I) \neq \emptyset$, so there are points $x, y \in I$ such that $(x,y) \in \hat{e}$. There is an integer k such that $e \in A_k$, $e_k \in \mathrm{Id}_k(x_k)$, and $e_k \in \mathrm{Id}_k(y_k)$. If f_k is the matrix unit in A_k for which $(x,y) \in \hat{f}_k$, then f_k is a subordinate of e. Since f_k generates $\mathrm{Id}_k(x_k) \cap \mathrm{Id}_k(y_k)$, we have $e_k \in \mathrm{Id}_k(f_k)$. This implies that $f_k \notin \mathscr{I}$ and hence $e \notin \mathscr{I}$, a contradiction. Thus, $\mathscr{I} \subseteq \mathscr{I}'$.

All that remains is to prove that $\mathscr{I}' \subseteq \mathscr{I}$. We observed earlier that, for each k, there is a point $(\tilde{p}, \tilde{q}) \in \hat{e}_k \cap (I \times I)$. Thus $e_k \notin \mathscr{I}'$, for all k. Suppose that e is a matrix unit which is not in $\mathscr{I}$. By the definition of $\mathscr{I}$, $e_k \in \mathrm{Id}_k(e)$ for some k. But this means that $e \notin \mathscr{I}$ for otherwise we would have $e_k \in \mathscr{I}'$, a contradiction. Thus $\mathscr{I}' \subseteq \mathscr{I}$.

For each meet irreducible ideal, we can use the associated interval I to construct a nest representation whose kernel is the ideal.

THEOREM 2.4. *With notation as above, let $\mathscr{I}$ be a meet irreducible ideal in A with associated interval I as in Theorem 2.3. Then there is a nest representation of A acting on the Hilbert space $\ell^2(I)$ whose kernel is $\mathscr{I}$. Furthermore, the nest of invariant subspaces for this representation is an atomic nest with one-dimensional atoms.*

PROOF. Let $\{\delta_x : x \in I\}$ be the standard orthonormal basis for $\ell^2(I)$. Define π on the matrix units in A by setting

$$\pi(e)\delta_y = \begin{cases} 0, & \text{if there is no } x \in I \text{ such that } (x,y) \in \hat{e}, \\ \delta_x, & \text{if there is } x \in I \text{ such that } (x,y) \in \hat{e}, \end{cases} .$$

for each matrix unit e and basis vector δ_y. Since $\hat{e}$ is a G-set, $\pi(e)\delta_y$ is well defined and is a partial isometry in $\mathscr{B}(\ell^2(I))$. It is straightforward to check that $\pi(ef)\delta_y = \pi(e)\pi(f)\delta_y$ for any two matrix units $e,f \in A$; so, the linear extension of π to the algebra (not closed) generated by the matrix units of A is an algebra homomorphism.

In order to show that π extends to a representation of A acting on $\ell^2(I)$, we shall prove that π has norm 1. Let J denote the equivalence class in X which contains the interval I. Identify $\ell^2(I)$ as a subspace of $\ell^2(J)$. Define $\tilde{\pi}$

on the matrix units of the C*-envelope, B, of A the same way as π is defined, except now allow x and y to be any elements of J. The linear extension of $\tilde{\pi}$ to the (non-closed) algebra generated by the matrix units of B is a *-representation. This algebra is, in fact, the union of the finite dimensional C*-algebras $B_k = C^*(A_k)$. The restriction of $\tilde{\pi}$ to each B_k is a representation of a C*-algebra and so has norm 1; consequently $\tilde{\pi}$ has norm 1. Since π is a compression of the restriction of $\tilde{\pi}$ to the algebra generated by the matrix units of A, π, too, has norm 1. Thus, π extends to a representation of A.

If M is an invariant subspace for π and if $\delta_y \in M$, then $\delta_x \in M$ for all $x \in I$ with $x \leq y$. This is immediate, since $x \leq y$ means that there is a matrix unit $e \in A$ with $(x, y) \in \hat{e}$. Thus, if M is an invariant subspace for π, there is an initial segment S of I such that M is generated by $\{\delta_x : x \in S\}$. This implies that the invariant subspaces for π are totally ordered by inclusion. Thus, π is a nest representation.

Recall from Theorem 2.3 that $\mathscr{I}$ has support set $P \backslash \overline{P \cap (I \times I)}$. If e is a matrix unit in A then $\pi(e) = 0$ if, and only if, $\hat{e} \cap (I \times I) = \emptyset$. If $\hat{e} \cap (I \times I) = \emptyset$, then $P \cap (I \times I)$ is disjoint from the open set $\hat{e}$; hence $\overline{P \cap (I \times I)}$ is disjoint from $\hat{e}$. Thus $\hat{e} \subseteq P \backslash \overline{P \cap (I \times I)}$ and so $e \in \mathscr{I}$. Since ideals are generated by the matrix units which they contain, it follows that $\ker \pi \subseteq \mathscr{I}$. On the other hand, if e is a matrix unit in $\mathscr{I}$, then we have $\hat{e} \subseteq P \backslash \overline{P \cap (I \times I)}$, whence $\hat{e} \cap (I \times I) = \emptyset$ and $e \in \ker \pi$. Thus $\mathscr{I} \subseteq \ker \pi$ and we have equality.

3. Ideal Sets for Trivially Analytic Algebras

In this and the next section, we shall focus primarily on TAF algebras which are analytic and whose C*-envelopes are simple. An analytic subalgebra of an AF C*-algebra is automatically strongly maximal triangular. So the results of the previous section apply in this setting. The class of trivially analytic subalgebras of AF C*-algebras is fairly extensive; it includes, for example, all full nest algebras. These are algebras with a presentation of the form

$$T_{n_1} \to T_{n_2} \to T_{n_3} \to \cdots \to A$$

subject to the requirement that each embedding carries the nest of invariant projections of T_{n_i} into the invariant projections of $T_{n_{i+1}}$. The well-known refinement algebras form a subfamily of the family of full nest algebras.

In this section we shall give a complete description of all the meet irreducible ideals in a trivially analytic TAF algebra with an injective 0-cocycle via a description of the meet irreducible ideal sets of the spectrum of the algebra. This is the setting most analogous to the finite dimensional context. It is the context with the most intuitive picture of meet irreducible ideal sets.

REMARK. The description of the meet irreducible ideal sets is actually valid in a somewhat more general context, which we outline in this remark. The basic properties that we need for the description of the meet irreducible ideal sets are the following:

1. Each equivalence class from G is countable and dense in X.
2. The two projection maps from $X \times X$ to X when restricted to G are open and continuous with respect to the groupoid topology on G.
3. There is a total order $\preceq$ on X which, on each equivalence class from G, agrees with the order induced by P. Furthermore, the order topology on X agrees with the original, Gelfand topology on X.

The first of these properties implies that the groupoid C*-algebra associated with G is simple. The second property is equivalent to G being r-discrete and admitting a left Haar system. See [R, Prop. 1.2.8].

The third property is the critical one for our purposes. The existence of a total order on X with these properties follows immediately from the existence of a trivial cocycle which is the coboundary of an injective 0-cocycle b: define $x \preceq y$ iff $b(x) \leq b(y)$.

The existence of a total order with property 3 is almost, but not quite equivalent to the existence of a trivial cocycle on X which is the coboundary of an injective function. Equivalence requires one additional property: the order $\preceq$ has at most countably many gaps. (A gap is a pair of elements from X with no intermediate elements from X.)

It is not difficult to construct an example of a triple (X, P, G) which meets all of the properties above except that it has uncountably many gaps with respect to the order on X. (Basically, construct a Cantor like set from the interval $[0, 1]$ doubling the irrational points instead of the rational points. For the groupoid G take the union of all sets of the form $\{(qx, x) : x \in A\}$, where A is some open interval from X and q is a positive rational number with the property that $qX \subseteq X$.) The C*-algebra built on such a groupoid will be inseparable and will fail to have most of the nice properties that groupoid C*-algebras usually enjoy, so this example is of dubious interest.)

If X does have countably many gaps, construct a one-to-one, continuous map $b : X \longrightarrow \mathsf{R}$ as follows. Let S be a countable dense subset of X which does not contain any points which have either an immediate successor or an immediate predecessor. Let $a : S \longrightarrow [0, 1]$ be a monotonically increasing map of S onto a countable, dense subset of $[0, 1]$. Extend a to a continuous map (also denoted by a) of X onto $[0, 1]$. The map a is increasing, but not necessarily one-to-one. In particular, if x is an immediate predecessor of y, then $a(x) = a(y)$. Let $\{(x_n, y_n)\}$ be an enumeration of all the gap pairs from X. For each x, let $\beta(x) = \{n : y_n \leq x\}$. Define $b : X \longrightarrow [0,2]$ by

$$b(x) = a(x) + \sum_{n \in \beta(x)} \frac{1}{2^n}.$$

The function b has all the desired properties.

The description of all the meet irreducible ideals in a trivially analytic TAF algebra with injective 0-cocycle can be verified making use of only the properties of the spectral triple listed above. It is not necessary to use the countability of the gap points nor the fact that the enveloping C*-algebra is AF. The argument, however, is long, tedious, and of little intrinsic interest. Consequently, we will instead make use of Theorems 2.1 and 2.3 to provide a much more palatable verification at the expense of a slight loss of generality.

For the following, assume that A is a trivially analytic TAF algebra with diagonal D and enveloping C*-algebra B, which is simple. Let (X, P, G) be the spectral triple for (D, A, B). Let $\preceq$ be a total order on X which agrees with P on equivalence classes from G and assume that the order topology agrees with the original (Gelfand) topology on X. If a point $a \in X$ has an immediate successor, we say that a has a *gap above*. Similarly, if b has an immediate predecessor, then b has a *gap below*.

NOTATION. For each pair of points $a, b \in X$ let

$$\sigma_{a,b} = \{(x, y) \in P : x \prec a \text{ or } b \prec y\}$$
$$\tau_{a,b} = \sigma_{a,b} \cup \{(a, b)\}.$$

Observe that the set $\sigma_{a,b}$ is an open subset of P which satisfies the ideal property. Thus, it is always the support set for an ideal in A. The set $\tau_{a,b}$ also satisfies the ideal property, but it need not be open. It will be an open subset of P precisely when $(a, b) \in P$ and there is a neighborhood, ν, of (a, b) such that $\nu \backslash \{(a, b)\} \subseteq \sigma_{a,b}$. When this is the case, $\tau_{a,b}$ is an ideal set. In a refinement algebra, $\tau_{a,b}$ is an ideal set for all $(a, b) \in P$. In a full nest algebra, there may be points (a, b) for which $\tau_{a,b}$ is not open. In the following, we shall always assume that $\tau_{a,b}$ is an ideal set.

If $b \prec a$, then $\sigma_{a,b} = P$. If $a = b$, then $\tau_{a,b} = P$ and $\sigma_{a,b}$ is a maximal ideal (with codimension 1). The ideal set P is meet irreducible by default and each $\sigma_{a,a}$ is trivially meet irreducible. Consequently, in the proof of Theorem 3.1, we always assume $a \prec b$.

All meet irreducible ideal sets for a trivially analytic algebra are described in the following theorem.

THEOREM 3.1. *Let A be a trivially analytic TAF algebra with an injective* 0-*cocycle such that the* C^*-*envelope of A is simple. Assume that (X, P, G) is the spectral triple associated with A and that $\preceq$ is a total order on X compatible with P. The following is a complete list of all the meet irreducible ideal sets in P:*

1. $\sigma_{a,b}$ *if* $(a, b) \in P$.
2. $\sigma_{a,b}$ *if* $(a, b) \notin P$ *and there is either no gap above for a or no gap below for b.*
3. $\tau_{a,b}$ *if* $(a, b) \in P$, *there is either no gap above for a or no gap below for b, and $\tau_{a,b}$ is open.*

PROOF. Let σ be a meet irreducible ideal set contained in P. By Theorem 2.3, there is an equivalence class, orb_z, from G and an interval $I \subseteq \mathrm{orb}_z$ such that $\sigma = P \backslash \overline{P \cap (I \times I)}$. Let $a = \inf I$ and $b = \sup I$. The inf and sup are taken in X with respect to the order $\preceq$; the compactness of X guarantees that the inf and sup exist.

We observe first that $\sigma_{a,b} \subseteq \sigma$. Indeed, suppose that $(x, y) \in P$ and $(x, y) \notin \sigma$. Then $(x, y) \in \overline{P \cap (I \times I)}$. Now, $\overline{P \cap (I \times I)} \subseteq P \cap (\bar{I} \times \bar{I})$ (the containment may be proper), so $a \preceq x \preceq b$ and $a \preceq y \preceq b$. But this shows that $(x, y) \notin \sigma_{a,b}$. Thus, $\sigma_{a,b} \subseteq \sigma$.

The next observation is that $\sigma \subseteq \tau_{a,b}$. Indeed, suppose that $(x, y) \in P \backslash \tau_{a,b}$. Then we know that $a \preceq x, y \preceq b$ and $(x, y) \neq (a, b)$. If both $x \neq a$ and $y \neq b$, then there is an open neighborhood, ν, of (x, y) which is contained in $P \backslash \tau_{a,b}$. We may further assume that the projection maps π_1 and π_2 are homeomorphisms on ν. In particular, $\pi_1(\nu)$ is an open set in X which contains a. Consequently, there is a point $u \in I \cap \pi_1(\nu)$. It follows that there is a point $(u, v) \in P \cap (I \times I)$ which is in ν. This shows that $(x, y) \in \overline{P \cap (I \times I)}$. If $x = a$ and a has no gap above, then $y \prec b$ and we may argue in much the same way to conclude that $(a, y) \in \overline{P \cap (I \times I)}$. If $x = a$ and a has a gap above, then $a \in I$. Since $y \prec b$, we also have $y \in I$; in particular, $(a, y) \in \overline{P \cap (I \times I)}$. The case in which $y = b$ is handled in an analogous fashion. This proves that $\sigma \subseteq \tau_{a,b}$.

Since $\sigma_{a,b}$ and $\tau_{a,b}$ differ by only one point, we have shown that every meet irreducible ideal set has one of the two forms $\sigma_{a,b}$ or $\tau_{a,b}$. To show that every meet irreducible ideal set is in the list in the theorem, we just have to show that the ideals of the form $\sigma_{a,b}$ and $\tau_{a,b}$ which are not on the list are not meet irreducible.

To that end, fix $(a, b) \in G \times G$ and let

$$\rho_1 = \{(x, y) \in P : x \preceq a \text{ or } b \prec y\}$$
$$\rho_2 = \{(x, y) \in P : x \prec a \text{ or } b \preceq y\}.$$

Suppose that $(a,b) \notin P$ and that a has a gap above and that b has a gap below. Since a has a gap above and b has a gap below, both ρ_1 and ρ_2 are open and therefore ideal sets. It is easy to check that $\sigma_{a,b}$ is unequal to either ρ_1 or ρ_2 and that $\sigma_{a,b} = \rho_1 \cap \rho_2$. Thus $\sigma_{a,b}$ is not meet irreducible when $(a,b) \notin P$, a has a gap above and b has a gap below.

Suppose that $(a,b) \in P$ and a has a gap above and b has a gap below. We also assume that $a \neq b$, since otherwise $\tau_{a,b} = P$. Again, the hypotheses insure that ρ_1 and ρ_2 are ideal sets which are unequal to $\tau_{a,b}$ and that $\tau_{a,b} = \rho_1 \cap \rho_2$. Thus, $\tau_{a,b}$ is not meet irreducible when a has a gap above or b has a gap below.

It remains only to show that the ideal sets on the list are in fact meet irreducible. This can be done by direct argument or with the help of Theorem 2.1. We will sketch the argument which employs Theorem 2.1.

Suppose that $(a,b) \in P$. Let $I = \{x \in \mathrm{orb}_a : a \preceq x \preceq b\}$. Then $\sigma_{a,b} = P \backslash \overline{P \cap (I \times I)}$ and is therefore meet irreducible. Note that this is the only choice for I which works in this case. In subsequent cases the choice of I will not be unique.

Suppose that $(a,b) \notin P$ and a does not have a gap above. In this case, let $I = \{x \in \mathrm{orb}_b : a \prec x \preceq b\}$. Then $\overline{P \cap (I \times I)} = \{(x,y) \in P : a \preceq x, y \preceq b\}$ and $\sigma_{a,b} = P \backslash \overline{P \cap (I \times I)}$ and so is meet irreducible.

Suppose that $(a,b) \notin P$ and b does not have a gap below. This time we let $I = \{x \in \mathrm{orb}_a : a \preceq x \prec b\}$. Then $\sigma_{a,b} = P \backslash \overline{P \cap (I \times I)}$ and is meet irreducible.

In the case in which a has no gap above and b has no gap below, we take $I = \{x \in \mathrm{orb}_z : a \prec x \prec b\}$, where z is an arbitrary element of X. Again we get $\sigma_{a,b} = P \backslash \overline{P \cap (I \times I)}$.

Ideal sets of the form $\tau_{a,b}$ remain. Suppose that $(a,b) \in P$ and that a has no gap above. Let $I = \{x \in \mathrm{orb}_b : a \prec x \preceq b\}$. Since $\tau_{a,b}$ is an ideal set, (a,b) lies in an open neighborhood N which is a subset of $\tau_{a,b}$ and therefore disjoint from $P \cap (I \times I)$. This shows that $(a,b) \notin \overline{P \cap (I \times I)}$. The rest of the argument needed to show that $\tau_{a,b} = P \backslash \overline{P \cap (I \times I)}$ is similar to what has been done before. Thus $\tau_{a,b}$ is meet irreducible when a as no gap above.

The argument that $\tau_{a,b}$ is meet irreducible when b has no gap below is analogous the the preceding one. As in the case for $\sigma_{a,b}$, when neither a has a gap above nor b has a gap below, there are many choices for the interval I which will yield $\tau_{a,b} = P \backslash \overline{P \cap (I \times I)}$.

4. Ideal Sets and the Extended Asymptotic Range

In the first part of this section, we gather some results about ideal sets in the spectrum of A, and then we give some further results in the case in which the extended asymptotic range of the cocycle (to be defined below) is $\{0, \infty\}$. Throughout this section A will be an analytic TAF algebra whose enveloping C*-algebra B is simple, and I will be an interval from an equivalence class from G. As before, we write $x \leq y$ when $(x, y) \in P$. We do not assume that there is an order on X which extends P. The cocycle c on G will, in general, not be a coboundary. The simplicity of B is equivalent to the density in X of each equivalence class from G.

DEFINITIONS. For any subset $E \subset X$, we say that E is *increasing* if $x \in E$ and $x \leq y$ imply $y \in E$. We define *decreasing* in an analogous fashion. For an interval I, if the restriction of the cocycle c to $I \times I$ is bounded, we say that I is *finite with respect to* c. If $c|I \times I$ is unbounded, we say that I is *infinite with respect to* c.

Note that, unlike I, the set E is not totally ordered by $\leq$ (i.e. by P). We also point out that infinite intervals exist only when the cocycle is not trivial. (Trivial cocycles are necessarily bounded.)

There is a considerable difference between the properties of finite intervals and the properties of infinite intervals. First we gather some results about infinite intervals. We shall learn shortly that infinite intervals are of little interest – they yield only the trivial 0-ideal.

LEMMA 4.1. *Suppose that I is an interval from an equivalence class which is infinite with respect to c. Then I is either increasing or decreasing.*

PROOF. Let orb_a be the equivalence class which contains I. Assume that I is neither increasing nor decreasing. Then there exist an element $y \in I$ and an element $z \in \mathrm{orb}_a$ such that $y < z$ and $z \notin I$. Also, there exist an element $x \in I$ and an element $w \in \mathrm{orb}_a$ such that $w < x$ and $w \notin I$. Since I is an interval, no element of I can be less than w nor greater than z. Thus, if $(s, t) \in I \times I$, we have either $w < s \leq t < z$ or $w < t \leq s < z$. In particular, the cocycle property implies that

$$\begin{aligned} 0 \leq c(s,t) \leq c(w,z) \qquad & \text{if } s \leq t, \text{ and} \\ 0 \leq c(t,s) \leq c(w,z) \qquad & \text{if } t \leq s. \end{aligned}$$

Thus, $|c(s, t)| \leq c(w, z)$ in all cases and I is finite with respect to c – contrary to assumption. This shows that I is either increasing or decreasing.

If ν is an open G-set contained in G, and if $x \in \pi_1(\nu)$, then there is a unique element $y \in X$ such that $(x, y) \in \nu$; in this situation, we shall often write $y = \nu(x)$. We thereby identify ν with a partial homeomorphism of X into X. In effect, we are using the same symbol for the partial homeomorphism and for its graph. If V is an open subset of X, we let $\nu(V)$ denote $\{\nu(x) : x \in V \cap \pi_1(\nu)\}$.

PROPOSITION 4.2. *Let I be an infinite interval from an equivalence class. Then I is dense in X.*

PROOF. Let $V = X \setminus \overline{I}$. We have to show that $V = \emptyset$. Suppose that V is not empty. We know from Lemma 4.1 that I is either increasing or decreasing. Assume that it is increasing. (If I is decreasing, a similar argument to the one below will also yield a contradiction.)

From the density of equivalence classes, it follows that $X = \bigcup \nu(V)$, where the union is taken over all compact, open G-sets ν (See [R]). However, X is a compact set, so there are finitely many compact, open G-sets $\nu_1, \dots, \nu_k$ so that $X = \bigcup_{j=1}^{k} \nu_j(V)$. Since each ν_j is compact and c is continuous, there is M such that $c|\nu_j < M$, for all j.

Since c is unbounded on I, there are points $t, x \in I$ such that $c(t,x) > M$. The $\nu_j(V)$ cover X, so there is j such that $x \in \nu_j(V)$; i.e., there is $v \in V$ such that $x = \nu_j(v)$. Now $v \notin I$ (since V is the complement of $\overline{I}$) and $t \in I$. The fact that I is increasing implies that $v < t$. Also, since $(v,x) \in \nu_j$ and $c < M$ on ν_j, we have $c(v,x) < M$. Thus we have $c(v,x) = c(v,t) + c(t,x)$ with $c(v,x) < M$ and $c(t,x) > M$. This implies that $c(v,t) < 0$. But that means that $t < v$, contradicting the observation above that $v < t$. Thus we conclude that $V =$ and $\overline{I} = X$.

The following proposition is false without the assumption that I is either increasing or decreasing.

PROPOSITION 4.3. *Let I be an interval contained in an equivalence class from G. Assume that I is either increasing or decreasing and that I is dense in X. Then $P \cap (I \times I)$ is dense in P. Consequently, the meet irreducible ideal associated with I is the* 0*-ideal.*

PROOF. We assume that I is increasing. The proof when I is decreasing is similar, as usual. Let $(x,y) \in P$. Let ν be an open G-set such that $(x,y) \in \nu \subset P$ and the coordinate projections are homeomorphisms on ν. Since $\overline{I} = X$, there is a sequence $x_k \in \pi_1(\nu) \cap I$ such that $x_k \to x$ in X. Let $y_k = \nu(x_k)$. Since $(x_k, y_k) \in P$ and I is increasing, $y_k \in I$ for all k. The coordinate projections are homeomorphisms on ν, so $y_k \to y$ in X and $(x_k, y_k) \to (x,y)$ in P. Thus, $(x,y) \in \overline{P \cap (I \times I)}$ and $\overline{P \cap (I \times I)} = P$.

COROLLARY 4.4. *With the same assumptions as above, $I \times I$ is dense in G.*

PROOF. This follows from the fact that $G = P \cup P^{-1}$.

COROLLARY 4.5. *If I is an infinite interval with respect to the cocycle c, then the meet irreducible ideal associated with I is* $\{0\}$.

PROOF. Combine Lemma 4.1 and Propositions 4.2 and 4.3.

Assume that the cocycle c is Z-valued. The standard algebras provide a class of examples with Z-valued cocycles. See [PW] for more on the relationship between Z-valued cocycles and standard embeddings.

Let I be an interval which is finite with respect to c. We claim that in the Z-valued cocycle case, the interval I is in fact a set with finite cardinality. Indeed, let x be any element from I and define a function $\phi : I \longrightarrow \mathsf{R}$ by $\phi(y) = c(x, y)$. Observe that ϕ is one-to-one. (If $\phi(y_1) = \phi(y_2)$, the $c(x, y_1) = c(x, y_2)$ and hence $c(y_1, y_2) = c(y_1, x) + c(x, y_2) = -c(x, y_1) + c(x, y_2) = 0$. Since $c^{-1}(\{0\})$ is the diagonal, $y_1 = y_2$.) Thus, ϕ is a bounded, integer valued, one-to-one map on I. It is now immediate that I is a finite set.

If I is a finite set, then of course $\overline{P \cap (I \times I)} = P \cap (I \times I)$. Thus the complement of the ideal set σ associated with I is a finite subset of P. If $\mathscr{I}$ is the ideal corresponding to σ and π is the nest representation given by Theorem 2.3, then the construction of π implies that π acts on a finite dimensional Hilbert space. Consequently, $\mathscr{I}$ has finite co-dimension in A. Thus, we have the following proposition:

PROPOSITION 4.6. *Let A be an analytic* TAF *algebra whose* C*-*envelope is simple and which has a* Z-*valued cocycle. Then any non-trivial meet irreducible ideal in A has finite co-dimension.*

The results about the complement of the ideal set for a meet irreducible ideal in an analytic algebra with a Z-valued cocycle can be extended in modified form to a broader class of algebras. For this we need the concept of *asymptotic range* from [,] and a modification of asymptotic range from [R, Definition 1.4.3] and a modification of asymptotic range from [S, p. 345].

DEFINITIONS. If c is a real valued cocycle, the *range* of c is $R(c) = \overline{c(G)}$. If U is a non-empty open subset of X, c_U will denote the restriction of c to $G \cap (U \times U)$. The *asymptotic range* of c is $R_\infty(c) = \bigcap R(c_U)$, where the union is taken over all non-empty open subsets of X. We say that ∞ is an *asymptotic value* of c if, for every $M > 0$ and every non-empty open subset $U \subseteq X$, $R(c_U) \cap [M, \infty) \neq \varnothing$. Finally, we define the *extended asymptotic range* of c to be

$$\begin{cases} R_\infty(c) & \text{if } \infty \text{ is not an asymptotic value of } c, \\ R_\infty(c) \cup \{\infty\} & \text{if } \infty \text{ is an asymptotic value of } c. \end{cases}$$

It is shown in [S] that the extended asymptotic value is an invariant for the algebra (with respect to isometric isomorphism) and that there are only four possible values for $\tilde{R}_\infty(c)$: the sets $\{0\}$, $\{0, \infty\}$, $\mathsf{R} \cup \{\infty\}$ and $\lambda\mathsf{Z} \cup \{\infty\}$ for some $\lambda \neq 0$. The first case, $\tilde{R}_\infty(c) = \{0\}$ occurs if, and only if, the cocycle c is trivial. On the other hand, the standard algebras satisfy $\tilde{R}_\infty(c) = \{0, \infty\}$.

THEOREM 4.7. *Assume that A is an analytic* TAF *algebra whose cocycle c has extended asymptotic range $\tilde{R}_\infty(c) = \{0, \infty\}$. Assume also that the* C*-*envelope of A is simple. Let I be an interval from an equivalence class of G which is finite with respect to c. Then $\overline{I}$ has empty interior. Consequently, $\overline{P \cap (I \times I)}$ has empty interior in P; i.e., the ideal set σ corresponding to I is dense in P.*

PROOF. Suppose $I \subseteq \mathrm{orb}_a$ and that $\overline{I}$ has non-empty interior. We first observe that we may as well assume, without loss of generality, that $\overline{I} = X$. Indeed, if the interior of $\overline{I}$ is non-empty, then there is a compact open subset $V \subseteq X$ such that $V \subseteq \overline{I}$. We can then simply replace G by G restricted to V. We just need to note that $\tilde{R}_\infty(c|G \cap (V \times V)) = \tilde{R}_\infty(c) = \{0, \infty\}$.

The assumption that I is finite with respect to c implies that there is a number M such that $|c(x, y)| \leq M$ for all $x, y \in I$. Since $\tilde{R}_\infty(c) = \{0, \infty\}$, for every $x \in X$ and every $\epsilon > 0$ we can find an open set $U(\epsilon, x)$ containing x such that

$$R\big(c \,|\, G \cap (U(\epsilon, x) \times U(\epsilon, x))\big) \cap (\epsilon, 2M) = \emptyset.$$

Consequently

$$R\big(c \,|\, (I \times I) \cap (U(\epsilon, x) \times U(\epsilon, x))\big) \subseteq [-\epsilon, \epsilon].$$

Suppose that $x_n \in I$ and $x_n \to x$. Then there is N such that for $n \geq N$, $x_n \in U(\epsilon, x)$. Hence, for $n, m \geq N$,

$$|c(a, x_n) - c(a, x_m)| = |c(x_m, x_n)| \leq \epsilon.$$

It follows that $(c(a, x_n))$ is a Cauchy sequence and therefore has a limit.

If y_n is another sequence from I such that $y_n \to x$, then by the same argument the "interwoven" sequence $c(a, x_1)$, $c(a, y_1)$, $c(a, x_2)$, $c(a, y_2)$, $\ldots$ is also Cauchy. Consequently, $\lim_{n\to\infty} c(a, x_n) = \lim_{n\to\infty} c(a, y_n)$.

We now define

$$g(x) = \lim_{n\to\infty} c(a, x_n), \quad \text{where } x_n \in I \text{ and } x_n \to x.$$

The argument above shows that g is well defined; we next show that g is continuous. Fix $x \in X$ and $\epsilon > 0$. We shall show that for any $y \in U(\epsilon, x)$, $|g(y) - g(x)| \leq \epsilon$, thereby verifying that g is continuous. There exist sequences $x_n \in U(\epsilon, x) \cap I$ and $y_n \in U(\epsilon, x) \cap I$ such that $x_n \to x$ and $y_n \to y$. But then $|c(a, x_n) - c(a, y_n)| = |c(y_n, x_n)| \leq \epsilon$. This holds for all n, so $|g(x) - g(y)| \leq \epsilon$.

Since $g(x) = c(a, x)$ for all $x \in I$, it follows that $c(x, y) = g(y) - g(x)$ on $I \times I$.

By assumption, the cocycle c is unbounded on G. Since $G \cap (\mathrm{orb}_a \times \mathrm{orb}_a)$ is dense in G, it follows that c is unbounded on $\mathrm{orb}_a \times \mathrm{orb}_a$.

Write $\mathrm{orb}_a = \bigcup_{n=1}^{\infty} I_n$, where $I \subset I_1 \subset I_2 \subset \ldots$ are intervals in orb_a and the restriction of c to $I_n \times I_n$ is bounded for every n. By the arguments above, for each n there is a continuous function g_n defined on X such that $c(x, y) = g_n(y) - g_n(x)$ on $I_n \times I_n$. From the definition of the g_n, it follows that, for $m < n$, $g_n|I_m = g_m|I_m$. This shows that $g_n = g_m$ for all n, m (since g_n and g_m are continuous and $\overline{I_n} = \overline{I_m} = X$).

In particular, for $x \in I_n$, $g(x) = g_n(x) = c(a, x)$. This holds for all n, so in fact $g(x) = c(a, x)$ for all $x \in \mathrm{orb}_a$. Therefore $c(x, y) = g(y) - g(x)$ on $\mathrm{orb}_a \times \mathrm{orb}_a$. By the continuity of c and g and the density of $\mathrm{orb}_a \times \mathrm{orb}_a$ in G, we have $c(x, y) = g(y) - g(x)$ for all $(x, y) \in G$. But this says that c is a trivial cocycle and hence that $\tilde{R}_\infty = \{0\}$, contrary to assumption.

We have now proved that $\overline{I}$ has empty interior in X. It follows that $P \cap (\overline{I} \times \overline{I})$ has empty interior in P. Since $\overline{P \cap (I \times I)} \subset P \cap (\overline{I} \times \overline{I})$, the remaining assertions of the theorem follow.

REMARK. If $\tilde{R}_\infty(c) = \{0, \infty\}$, then every interval contained in an equivalence class is either dense in X or else nowhere dense. It follows that if $\mathscr{I}$ is a proper meet irreducible ideal in A and if e is any matrix unit from A, then there is a diagonal projection q such that $qe \in \mathscr{I}$.

EXAMPLE 4.8. The next section studies completely meet irreducible ideals in strongly maximal TAF algebras. It is convenient to observe here that the meet irreducible ideal associated with the interval I in the previous theorem need not be completely meet irreducible. Consider an alternating TAF algebra, which can be described by the cocycle

$$c(x, y) = \sum_{n=-\infty}^{\infty} 2^n (y_n - x_n),$$

where $X = \prod_{n=-\infty}^{\infty} \{0, 1\}$ and $x \sim y$ if $x_n = y_n$ for $|n| > N$. One can verify that $\tilde{R}_\infty(c) = \{0, \infty\}$ by an argument similar to [S, Example 7.3]. Now set

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

and $\omega = (\omega_n)_n \in X$. Also, let

$$z_k^{(n)} = \begin{cases} \omega_k & \text{if } k \geq 0 \text{ or } k < -2n, \\ 1 & \text{otherwise}. \end{cases}$$

and

$$y_k^{(n)} = \begin{cases} \omega_k & \text{if } k \geq 0 \text{ or } k < -2n, \\ 0 & \text{otherwise.} \end{cases}$$

Then if $I = \{y \in [\omega] : y_n = \omega_n \text{ for } n \geq 0\}$, we have $z^{(n)} = \left(z_k^{(n)}\right)_k \in I$ and $y^{(n)} = \left(y_k^{(n)}\right)_k \in I$. Moreover, observe that

$$0 \leq c(\omega, z^{(n)}) \longrightarrow \tfrac{1}{3} \qquad \text{and} \qquad 0 \geq c(\omega, y^{(n)}) \longrightarrow -\tfrac{2}{3}.$$

If we write c_n for $c(\omega, z^{(n)})$ and d_n for $c(\omega, y^{(n)})$, then we have $c_{n+1} > c_n$ and $d_{n+1} < d_n$, for all n. Thus, for all $z \in I$ we have $-\frac{2}{3} < c(\omega, z) < \frac{1}{3}$.

Now set $I_n = \{y \in I : d_n \leq c(\omega, y) \leq c_n\}$. Clearly,

$$P \cap (I \times I) = \bigcup_n (P \cap (I_n \times I_n))$$

so that, if we write $J(I)$ and $J(I_n)$ for the corresponding ideals, then

$$J(I) = \bigcap_n J(I_n).$$

If $J(I) = J(I_n)$ for some n, then

$$P \cap (I \times I) \subseteq \overline{P \cap (I_n \times I_n)}.$$

But

$$P \cap (I_n \times I_n) \subseteq c^{-1}([0, c_n - d_n]),$$

hence also

$$\overline{P \cap (I_n \times I_n)} \subseteq c^{-1}([0, c_n - d_n]).$$

But

$$c\left(y^{(n+1)}, z^{(n+1)}\right) = c\left(y^{(n+1)}, \omega\right) + c\left(\omega, z^{(n+1)}\right) = c_{n+1} - d_{n+1} > c_n - d_n.$$

Hence

$$\left(y^{(n+1)}, z^{(n+1)}\right) \in [P \cap (I \times I)] \setminus \left[\overline{P \cap (I_n \times I_n)}\right],$$

showing that $J(I) \neq J(I_n)$ for any n.

Summarizing, $J(I)$ is the intersection of the $J(I_n), n \in \mathbb{Z}$ but is not equal to any $J(I_n)$ and so $J(I)$ is not completely meet irreducible. In contrast, if the TAF algebra is the standard embedding algebra, then all intervals associated with non-zero ideals are finite; it is easy to see that the associated ideals are completely meet irreducible, since they have finite co-dimension.

5. Completely Meet Irreducible Ideals

Here, we consider completely meet irreducible ideals, an important subset of the meet irreducible ideals. In particular, we obtain analogues of Theorem 1.2 and 3.1 for these ideals. This allows us to show that $\mathrm{MIC}(A)$, the essential tool used in [DH] to study TAF algebras with isomorphic lattices of ideals, is isomorphic, as a set, to the family of all completely meet irreducible ideals. In the case in which an algebra is generated by its order preserving normalizer, there is a natural bijection between the spectrum of the algebra and the family of completely meet irreducible ideals.

DEFINITION. An ideal $\mathscr{I}$ is said to be *completely meet irreducible* provided that, whenever $\mathscr{I} = \bigcap_{\lambda\in\Lambda} \mathscr{I}_\lambda$, we have $\mathscr{I} = \mathscr{I}_\mu$, for some $\mu \in \Lambda$.

DEFINITION. A sequence $(e_k)_{k\geq N}$ of matrix units from A will be called a *CMI-chain* if the following three conditions are satisfied for all $k \geq N$:
(A) $e_k \in A_k$.
(B) $e_{k+1} \in \mathrm{Id}_{k+1}(e_k)$.
(C) The ideal in A generated by $e_k - e_{k+1}$ does not contain e_j, for any $j \geq N$.

Of course, conditions (A) and (B) are just the conditions for the sequence to be an MI-chain.

REMARK. The three conditions above imply that e_{k+1} is a subordinate of e_k, for each k. Indeed, suppose that e_{k+1} is not a subordinate of e_k. Let f be any subordinate of e_k and let $r = ff^*$ be the range projection of f and $s = f^*f$ the initial projection of f. Observe that $f = re_ks \in \mathrm{Id}_{k+1}(e_k)$ and $re_{k+1}s = 0$. Since e_{k+1} is not in the ideal in A generated by $e_k - e_{k+1}$ and $f = r(e_k - e_{k+1})s$, we conclude that e_{k+1} is not in the ideal in A generated by f. In particular, $e_{k+1} \notin \mathrm{Id}_{k+1}(f)$. Since this is true for each subordinate of e_k, it follows that $e_{k+1} \notin \mathrm{Id}_{k+1}(e_k)$. But this contradicts condition (B).

THEOREM 5.1. *Let A be a strongly maximal* TAF *algebra with some presentation. If $\mathscr{I}$ is an ideal in A, then $\mathscr{I}$ is completely meet irreducible if, and only if, $\mathscr{I}$ is the ideal corresponding to a* CMI-*chain of matrix units in the presentation.*

PROOF. Suppose that $\mathscr{I}$ is completely meet irreducible. If $\mathscr{J}$ is the intersection of all ideals of A that properly contain $\mathscr{I}$, then, by hypothesis, $\mathscr{J}$ properly contains $\mathscr{I}$. By the choice of $\mathscr{J}$, there is no ideal, $\mathscr{K}$, such that $\mathscr{I} \subsetneq \mathscr{K} \subsetneq \mathscr{J}$.

If $A_1 \to A_2 \to A_3 \to \cdots \to A$ is a presentation for A, then there is some N such that $\mathscr{I} \cap A_N \neq \mathscr{J} \cap A_N$. It is easy to see that for each $j \geq N$, there is exactly one matrix unit, call it e_j, that is in $\mathscr{J} \cap A_j$ but not in $\mathscr{I} \cap A_j$. By construction, $\mathscr{I}$ is the largest ideal that does not contain e_j for any $j \geq N$. Also, $e_j - e_{j+1}$ must be in $\mathscr{I}$, since it is in $\mathscr{J} \cap A_{j+1}$ and e_{j+1} is not a subordinate of it. Thus, $(e_j)_{j \geq N}$ is a CMI-chain. (Note: this can also be proved by appealing to [Dh, Lemma 2].)

Suppose that $\mathscr{I}$ is the ideal corresponding to a CMI-chain $(e_k)_{k \geq N}$. First observe that condition (C) implies that $e_k - e_{k+1} \in \mathscr{I}$, for each k. Let $\mathscr{J}$ be an ideal that properly contains $\mathscr{I}$. By the definition of $\mathscr{I}$, there is some $j \geq N$ so that $e_{j+1} \in \mathscr{J}$. But since $e_j - e_{j+1} \in \mathscr{I} \subseteq \mathscr{J}$, this implies that $e_j \in \mathscr{J}$. It follows that $\mathscr{J}$ contains e_N. Hence, for a set of ideals each of which properly contains $\mathscr{I}$, the intersection will contain e_N, and thus will properly contain $\mathscr{I}$. This shows that $\mathscr{I}$ is completely meet irreducible.

Unlike MI-chains, CMI-chains are essentially unique; in fact, we can prove slightly more.

PROPOSITION 5.2. *If $\mathscr{I}$ is a completely meet irreducible ideal with* CM*I-chain (e_i) and (f_i) is an* MI-*chain that also gives $\mathscr{I}$, then there is some K so that for all $k \geq K$, $f_k = e_k$.*

PROOF. From the proof of Theorem 5.1, we have a unique ideal $\mathscr{J}$, properly containing $\mathscr{I}$, that is the meet of all ideals properly containing $\mathscr{I}$. Further, there is some integer L so that for all $k \geq L$, e_k is the unique matrix unit of the k-th algebra that is in $\mathscr{J}$ but not in $\mathscr{I}$. If (f_i) is an MI-chain associated to $\mathscr{I}$, then $\mathscr{I}$ is the largest ideal which does not contain any f_i. Thus, $\mathscr{J}$ contains some f_J (and hence all f_i for all $i \geq J$, by the condition that f_{i+1} be in the ideal generated by f_i). Since no f_i is in $\mathscr{I}$, it follows that for all $k \geq K = \max(L, J), f_k$ is a matrix unit in $\mathscr{J}$ but not in $\mathscr{I}$, and hence f_k equals e_k.

In order to show the connection between completely meet irreducible ideals and the theory in [DH], we need some definitions and notation from Section 2 of that paper.

DEFINITION. If $\mathscr{I}$ and $\mathscr{J}$ are ideals in A, we call $[\mathscr{I}, \mathscr{J}]$ a *minimal interval* if $\mathscr{I} \nsubseteq \mathscr{J}$ and if, whenever $\mathscr{K}$ is an ideal in A with $\mathscr{I} \subseteq \mathscr{K} \subseteq \mathscr{J}$, then either $\mathscr{K} = \mathscr{I}$ or $\mathscr{K} = \mathscr{J}$.

DEFINITION. If $[\mathscr{I}, \mathscr{J}]$ is a minimal interval, its cone is the set $\{\mathscr{K} : \mathscr{J} \subseteq \mathscr{K} \vee \mathscr{I}\}$. Let MIC$(A)$ denote the set of all equivalence classes of minimal intervals under the equivalence relation of equal cones.

REMARK. Each equivalence class of minimal intervals contains a maximal representative, $[\mathscr{I}, \mathscr{J}]$. Just take $\mathscr{I} = \bigvee \mathscr{I}_\lambda$ and $\mathscr{J} = \bigvee \mathscr{J}_\lambda$, where both spans are taken over all minimal intervals $[\mathscr{I}_\lambda, \mathscr{J}_\lambda]$ in the equivalence class. Thus, we could equally well define $\mathrm{MIC}(A)$ to be the set of all maximal representatives.

PROPOSITION 5.3. *Let A be a strongly maximal* TAF *algebra. The set of all completely meet irreducible ideals in A coincides with* $\mathrm{MIC}(A)$.

PROOF. Let $\mathscr{I}$ be completely meet irreducible. Set $\mathscr{J}$ equal to the intersection of all ideals in A which properly contain $\mathscr{I}$. By the complete meet irreducibility of $\mathscr{I}$, $\mathscr{J}$ properly contains $\mathscr{I}$. So, $[\mathscr{I}, \mathscr{J}]$ is a minimal interval and hence gives an element of $\mathrm{MIC}(A)$.

Fix an element of $\mathrm{MIC}(A)$ and let $[\mathscr{I}, \mathscr{J}]$ be the maximal representative of the equivalence class. If $[\mathscr{I}', \mathscr{J}']$ is any element of the equivalence class, then $\mathscr{I}$ is the join of all ideals $\mathscr{K}$ for which $\mathscr{K} \vee \mathscr{I}'$ does not contain $\mathscr{J}'$ and $\mathscr{J}$ is the join of $\mathscr{J}'$ and $\mathscr{I}$. Repeat the argument of Proposition 5.1 or invoke [DH, Lemma 2] to see that there is a CMI-chain of matrix units $(e_j)_{j \geq N}$ for which $\mathscr{I}$ is the associated ideal. By Proposition 5.1, $\mathscr{I}$ is completely meet irreducible.

The next result establishes a bijection between the spectrum of an algebra and the set of completely meet irreducible ideals, provided that the algebra is generated by its order preserving normalizer. Again, we need a few definitions. The first is from [PPW].

DEFINITION. The *diagonal order* is the partial order defined on the collection of all projections in the diagonal, D, of A as follows: $e \preceq f$ if there is a normalizing partial isometry, w, in A such that $e = ww^*$ and $f = w^*w$.

DEFINITION. If w is a normalizing partial isometry, then the map, $x \longrightarrow w^*xw$, induces a bijection between the diagonal projections which are subprojections of ww^* and the diagonal projections which are subprojections of w^*w. We say that w is *order preserving* if this map preserves the diagonal order restricted to its range and domain. We define the *order preserving normalizer* of A to be the set of all normalizing partial isometries which are order preserving.

REMARK. If is the graph of an order preserving partial isometry (i.e., τ is an order preserving G-set), then there cannot be distinct points $(x, y) \in \tau$ and $(u, v) \in \tau$ such that $x \leq u \leq v \leq y$, where, as usual, $x \leq u$ means $(x, u) \in P$. This can be easily seen by looking at the action of τ on the sequences of diagonal matrix units which correspond to the points x, u, v, y in X. As in the previous section, when $(x, y) \in \tau$ and $(u, v) \in \tau$, we write $y = \tau(x)$ and $v = \tau(u)$; thus τ order preserving says that we cannot have $x < u$ and $\tau(u) < \tau(x)$.

The concept of an order preserving normalizer first appeared in [MS1] in a groupoid context; the term used there for the graph of an order preserving normalizing partial isometry is *monotone G-set*. The order preserving normalizer was studied by Power in [P3], where it was called the *strong normalizer*. Note that a sum of order preserving elements which is again a partial isometry is order preserving if, and only if, the ideal generated by each summand contains none of the other summands.

For the remainder of this section, we assume that the algebra A is generated by its order preserving normalizer. Algebras with this property were characterized in terms of their presentations in [DHo]. The characterization involves embeddings which are locally order preserving.

Definition. Let A_1 and A_2 be triangular subalgebras of finite dimensional C^*-algebras. An embedding $\phi : A_1 \longrightarrow A_2$ is *locally order preserving* if $\phi(e)$ is order preserving for each matrix unit $e \in A$.

An algebra A is generated by its order preserving normalizer if, and only if, there is a presentation for A such that for any contraction of the presentation, the embeddings in the contraction are locally order preserving [DHo, Theorem 18]. Another way to put this is that each matrix unit in A_j is order preserving in A_k when it is viewed as an element of A_k, for any $k > j$. This is, of course, equivalent to saying that there is a system of matrix units such that each matrix unit is an order preserving partial isometry in A.

Theorem 5.4. *Let A be a strongly maximal* TAF *algebra which is generated by its order preserving normalizer. Then there is a bijection between the spectrum, P, of A and the set of completely meet irreducible ideals in A.*

Although one way to prove this theorem is to combine Proposition 5.3 and [DH, Theorem 7], we give two self-contained proofs. All three arguments use essentially the same underlying map, but the first proof below uses the inductive limit structure while the second uses the groupoid structure. In particular, the second proof is not limited to subalgebras of AF C^*-algebras.

Proof 1. Fix a presentation for A with the property that every embedding is locally order preserving. By Proposition 5.1, there is a one-to-one correspondence between completely meet irreducible ideals in A and CMI-chains. But when every embedding is locally order preserving, the CMI conditions are satisfied by every chain (e_j) for which each e_{j+1} is a subordinate of e_j. The proof is completed by observing that the collection of all such chains is in natural one-to-one correspondence with the spectrum, P, of A.

Proof 2. Given $(x, y) \in P$, let $I = [x, y] = \{u : x \leq u \leq y\}$ be a closed interval in an equivalence class and let $Q(x, y) = \overline{P \cap (I \times I)}$. Let $J(x, y)$ be the (meet irreducible) ideal whose support is $P\backslash\overline{P\cap(I\times I)}$. We shall show that the map $(x,y)\longrightarrow J(x,y)$ is a bijection from P onto the collection of completely meet irreducible ideals.

First, we make a useful observation. If τ is an order preserving G-set which contains (x,y) then $\tau\cap Q(x,y)=\{(x,y)\}$. To see this, first note that $\tau\cap P\cap(I\times I)=\{(x,y)\}$ Å this is just the remark after the definition of order preserving partial isometry. Secondly, if $(w,z)\in\tau\cap Q(x,y)$, then there is a sequence $(x_n,y_n)\in P\cap(I\times I)$ such that $(x_n,y_n)\longrightarrow(u,v)$. For large n, $(x_n,y_n)\in\tau$; therefore, $(x_n,y_n)\in\tau\cap Q(x,y)$. Thus, $x_n=x$ and $y_n=y$ for large n; this shows that $u=x$ and $v=y$, verifying the observation.

Next we show that each ideal, $J(x,y)$, is completely meet irreducible. It is convenient to work with the complements of ideal sets, so suppose that $Q(x,y)=\overline{\bigcup F_\alpha}$, where each F_α is the complement in P of an ideal set. Since $I=[x,y]$ is a closed interval, $(x,y)\in Q(x,y)=\overline{\bigcup F_\alpha}$; hence, there is a sequence of points $(x_n,y_n)\in F_{\alpha_n}$ such that $(x_n,y_n)\longrightarrow(x,y)$. Let τ be an order preserving G-set which contains (x,y). Since τ is open, there exists k such that $(x_k,y_k)\in\tau$. Thus, $(x_k,y_k)\in\tau\cap Q(x,y)=\{(x,y)\}$. So $x_k=x$ and $y_k=y$ and, hence, $F_{\alpha_k}=Q(x,y)$.

To see that the map $(x,y)\longrightarrow J(x,y)$ is onto the family of completely meet irreducible ideals, let $\mathscr{I}$ be such an ideal. Let σ be the support set for $\mathscr{I}$. Observe that if $(x,y)\in P\backslash\sigma$, then $Q(x,y)\subseteq P\backslash\sigma$. Thus

$$\bigcup_{(x,y)\notin\sigma} Q(x,y)=P\backslash\sigma.$$

(Equality follows from the fact that each $(x,y)\in Q(x,y)$.) Thus $\sigma=\bigcap_{(x,y)\notin\sigma}P\backslash Q(x,y)$ and hence $\mathscr{I}=\bigcap_{(x,y)\notin\sigma}J(x,y)$. By the complete irreducibility of $\mathscr{I}$, $\mathscr{I}=J(x,y)$, for some (x,y).

It remains to show that the mapping is one-to-one. Assume that $J(x,y)=J(u,v)$ for points $(x,y),(u,v)\in P$. Then $Q(x,y)=Q(u,v)$. Let τ_1 and τ_2 be order preserving G-sets such that $(x,y)\in\tau_1$ and $(u,v)\in\tau_2$. Since $(x,y)\in Q(u,v)$, there is a sequence $(x_n,y_n)\in\tau_1$ such that $(x_n,y_n)\longrightarrow(x,y)$ and $u\le x_n\le y_n\le v$. For every n, $(x_n,y_n)\in Q(u,v)$; hence

$$\bigcup_n Q(x_n,y_n)\subseteq Q(u,v)=Q(x,y).$$

Since $(x,y)=\lim_n(x_n,y_n)\in\overline{\bigcup Q(x_n,y_n)}$, we have $Q(x,y)\subseteq\overline{\bigcup Q(x_n,y_n)}$. Thus $Q(x,y)=Q(u,v)=\overline{\bigcup Q(x_n,y_n)}$. Since $Q(x,y)$ is completely meet irreducible, there is m such that $Q(x,y)=Q(u,v)=Q(x_m,y_m)$. We have $u\le x_m\le y_m\le v$ and, also, $(u,v)\in Q(x_m,y_m)$; hence, there are z_k, w_k such that $(z_k,w_k)\longrightarrow(u,v)$ and $x_m\le z_k\le w_k\le y_m$, for all k. Without loss of generality, we may assume that $(z_k,w_k)\in\tau_2$, for all k. But then $u\le x_m\le z_k\le$

$w_k \le y_m \le v$. Since τ_2 is order preserving, we must have $u = z_k$, for every k. Thus, $u \le x_m \le u$; i.e., $u = x_m$. We can replace the sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ by $\{(x_n, y_n)\}_{n=N}^{\infty}$ for every $N \in$; hence we can find a subsequence (x_{m_k}) such that $x_{m_k} = u$, for all k. Therefore, $x = \lim x_{m_k} = u$. This shows that $(x, y) = (u, v)$ and the mapping is one-to-one.

In Theorems 2.3 and 2.4, we construct, for each meet irreducible ideal, an interval in the spectrum and a nest representation. If the ideal is completely meet irreducible, then it seems plausible that the interval and the nest may have a special form. We establish this in the next theorem. First we need the following fact, which is implicit in the proofs of Theorem 5.4.

Lemma 5.5. *If is the support set of a completely meet irreducible ideal $\mathscr{I}$ and τ is the support set of $\mathscr{J} = \bigcap\{\mathscr{K} \in \mathrm{Id}(A) : \mathscr{I} \not\subseteq \mathscr{K}\}$, then $\tau \backslash \sigma = \{(x, y)\}$, for some point $(x, y) \in P$. Moreover, if A is generated by its order preserving normalizer, then (x, y) is precisely the point associated to $\mathscr{I}$ by Theorem* 5.4.

Proof. Observe that since $\mathscr{I}$ is completely meet irreducible, $\mathscr{J}$ properly contains $\mathscr{I}$. As in the proof of Theorem 5.1, there is a sequence of matrix units, call it (e_k), so that e_k is the unique matrix unit of A_k that is in $\mathscr{J}$ but not in $\mathscr{I}$. It follows that, for each k, the support set of $\mathscr{J}/\mathscr{I}$ is contained inthe support of e_k in the spectrum. By the proof of Theorem 5.1, we know the e_k form a CMI-chain, and so determine a point (x, y) in the spectrum. Moreover, the support sets of the e_k form a neighbourhood base for (x, y), so the intersection of the support sets is exactly the singleton $\{(x, y)\}$.

Looking at the first proof of Theorem 5.4, the point in P associated to $\mathscr{I}$ is precisely the point determined by (e_k), i.e., (x, y).

Theorem 5.6. *Suppose $\mathscr{I}$ is a completely meet irreducible ideal, $\mathscr{J}$ is the meet of all ideals properly containing $\mathscr{I}$ and (x, y) is the one point difference of their support sets. If I is an interval in X so that $P \backslash \overline{P \cap (I \times I)}$ is the support of $\mathscr{I}$, then $I = [x, y] = \{z \in X : (x, z), (z, y) \in P\}$.*

Before proving this result, we note the following immediate corollary:

Corollary 5.7. *If $\mathscr{I}$ is a completely meet irreducible ideal and $\mathscr{N}$ is the nest of the nest representation with kernel $\mathscr{I}$ constructed in Theorem* 2.4*, then* 0 *has an immediate successor in $\mathscr{N}$ and* 1 *has an immediate predecessor.*

Proof of Theorem 5.6. If $P \cap (I \times I)$ does not contain (x, y) then, by Lemma 5.5, $P \cap (I \times I)$ is contained in the complement of the support of $\mathscr{J}$, a closed set. Hence $P \backslash \overline{P \cap (I \times I)}$ contains the support set of $\mathscr{J}$, contradicting our hypothesis. This shows that $P \cap (I \times I)$ contains (x, y) and so I contains at least the interval from x to y.

However, every point (a, y) where a is not in the interval from x to y, is in the support set of $\mathscr{I}$–to see this, observe that since (x, y) is in the support of $\mathscr{J}$, an ideal set, $(a, y) = (a, x) \circ (x, y)$ is in $\mathscr{J}$ and as $(a, y) \neq (x, y)$, it must also be in the support of $\mathscr{I}$. Thus, I cannot contain any points in the equivalence class of x and y that are less than x. Similarly, it cannot contain any points greater than y, and so I must be exactly the specified interval.

Next, we show that the sufficient condition of Corollary 5.7 is not necessary and, moreover, that there is no neccessary and sufficient condition that characterizes nests arising from completely meet irreducible ideals.

EXAMPLE 5.8. We give two ideals in different full nest algebras that are represented on the same nest algebra, yet one ideal is completely meet irreducible and the other is not. It follows that there is no condition on the nest of a nest representation that characterizes when the inducing ideal is completely meet irreducible.

Consider the 3^∞ UHF C*-algebra. Our first full nest algebra is the refinement embedding TAF algebra and the ideal is that associated to the CMI-chain given by

$$e^{(2)}_{2,8}, e^{(3)}_{5,23}, e^{(4)}_{14,68}, e^{(5)}_{41,203}, \ldots$$

where $(e^{(k)}_{i,j})_{i,j}$ are the usual matrix units for T_{3^k}. It is straightforward to check that this sequence of matrix units, call it $(f_i)_{i \geq 2}$, is a CMI-chain.

The second full nest algebra is also a triangular subalgebra of the 3^∞ UHF C*-algebra given by

$$T_3 \to T_9 \to T_{27} \to \cdots \to \mathscr{T}$$

The difference is that the inclusions from T_{3^n} to $T_{3^{n+1}}$ is the multiplicity three refinement embedding followed by conjugation by the unitary $I_3 \oplus U \oplus U^2 \oplus \cdots \oplus U^{n/3-1}$, where

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that, as $U^2 = I_3$, the conjugating unitary simplifies to $I_3 \oplus U \oplus I_3 \oplus \cdots \oplus U \oplus I_3$.

The ideal is given by the same sequence of matrix units as before. However, this time $(f_i)_{i \geq 2}$ is only an MI-chain, not a CMI-chain. For example, $f_2 = e^{(2)}_{2,8}$ is sent to

$$e^{(3)}_{4,22} + e^{(3)}_{5,23} + e^{(3)}_{6,24}$$

and so f_3 is in the ideal generated by $f_2 - f_3$. Thus, we cannot build a CMI-chain starting at f_2. Similar arguments apply to each f_i, and so the sequence does not give a CMI-chain.

However, each α_n agrees with the refinement embedding on the diagonal, so the diagonals of the two algebras are isomorphic. Since the initial and final projections of the f_i are the same in the two diagonals, it follows that the construction of Theorem 2.3 gives the same interval for both ideals. Thus, the nests constructed as in Theorem 2.4 will also be the same, as claimed.

Theorem 3.1 gives a complete description of all the meet irreducible ideal sets in the case when the spectrum, X, of the diagonal carries a total order compatible with the partial order induced by the spectrum, P, of the TAF algebra. (Note that the algebras of Example 5.8 fit into this framework.) It is not difficult to determine which of these meet irreducible ideal sets is, in fact, completely meet irreducible. First, we need a lemma identifying the ideal set associated with an arbitrary intersection of ideals. (The family of ideal sets is not a complete lattice with respect to set intersection and union. It is, of course, a lattice and this lattice is isomorphic to the lattice of ideals via the association of ideals with ideal sets.) In what follows, (S) denotes the topological interior of S.

Lemma 5.9. *Let A be a* TAF *algebra with spectral triple (X, P, G). Let $\mathscr{I}_\nu$ be a family of ideals in A and let $\mathscr{I} = \bigcap_\nu \mathscr{I}_\nu$. Let σ_ν and σ be the ideal sets in P associated with $\mathscr{I}_\nu$ and $\mathscr{I}$, respectively. Then $\sigma = \operatorname{int}(\cap_\nu \sigma_\nu)$.*

Proof. Since $\mathscr{I} \subseteq \mathscr{I}_\nu$, we have $\sigma \subset \sigma_\nu$, for all ν. Thus $\sigma \subseteq \cap_\nu \sigma_\nu$. But σ is an open set, so $\sigma \subseteq \operatorname{int}(\cap_\nu \sigma_\nu)$. For the reverse containment, let $(x, y) \in \operatorname{int}(\cap_\nu \sigma_\nu)$. Then there is a compact, open G-set, N, such that $(x, y) \in N \subseteq \operatorname{int}(\cap_\nu \sigma_\nu)$. In particular, $N \subseteq \sigma_\nu$, for all ν. Let e_N be a partial isometry in A whose graph is N. Then $e_N \in \mathscr{I}_\nu$, for all ν; hence, $e_N \in \mathscr{I}$. This shows that $N \subseteq \sigma$; in particular, $(x, y) \in \sigma$.

Recall that $\sigma_{a,b}$ denotes the ideal set $\{(x, y) \in P : x \prec a \text{ or } b \prec y\}$. When $(a, b) \in P$, the set $\sigma_{a,b} \cup \{(a, b)\}$ may or may not be an open subset of P. When it is open, it is an ideal set and we denote it by $\tau_{a,b}$.

Proposition 5.10. *Let A be a trivially analytic* TAF *algebra with an injective 0-cocycle such that the* C^**-envelope of A is simple. Assume that (X, P, G) is the spectral triple associated with A and that $\preceq$ is a total order on X compatible with the spectral triple. If $\sigma_{a,b} \cup \{(a, b)\}$ is open, then the ideal associated with $\sigma_{a,b}$ is completely meet irreducible. No other ideal is completely meet irreducible.*

Proof. First assume that $(a, b) \in P$ and that $\sigma_{a,b} \cup \{(a, b)\}$ is open. To prove that $\sigma_{a,b}$ is completely meet irreducible (in the sense that it is not the interior of the intersection of all ideal sets which strictly contain $\sigma_{a,b}$), it suffices to show that if τ is an ideal set which strictly contains $\sigma_{a,b}$, then $(a,b) \in \tau$. Let (x,y) be an element of τ which is not in $\sigma_{a,b}$. Then we must have $a \preceq x$ and $y \preceq b$. If either $x = a$ or $y = b$, then x and y are in the same equivalence class as a and b and the ideal property for τ implies that $(a,b) \in \tau$. If $a \prec x$ and $y \prec b$, then the only problem is that x and y might not be in the equivalence class of a and b. But the fact that the two coordinate projection maps are local homeomorphisms on G guarantees the existence of a neighborhood, N, of (x,y) such that $N \subseteq \tau$ and such that $(s,t) \in N$ implies $a \prec s$ and $t \prec b$. The density of equivalence classes (the simplicity of the C*-envelope of A) implies that there is an element $(s,t) \in N$ such that s and t are in the same equivalence class as a and b. The ideal property now implies that $(a,b) \in \tau$.

It remains to show that no other ideal set is completely meet irreducible. Since completely meet irreducible ideal sets are meet irreducible, we need only show that all of the ideal sets listed in Theorem 3.1 except for the $\sigma_{a,b}$ with $\sigma_{a,b} \cup \{(a,b)\}$ open are not completely meet irreducible. We claim that it suffices to prove the following property: $(*)$ let $a,b \in X$; suppose that $(x,y) \in P$, that $(x,y) \notin \sigma_{a,b}$, and that $(x,y) \neq (a,b)$. Then there is an ideal set τ such that $\sigma_{a,b} \subseteq \tau$ and $(x,y) \notin \tau$.

To see that property $(*)$ yields the proposition, consider first the case when $(a,b) \in P$ but $\sigma_{a,b} \cup \{(a,b)\}$ is not open. From property $(*)$, the intersection of all ideal sets which properly contain $\sigma_{a,b}$ is $\sigma_{a,b} \cup \{(a,b)\}$; the interior of the latter set is $\sigma_{a,b}$. Thus $\sigma_{a,b}$ is the interior of the intersection of all ideal sets which properly contain it; i.e. $\sigma_{a,b}$ is not completely meet irreducible. Next consider $\sigma_{a,b}$ when $(a,b) \notin P$. In this case, property $(*)$ implies that $\sigma_{a,b}$ is the intersection of all ideal sets which properly contain it; again $\sigma_{a,b}$ is not completely meet irreducible. Finally consider ideal sets of the form $\tau_{a,b}$. Then property $(*)$ implies that $\tau_{a,b}$ is the intersection of all ideal sets which strictly contain it and hence is not completely meet irreducible.

It remains to verify property $(*)$. If $a = b$, then property $(*)$ is vacuous. (And, in any event, $\sigma_{a,a}$ is a maximal ideal and hence is trivially completely meet irreducible.) So assume that $a \prec b$. Suppose that $(x,y) \in P$, $(x,y) \notin \sigma_{a,b}$, and $(x,y) \neq (a,b)$. Since $(x,y) \notin \sigma_{a,b}$, we have $a \preceq x$ and $y \preceq b$. Since $(x,y) \neq (a,b)$, we further have that either $a \prec x$ or $y \prec b$. This means that at least one of the two points (a,a) and (b,b) is an element of $\tau = \sigma_{x,y}$. But neither (a,a) nor (b,b) is an element of $\sigma_{a,b}$, so τ is an ideal set which properly contains $\sigma_{a,b}$ and which does not contain (x,y).

6. A Distance Formula

In this section we prove a distance formula for ideals in strongly maximal TAF algebras which is analogous to the distance formula for a nest algebra. First we prove the distance formula for the special case of an elementary groupoid of type n [R, III.1.1], i.e., the groupoid corresponding to $M_n(C(X))$ where X is a suitable topological space. Recall that we use $[i,j]$ for the set $\{i, i+1, \ldots, j\}$.

Proposition 6.1. *Let X be a locally compact, second countable Hausdorff topological space, let $H = X \times [1,n] \times [1,n]$ and suppose that $Y \subseteq H$ satisfies $(x,(i,j)) \in Y$ implies $(x,(i',j')) \in Y$ for all i', j' with $i' \le i$, $j' \ge j$. If $f \in C(H)$ satisfies*

$$(*) \qquad \sup\Big\{ \Big\| f|_{\{x\}\times[i_0,n]\times[1,j_0]} \Big\| : \{x\} \times [i_0, n] \times [1, j_0] \subseteq Y \Big\} \le 1,$$

where the norm is the matrix norm of the restriction of f, then there is $g \in C(H)$ so that $g = f$ on Y and, for each $x \in X$,

$$\Big\| g|_{\{x\}\times[1,n]\times[1,n]} \Big\| \le 1.$$

Proof. First we order the n^2 coordinates of $[1,n] \times [1,n]$ in such a way that $(n,1)$ is first, $(1,n)$ is last and, if $i_1 \ge i_2$ and $j_1 \le j_2$, then (i_1, j_1) precedes (i_2, j_2). There are clearly many ways to do this. Write Z_m for the first m coordinates in this ordering. Let $g_0 = f$. We define, inductively, $g_m \in C(H)$ so that

(1) $g_m = f$ on Y, and

(2) condition $(*)$ is satisfied for g_m in place of f and $Y \cup (X \times Z_m)$ in place of Y.

Setting $g = g_{n^2}$ then completes the proof.

We start by defining, for $a \ge 0$ and $b \in \mathsf{C}$,

$$h(a,b) = \begin{cases} 0, & \text{if } b = 0, \\ \dfrac{b}{|b|}\min(|b|, a), & \text{if } b \ne 0. \end{cases}$$

We have the following three properties: (a) $|h(a,b)| \le a$, (b) if $|b| \le a$, then $h(a,b) = b$, and (c) for continuous functions $a(x)$, $b(x)$ with $a(x) \ge 0$, $b(x) \in \mathsf{C}$, the map $x \mapsto h((a(x), b(x))$ is continuous.

By (c), we have $g_1 \in C(H)$ where g_1 is defined by

$$g_1(x,(i,j)) = \begin{cases} h(1, f(x,(n,1))), & \text{if } (i,j) = (n,1), \\ f(x,(i,j)), & \text{if } (i,j) \neq (n,1). \end{cases}$$

Also, if $(x,(n,1)) \in Y$, then $(*)$ above implies that $|f(x,(n,1))| \leq 1$ and (b) shows that $g_1(x,(n,1)) = f(x,(n,1))$. Hence we get $g_1 = f$ on Y. If $(x,(n,1)) \notin Y$, then we get $|g_1(x,(n,1))| \leq 1$ (by (a)) and, thus, $(*)$ holds for g_1 and $Y \cup \{(x,(n,1)) : x \in X\}$. This completes the initial induction step.

Assume that $g_1, \ldots, g_{m-1}$ are defined satisfying (1) and (2). To define g_m we change g_{m-1} only on $X \times (Z_m \backslash Z_{m-1})$. Write $Z_m \backslash Z_{m-1} = \{(p,q)\}$. For brevity, we use g in place of g_{m-1}. For each $x \in X$, we obtain matrices by restricting g as follows:

$$\begin{aligned} A(x) &= g|_{\{x\}\times[p+1,n]\times[1,q-1]}, \\ B(x) &= g|_{\{x\}\times\{p\}\times[1,q-1]}, \\ C(x) &= g|_{\{x\}\times[p+1,n]\times\{q\}}. \end{aligned}$$

If one of the intervals is empty, the appropriate matrices are zero; e.g., if $(p,q) = (n,2)$, $C(x) = A(x) = 0$.

For every $x \in X$, we set

$$\begin{aligned} K(x) &= B(x)(I - A^*(x)A(x))^{-1/2} \in M_{1,q-1}, \\ L(x) &= (I - A(x)A(x)^*)^{-1/2} C(x) \in M_{n-p,1}, \\ t(x) &= (I - K(x)K(x)^*)^{-1/2}(I - L(x)^*L(x))^{-1/2} \in, \quad t(x) \geq 0, \\ s(x) &= -K(x)A(x)^*L(x) \in \mathsf{C} \end{aligned}$$

Then, by [DKW, Theorem 1.2], for every number w with $|w| \leq t(x)$, the matrix

$$\begin{pmatrix} B(x) & w + s(x) \\ A(x) & C(x) \end{pmatrix}$$

has norm less than or equal to 1. In fact, this is also a necessary condition. We now define

$$g_m(x,(i,j)) = \begin{cases} g_{m-1}(x,(i,j)), & \text{if } (i,j) \neq (p,q), \\ s(x) + h(t(x), g_{m-1}(x,(p,q)) - s(x)), & \text{if } (i,j) = (p,q). \end{cases}$$

For $(x,(i,j)) \in Y$, if $(i,j) \neq (p,q)$, then $g_m(x,(i,j)) = g_{m-1}(x,(i,j)) = f(x,(i,j))$. If $(x,(p,q)) \in Y$, then $\{x\} \times [p,n] \times [1,q] \subseteq Y$. As condition $(*)$ is satisfied for g_{m-1} in place of f, we see that

$$\left\| \begin{pmatrix} B(x) & [g_{m-1}(x,(p,q)) - s(x)] + s(x) \\ A(x) & C(x) \end{pmatrix} \right\| \leq 1.$$

Thus, by the result of [DKW] quoted above,

$$|g_{m-1}(x,(p,q)) - s(x)| \le t(x).$$

It now follows from the properties of h that

$$h(t(x), g_{m-1}(x,(p,q)) - s(x)) = g_{m-1}(x,(p,q)) - s(x),$$

and therefore, $g_m(x,(p,q)) = g_{m-1}(x,(p,q))$. But $g_{m-1}(x,(p,q)) = f(x,(p,q))$ (since $(x,(p,q)) \in Y$ and g_{m-1} satisfies condition (1)). Hence $g_m(x,(p,q)) = f(x,(p,q))$ in this case. This shows that g_m satisfies (1). To prove (2), we have to prove $(*)$ for g_m and the point $(i_0, j_0) = (p,q)$. But this follows from [DKW, Theoem 1.2].

The following theorem takes place in the context of a strongly maximal TAF algebra, A, with C*-envelope, B and spectral triple (X, P, G). Elements of B will be viewed as continuous functions on G in the usual way for groupoid C*-algebras. Also $\mathscr{M}$ will denote the collection of all "finite rectangles" in G; i.e., $Q \in \mathscr{M}$ if $Q = \{(x_i, y_j) : 1 \le i \le n, 1 \le j \le m\}$ for some x_i, y_j in some equivalence class. For such a $Q \in \mathscr{M}$, $T[Q]$ is the matrix obtained by restricting T to Q and the norm is the usual matrix norm.

Theorem 6.2. *If $\mathscr{U}$ is a closed A-module contained in B with support set σ, then, for any $T \in B$,*

$$\operatorname{dist}(T, \mathscr{U}) = \sup\big\{\|T[Q]\| : Q \in \mathscr{M}, Q \cap \sigma = \varnothing\big\}$$

Proof. For every $Q \in \mathscr{M}$ and $T \in B$, $\|T[Q]\| \le \|T\|$; hence

$$\operatorname{dist}(T, \mathscr{U}) \ge \sup\{\|T[Q]\|| : Q \in \mathscr{M}, Q \cap \sigma = \varnothing\}.$$

For the other direction, the proof is as in [MS2, Theorem 4.1] because the previous proposition proves the distance formula for elementary groupoids of type n and the argument of [MS2, Lemma 4.2] still works even though the collection $\mathscr{M}$ here is different from the corresponding collection there. (The important property of sets belonging to $\mathscr{M}$ is that $T \longrightarrow T[Q]$ is norm reducing.)

Corollary 6.3. *For every ideal $\mathscr{J} \subseteq A$ and for every $T \in B$,*

$$\operatorname{dist}(T, \mathscr{J}) = \sup\{\operatorname{dist}(T, \mathscr{I}) : \mathscr{I} \text{ is a meet irreducible ideal in } A \text{ and } \mathscr{I} \supseteq \mathscr{J}\}.$$

Proof. Clearly, $\ge$ holds. For the reverse inequality, note that for the left hand side we have

$$\operatorname{dist}(T, \mathscr{J}) = \sup\{\|T[Q]\| : Q \in \mathscr{M}, Q \cap \sigma(\mathscr{J}) = \varnothing\}$$

while the right hand side equals

$$\sup\{\|T[Q]\| : Q \in \mathscr{M}, Q \cap \sigma(\mathscr{I}) = \varnothing \text{ with } \mathscr{I} \supseteq \mathscr{J}, \mathscr{I} \text{ a meet irreducible ideal}\}.$$

Hence, it is enough to show that if $Q \in \mathscr{M}$ satisfies $Q \cap \sigma(\mathscr{J}) = \emptyset$, then there is some meet irreducible ideal $\mathscr{I} \supseteq \mathscr{J}$ such that $Q \cap \sigma(\mathscr{I}) = \emptyset$. For this, just assume that

$$Q = \{(x_i, y_j) : x_1 \leq x_2 \leq \ldots \leq x_n, y_1 \leq y_2 \leq \ldots \leq y_m\} \subseteq [u] \times [u]$$

and let I be the interval $[x_1, y_m] \subseteq [u]$. The meet irreducible ideal $\mathscr{I}$ associated with I satisfies $Q \cap \sigma(\mathscr{I}) = \emptyset$. Since $Q \cap \sigma(\mathscr{J}) = \emptyset$ we have also $\sigma(\mathscr{I}) \supseteq \sigma(\mathscr{J})$.

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MATH. SCAND. 87 (2000), 64–72

COMPLETE ORDER ISOMORPHISMS BETWEEN NON-COMMUTATIVE L^2-SPACES

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Abstract

In this article we shall study the completely positive maps between non-commutative L^2-spaces. Especially, we deal with a complete order isomorphism and a completely o.d. homomorphism between the Hilbert spaces associated with the matrix ordered standard forms of von Neumann algebras, and we investigate the relationship between these maps and the homomorphisms of von Neumann algebras.

1. Introduction

Many authors have studied the positive maps on an ordered Hilbert space defined by a selfdual cone The linear map preserving the order and the orthogonal decomposition in a selfdual cone (called o.d. homomorphism) is introduced as the non-commutative version of the lattice homomorphism in orthogonally decomposable spaces by Yamamuro [Y1], and we have interesting results (see for example [DY],[D], [Y2], [I]). We consider here such a map and a more general map between non-commutative L^2-spaces from the point of view of complete positivity in the category of matrix ordered standard forms of von Neumann algebras introduced by Schmitt and Wittstock [SW].

Let $(H, H_n^+, n \in \mathsf{N})$ and $(\hat{H}, \hat{H}_n^+, n \in \mathsf{N})$ be matrix ordered Hilbert spaces. A (bounded) linear map h of H into $\hat{H}$ is said to be n-positive when the multiplicity map $h_n = h \otimes 1_n$ maps H_n^+ into $\hat{H}_n^+$. If h is n-positive for every natural number n, then h is called a completely positive map (or a complete order homomorphism). A bijective linear map h of H onto $\hat{H}$ is called an order isomorphism if $hH^+ = \hat{H}^+$. We call h a complete order isomorphism if $h_n H_n^+ = \hat{H}_n^+$ for every $n \in \mathsf{N}$. We call h an o.d.(orthogonal decomposition) homomorphism if h is 1-positive and $(h\xi, h\eta) = 0$ whenever $\xi, \eta \in H^+$ and $(\xi, \eta) = 0$. If h is completely positive and h_n is an o.d. homomorphism for

* Partially supported by the Grants-in-Aid for Scientific Research, Ministry of Education, Japan.

Received November 25, 1997.

every $n \in \mathsf{N}$, we call h is a complete o.d. homomorphism. A bijective map h is called a complete o.d. isomorphism if both h and h^{-1} are complete o.d. homomorphisms.

We shall use the notation as introduced in [SW] with respect to matrix ordered standard forms and their construction.

From now on, let $(M, H, H_n^+, n \in \mathsf{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathsf{N})$ be matrix ordered standard forms of σ-finite von Neumann algebras. We use here the notation

$$\mathrm{Ad}(h) : x \in B(H) \mapsto hxh^{-1} \in B(\hat{H})$$

for the invertible map $h : H \to \hat{H}$.

2. Complete order isomorphism between matrix ordered Hilbert spaces

This section is devoted to the study of the complete order isomorphism between two matrix ordered Hilbert spaces. At first, we shall consider that an isomorphism between von Neumann algebras induces a complete order isomorphism between the related Hilbert spaces. We need a lemma.

Lemma 2.1. *For $(M, H, H_n^+, n \in \mathsf{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathsf{N})$, if h is a completely positive map of H onto $\hat{H}$ such that $hH^+ = \hat{H}^+$, then h is a complete order isomorphism.*

Proof. Consider the polar decomposition $h = u|h|$ of h. There exists by [C, Theorem 3.3] a positive invertible operator k such that $|h| = kJ_{H^+}kJ_{H^+}$. Since H_n^+ is generated by all elements $[x_i J_{H^+} x_j J_{H^+} \xi], x_i \in M, \xi \in H^+$ by [SW, Lemma 1.1], it follows that $|h|_n H_n^+ = H_n^+$, so $h_n H^+ = \hat{H}_n^+$.

Proposition 2.2. *For $(M, H, H_n^+, n \in \mathsf{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathsf{N})$, if ρ is an (not necessarily $*$-preserving) isomorphism of M onto $\hat{M}$, then there exists a complete order isomorphism h of H onto $\hat{H}$ satisfying $\rho = \mathrm{Ad}(h)|_M$.*

Proof. Suppose that ρ is an isomorphism of M onto $\hat{M}$. By [C, Theorem 3.1] there exists a bijection h of H^+ onto $\hat{H}^+$ satisfying $\rho(x) = hxh^{-1}$, $\forall x \in M$. If $x_1, \cdots, x_n \in M$ and $\xi \in H^+$, then we have

$$h_n[x_i J x_j J \xi] = [\rho(x_i)\hat{J}\rho(x_j)\hat{J}h\xi].$$

Note that $hJh^{-1} = \hat{J}$ because of $hJh^{-1}\xi = \xi$ for every $\xi \in \hat{H}^+$. It follows from Lemma 2.1 that $h_n H_n{}^+ = H_n^+$.

Lemma 2.3. *For $(M, H, H_n^+, n \in \mathsf{N})$, if u is a 1-positive unitary on H with $u \in M \cup M'$, then $u = 1$.*

Proof. By symmetry it suffices to prove in the case $u \in M'$. Take an arbitrary element $\xi \in H$. Then ξ is written as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4)$ such that $\xi_1 \perp \xi_2$ and $\xi_3 \perp \xi_4$, $\xi_i \in H^+$. Since $u\xi = JuJ\xi$, we have $u = JuJ$. Hence $u \in M \cap M'$ and $u = u^*$. In addition, since $s(\xi_1) \perp s(\xi_2)$ and $s(\xi_3) \perp s(\xi_4)$, where $s(\xi)$ denotes the support projection of a vector functional ω_ξ on M, and $uH^+ = H^+$, we have

$$(u\xi, \xi) = \sum_{i=1}^{4} (u\xi_i, \xi_i) \geq 0.$$

Hence $u \geq 0$, and so $u = 1$.

Proposition 2.4. *For $(M, H, H_n^+, n \in \mathsf{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathsf{N})$, if ρ is a $*$-isomorphism of M onto $\hat{M}$, then there exists uniquely a completely positive isometry u of H onto $\hat{H}$ satisfying $\rho = \mathrm{Ad}(u)|_M$.*

Proof. Suppose that ρ is a $*$-isomorphism of M onto $\hat{M}$. Then there exists by [H, Theorem 2.3] a 1-positive unitary operator of H onto $\hat{H}$ satisfying

$$\rho(x) = uxu^{-1}, x \in M.$$

Then u is completely positive by the proof as in Proposition 2.2. The unicity of u follows from Lemma 2.3.

Proposition 2.5. *For $(M, H, H_n^+, n \in \mathsf{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathsf{N})$, suppose that h is a complete order isomorphism of H onto $\hat{H}$ with the polar decomposition $h = u|h|$. Then u is a completely positive isometry of H onto $\hat{H}$. Furthermore, if h as above is an o.d. homomorphism, then h is a complete o.d. isomorphism of H onto $\hat{H}$.*

Proof. By [C, Theorem 3.3] there exists an invertible operator $k \in M^+$ such that $|h| = kJ_{H^+}kJ_{H^+}$. Therefore, $|h|$ is a complete order isomorphism, so is u. Moreover, if h is an o.d. homomorphism, then $|h|$ is an o.d. homomorphism on H. By [DY, (2.1)] $|h|$ belongs to $M \cap M'$. Since $|h| \otimes 1_n \in M \otimes M_n \cap M' \otimes I_n \subset M \otimes M_n \cap M' \otimes M_n' = M \otimes M_n \cap (M \otimes M_n)'$, where M_n' operates on M_n by matrix multiplication from the right, one sees that h_n is an o.d. homomorphism. Therefore, by [DY, (3.1)] h_n is an o.d. isomorphism. This completes the proof.

We shall next consider that a complete order isomorphism between the matrix ordered Hilbert spaces induces an isomorphism between the corresponding algebras. In [SW] Schmitt and Wittstock constructed the multiplier algebra in a matrix ordered Hilbert space as follows:

Let $(H, H_n^+, n \in \mathsf{N})$ be a matrix ordered Hilbert space. Put

$$\mathscr{M} = \{x \in B(H) | \{\mathrm{diag}(x, 1, \cdots, 1) \Xi \ \mathrm{diag}(x, 1, \cdots, 1)^J\} \in H_n^+ \quad \text{for every } \Xi \in H_n^+ \text{ and all } n \in \mathsf{N}\}.$$

Here $\mathrm{diag}(x_1, x_2, \cdots, x_n)$ denotes the $n \times n$ matrix with entries $a_{i,j} = \delta_{i,j} x_i (x_i \in B(H))$ and $\{\ \}$ denotes the Jordan product

$$\{x \xi y^J\} = \tfrac{1}{2}(xJyJ\xi + JyJx\xi), x, y \in B(H), \xi \in H.$$

It is shown in [SW] that if the completed face $(F_{\{\xi\}})^{\perp\perp}$ generated by $\xi \in H_n^+$ is projectable for all $\xi \in H_n^+, n \in \mathsf{N}$, then $\mathscr{M}$ is a von Neumann algebra and $(M, H, H_n^+, n \in \mathsf{N})$ is a matrix ordered standard form.

Proposition 2.6. *For* $(M, H, H_n^+, n \in \mathsf{N})$ *and* $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathsf{N})$*, if h is a complete order isomorphism of H onto $\hat{H}$, then* $\mathrm{Ad}(h)|_M$ *is an isomorphism of M onto $\hat{M}$.*

Proof. Suppose that h is a complete order isomorphism of H onto $\hat{H}$. We show that $hMh^{-1} \subset \hat{M}$. Choose an element $x \in M$. We then obtain for all

$$\Xi = \begin{bmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{bmatrix} \in \hat{H}_n^+$$

$$\{\mathrm{diag}(hxh^{-1}, 1, \cdots, 1) \Xi \ \mathrm{diag}(hxh^{-1}, 1, \cdots, 1)^{\hat{J}}\} =$$

$$= \frac{1}{2}\left(\begin{bmatrix} hxh^{-1}\hat{J}hxh^{-1}\hat{J}\xi_{11} & hxh^{-1}\xi_{12} & \dots & hxh^{-1}\xi_{1n} \\ \hat{J}hxh^{-1}\hat{J}\xi_{21} & \xi_{22} & \dots & \xi_{2n} \\ \vdots & \vdots & & \vdots \\ \hat{J}hxh^{-1}\hat{J}\xi_{n1} & \xi_{n2} & \dots & \xi_{nn} \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} \hat{J}hxh^{-1}\hat{J}hxh^{-1}\xi_{11} & hxh^{-1}\xi_{12} & \dots & hxh^{-1}\xi_{1n} \\ \hat{J}hxh^{-1}\hat{J}\xi_{21} & \xi_{22} & \dots & \xi_{2n} \\ \vdots & \vdots & & \vdots \\ \hat{J}hxh^{-1}\hat{J}\xi_{n1} & \xi_{n2} & \dots & \xi_{nn} \end{bmatrix} \right)$$

$$= \begin{bmatrix} hxJxJh^{-1}\xi_{11} & hxh^{-1}\xi_{12} & \dots & hxh^{-1}\xi_{1n} \\ hJxJh^{-1}\xi_{21} & \xi_{22} & \dots & \xi_{2n} \\ \vdots & \vdots & & \vdots \\ hJxJh^{-1}\xi_{n1} & \xi_{n2} & \dots & \xi_{nn} \end{bmatrix}$$

$$= h_n \ \mathrm{diag}(x, 1, \cdots, 1) J_n \ \mathrm{diag}((x, 1, \cdots, 1) J_n h_n^{-1} \Xi,$$

which belongs to $\hat{H}_n^+$ because h and h^{-1} are completely positive. This implies $hMh^{-1} \subset \hat{M}$. By the symmetric argument we obtain the converse inclusion.

THEOREM 2.7. *With* $(M, H, H_n^+, n \in \mathsf{N})$, *let* $(\hat{H}, \hat{H}_n^+, n \in \mathsf{N})$ *be a matrix ordered Hilbert space. Suppose that h is an order isomorphism of H onto* $\hat{H}$. *Then the following conditions are equivalent:*

1) *h is a complete o.d. isomorphism.*

2) *There exists a von Neumann algebra* $\hat{M}$ *such that* $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathsf{N})$ *is a matrix ordered standard form, and* $\mathrm{Ad}(h)|_M$ *is a* $*$*-isomorphism of M onto* $\hat{M}$.

PROOF. 1) $\Rightarrow$ 2): We show that every completed face G in $\hat{H}_n^+$ is projectable for each n. Since h is completely positive, there exists a closed face F in H_n^+ such that $h_n F = G$. By virtue of the matrix ordered standard form F is a selfdual cone in the closed linear span $[F]$ of F. Since h_n is an o.d. homomorphism, G is a selfdual cone in $[G]$. Hence G is projectable. Indeed, every element $\eta = P_G \xi \in P_G \hat{H}_n^+ (\xi \in \hat{H}_n^+)$ has the form

$$\eta = \eta_1 - \eta_2 + i(\eta_3 - \eta_4)$$

such that $\eta_1 \perp \eta_2, \eta_3 \perp \eta_4, \eta_i \in G$. If $i \geq 2$ then $(\eta, \eta_i) = (\xi, \eta_i) \geq 0$, so $\eta_i = 0$. Hence $\eta = \eta_1$. Thus we see the existence of $\hat{M}$ by [SW, Theorem 4.3]. Consider the polar decomposition $h = u|h|$. By assumption $|h|$ is an o.d. homomorphism, it follows from [DY, (2.1)] that $|h|$ belongs to $M \cap M'$. Then we have $\mathrm{Ad}(h) = \mathrm{Ad}(u)$ on M. Applying Proposition 2.6, we obtain the desired result.

2) $\Rightarrow$ 1): Suppose that $\mathrm{Ad}(h)|_M$ is a $*$-isomorphism of M onto $\hat{M}$. Since h is an order isomorphism, we have $hJh^{-1} = \hat{J}$. It follows from the proof of Proposition 2.2 that h is completely positive. Then $hxh^{-1} = (h^{-1})^* x h^*$, $x \in M$. Hence h^*h belongs to M', so does $|h|$. Since $|h| = kJ_{H^+}kJ_{H^+}$ for some invertible positive element $k \in M$, we have $k \in M'$. Therefore, [DY, (2,1)] shows that h is an o.d. homomorphism, so h is a complete o.d. isomorphism by Proposition 2.5. This completes the proof.

For a matrix ordered standard form $(M, H, H_n^+, n \in \mathsf{N})$, let A be a von Neumann subalgebra of M. Put for $n \in \mathsf{N}$

$$P_n(A) = \{[\xi_{i,j}] \in H_n | \sum_{i,j=1}^{n} a_i J_{H^+} a_j J_{H^+} \xi_{i,j} \in H^+ \text{ for } a_1, \cdots, a_n \in A\}.$$

One easily sees that $H_n^+ = P_n(M), n \in \mathsf{N}$. We also have that if $P_n(M) = P_n(A)$ for a subalgebra A of M and $n \in \mathsf{N}$ then

$$P_n(A) = \overline{\mathrm{co}}\{[a_i J_{H^+} a_j J_{H^+} \xi] \in H_n | a_1, \cdots, a_n \in A, \xi \in H^+\}.$$

Here $\overline{\mathrm{co}}$ denotes the closed convex hull. We obtain the following theorem:

THEOREM 2.8. *For* $(M, H, H_n^+, n \in \mathsf{N})$ *and* $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathsf{N})$*, let* u *be a* 1-*positive isometry of* H^+ *onto* $\hat{H}^+$*. Suppose that* A *is a von Neumann subalgebra of* M *satisfying* $uAu^{-1} \subset \hat{M}$ *and* $P_n(A) = H_n^+$ *for all* $n \in \mathsf{N}$*. Then* u *is completely positive, and* $\mathrm{Ad}(u)$ *is a* $*$*-isomorphism of* M *onto* $\hat{M}$*.*

PROOF. Let $a_i \in A, \xi \in H^+$. We have

$$u_n[a_iJa_jJ\xi] = [ua_iJa_jJ\xi] = [ua_iu^{-1}\hat{J}ua_ju^{-1}\hat{J}u\xi],$$

which belongs to $\hat{H}_n^+$ by assumption. Hence u is completely positive, so we get the proof applying Proposition 2.4.

As an example of the above theorem we have obtained the following fact:

EXAMPLE. For $(M, H, H_n^+, n \in \mathsf{N})$, let M be an injective factor (or a semifinite injective von Neumann algebra) and let H be separable. Then there exists an abelian von Neumann subalgebra A of M such that $H_n^+ = P_n(A)$ for every $n \in \mathsf{N}$ (see [M1, Theorem 2.4]).

For a matrix ordered Hilbert space $(H, H_n^+, n \in \mathsf{N})$, we shall write $P(H^+)$ for the 1-positive maps on H. Put

$$CPU(H^+) = \{u \in P(H^+) | u \text{ is a completely positive unitary}\}.$$

Moreover, for a matrix ordered standard form $(M, H, H_n^+, n \in \mathsf{N})$, put

$$CPU^\circ(H^+) = \{uJuJ | u \text{ is a unitary in } M\}.$$

One easily sees that $CPU(H^+)$ is a topological group under the strong operator topology. Since H_n^+ is generated by the elements $[a_iJa_jJ\xi]_{i,j=1}^n \times (a_1, \cdots, a_n \in M, \xi \in H^+)$, $uJuJ$ is completely positive. One then sees that $CPU^\circ(H^+) \subset CPU(H^+)$. In the following proposition we shall show that there exists a one-to-one correspondence between $CPU(H^+)$ (resp. $CPU^\circ(H^+)$) and a group of the automorphisms Aut(M) of M (resp. the inner automorphisms $\mathrm{Int}(M)$).

PROPOSITION 2.9. *Keep the notation above. The map:* $u \mapsto \mathrm{Ad}(u)$ *is a homeomorphism of* $CPU(H^+)$ *onto* $\mathrm{Aut}(M)$*. In addition,* $CPU^\circ(H^+)$ *is homeomorphic to* $\mathrm{Int}(M)$*.*

PROOF. Applying Proposition 2.4, Theorem 2.7 and [H, Proposition 3.5] we have that $CPU(H^+)$ is homeomorphic to $\mathrm{Aut}(M)$. It is now clear that $CPU^\circ(H^+)$ is isomorphic to $\mathrm{Int}\, M)$. This completes the proof.

In the above discussion, if $uJuJ = vJvJ$ for unitaries $u, v \in M$, then $v^*u = Jvu^*J \in M'$. Then there exists a unitary w in the center of M such that $u = vw$.

In the rest of this section, we examine the results of Dang and Yamamuro [D, DY, Y2] in the framework of the completely positive maps.Using their results,we immediately obtain the similar properties.

For $\xi, \eta \in H$, put

$$P(H^+, \xi, \eta) = \{h \in P(H^+) | h\xi = \eta\}.$$

Proposition 2.10 (see [D, (2)]). *For* $(M, H, H_n^+, n \in \mathsf{N})$*, the following conditions are equivalent:*

1) *For all cyclic and separating vectors* $\xi, \eta \in H^+$*, the set of all complete o.d. homomorphisms in* $CP(H^+, \xi, \eta)$ *coincides with the set of all extreme points in* $CP(H^+, \xi, \eta)$*.*

2) *For all cyclic and separating vectors* $\xi, \eta \in H^+$*, the set of all o.d. homomorphisms in* $P(H^+, \xi, \eta)$ *coincides with the set of all extreme points in* $P(H^+, \xi, \eta)$*.*

3) M *is abelian.*

Proof. That 1) $\Rightarrow$ 2) and 2) $\Leftrightarrow$ 3) follow from [D, (1) and (2)]. If M is abelian, then every 1-positive map on H is completely positive by [M1, Corollary 1.6]. Hence 3) $\Rightarrow$ 1).

Proposition 2.11 (see [DY, (3.4)]). *For* $(M, H, H_n^+, n \in \mathsf{N})$*, the following conditions are equivalent:*

1) *Every complete o.d. isomorphism on* H *is normal.*

2) *Every* $*$*-automorphism of* M *is identical on the center of* M*.*

Proof. 1) $\Rightarrow$ 2): Let ρ be a $*$-automorphism of M. By Proposition 2.4 there exists a completely positive unitary u on H satisfying $\rho(x) = uxu^*, x \in M$. For an invertible positive element $a \in M \cap M'$, put $h = ua$. Then h is a complete o.d. isomorphism by [Y1, (3.4)]. By assumption we have $a^2 = ua^2u^*$. It follows that $ux = xu$ for every $x \in M \cap M'$.

2) $\Rightarrow$ 1): Let h be a complete o.d. isomorphism on H with the polar decomposition $h = u|h|$. We then have by Theorem 2.7 that $\mathrm{Ad}(u)|_M$ is a $*$-automorphism of M. Since $|h|$ belongs to the center of M, we have $u|h|u^* = |h|$ by assumption. Therefore,

$$h^*h = |h|u^*u|h| = |h|^2 = u|h|^2u^* = hh^*.$$

This completes the proof.

Proposition 2.12 (see [Y2, Theorem]). *For* $(M, H, H_n^+, n \in \mathsf{N})$*, suppose that* $H_n^+ (n \in \mathsf{N})$ *is a selfdual cone related to a cyclic and separating vector* ξ_0 *in* H *for* M*, and* $J = J_{\xi_0}$*. Then the following conditions are equivalent:*

1) *Every complete order isomorphism h such that* $h\xi_0 = \xi_0$ *is a complete o.d. isomorphism.*

2) *Every order isomorphism h such that* $h\xi_0 = \xi_0$ *is an o.d. isomorphism.*

3) ξ_0 *is a trace vector.*

PROOF. We remark the following fact:

For $x \in M$, put $\delta = x + JxJ$. Applying the theorem of a derivation on a homogeneous cone (see [C, Theorem 3.4]), we have for all $n \in \mathsf{N}$ and $t \in \mathsf{R}$

$$(e^{t\delta} \otimes 1_n)H_n^+ = (e^{t\delta\otimes 1_n})H_n^+ \subset H_n^+$$

by virtue of the standard form $(M_n(M), H_n, J_n, H_n^+)$ and

$$\delta \otimes 1_n = x \otimes 1_n + J_n(x \otimes 1_n)J_n.$$

This means $e^{t\delta}$ is a complete o.d. isomorphism. Then by the proof of [Y2, Theorem] we obtain the desired result.

3. Completely order homomorphism between matrix ordered Hilbert spaces

In this section we shall describe what happens with homomorphisms which are not bijective. To do this, we need the results of [M2]. We considered the relationship between a completely positive projection on a matrix ordered Hilbert space and a normal conditional expectation with respect to a faithful normal state on the related von Neumann algebra, and showed that each of them induces the other. We immediately obtain the following property by [M2, Lemma 1]:

(1) *For matrix ordered standard forms* $(M, H, H_n^+, n \in \mathsf{N})$ *and* $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathsf{N})$, *suppose that h is a complete order homomorphism of H into* $\hat{H}$ *with the support projection e and the range projection f, and* $h_n H_n^+$ *is a selfdual cone in the range space of* h_n *for every* $n \in \mathsf{N}$. *If e and f are completely positive, then there exist von Neumann algebras A and B such that* $(A, eH, e_n H_n^+, n \in \mathsf{N})$ *and* $(B, f\hat{H}, f_n\hat{H}_n^+, n \in \mathsf{N})$ *are matrix ordered standard forms, and* $h|_{eH}$ *is a complete order isomorphism of eH onto* $f\hat{H}$.

The next property follows from Proposition 2.6 and [M2, Theorem 3].

(2) *With the notations as in* (1), *suppose that e and f contain cyclic and separating fixed vectors in* H^+ *and* $\hat{H}^+$*for M and* $\hat{M}$, *respectively. If* $N = M \cap \{e\}'$ *and* $\hat{N} = \hat{M} \cap \{f\}'$, *then* $N|_{eH} = eM|_{eH} = A$ *and* $\hat{N}|_{f\hat{H}} = f\hat{M}|_{f\hat{H}} = B$, *and there exists uniquely an isomorphism* ρ *of N onto* $\hat{N}$ *such that* $\rho(x)|_{f\hat{H}} = \mathrm{Ad}(h|_{eH})(x|_{eH})$ *for all* $x \in N$.

ACKNOWLEDGEMENT. The author would like to express his gratitude to the members of Sendai Seminar for Functional Analysis for their useful suggestions. Thanks are also due to the referee for useful comment.

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