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COMPLEX INTERPOLATION OF A BANACH SPACE WITH ITS DUAL

FRÉDÉRIQUE WATBLED

Abstract

Let X be a Banach space compatible with its antidual $\overline{X^\star}$, where $\overline{X^\star}$ stands for the vector space $X^\star$ where the multiplication by a scalar is replaced by the multiplication $\lambda \odot x^\star = \overline{\lambda}x^\star$. Let H be a Hilbert space intermediate between X and $\overline{X^\star}$ with a scalar product compatible with the duality $(X, X^\star)$, and such that $X \cap \overline{X^\star}$ is dense in H. Let F denote the closure of $X \cap \overline{X^\star}$ in $\overline{X^\star}$ and suppose $X \cap \overline{X^\star}$ is dense in X. Let K denote the natural map which sends H into the dual of $X \cap F$ and for every Banach space A which contains $X \cap F$ densely let A' be the realization of the dual space of A inside the dual of $X \cap F$. We show that if $|\langle K^{-1}a, K^{-1}b\rangle_H| \leq \|a\|_{X'}\|b\|_{F'}$ whenever a and b are both in $X' \cap F'$ then $(X, \overline{X^\star})_{\frac{1}{2}} = H$ with equality of norms. In particular this equality holds true if X embeds in H or H embeds densely in X. As other particular cases we mention spaces X with a 1-unconditional basis and Köthe function spaces on Ω intermediate between $L^1(\Omega)$ and $L^\infty(\Omega)$.

I. Introduction

We first recall the basic definitions of the Calderón complex interpolation method, which can be found in [4], [3] (Cf. also [7], [10]). We say that two Banach spaces A_0, A_1 are compatible if there exists a Hausdorff topological vector space $\mathscr{U}$ and continuous linear injections i_0 of A_0 into $\mathscr{U}$ and i_1 of A_1 into $\mathscr{U}$ which allow us to identify A_0 and A_1 with vector subspaces of $\mathscr{U}$. We can then give sense to the intersection and the sum of A_0 and A_1 which become Banach spaces equipped with the following norms:

$$\|a\|_{A_0\cap A_1} = \max(\|a\|_{A_0}, \|a\|_{A_1}),$$

$$\|a\|_{A_0+A_1} = \inf(\|a_0\|_{A_0} + \|a_1\|_{A_1}, a = a_0 + a_1, a_j \in A_j).$$

If $A_0 \cap A_1$ is dense in A_0 and A_1 then the dual of $A_0 \cap A_1$ can be identified with $A_0^\star + A_1^\star$ and the dual of $A_0 + A_1$ can be identified with $A_0^\star \cap A_1^\star$, which provides a scheme where $A_0^\star$ and $A_1^\star$ are compatible. We say that a space A is intermediate between A_0 and A_1 if $A_0 \cap A_1 \subset A \subset A_0 + A_1$ with continuous inclusions. Let $S = \{z \in \mathsf{C}, 0 \leq \Re z \leq 1\}$, $S_0 = \{z \in \mathsf{C}, 0 < \Re z < 1\}$. If (A_0, A_1) is a compatible couple of complex Banach spaces, $\mathscr{F}(A_0, A_1)$ de-

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notes the family of functions defined on S, continuous and bounded with values in $A_0 + A_1$, holomorphic on S_0, such that the functions $t \mapsto f(j + it)$, $j = 0, 1$, are continuous functions from R to A_j which tend to 0 as $|t| \to +\infty$. The space $\mathscr{F}(A_0, A_1)$ is a Banach space under the norm

$$\|f\|_{\mathscr{F}(A_0,A_1)} = \max_{j=0,1} \sup_{t \in \mathsf{R}} \|f(j + it)\|_{A_j},$$

and the complex interpolation spaces are defined for $\theta \in [0, 1]$ by

$$(A_0, A_1)_\theta = \{f(\theta), f \in \mathscr{F}(A_0, A_1)\},$$

which are Banach spaces under the norm

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathscr{F}(A_0,A_1)}, f \in \mathscr{F}(A_0, A_1), f(\theta) = a\}.$$

Let us denote by $\mathscr{F}_0(A_0, A_1)$ the family of functions in $\mathscr{F}(A_0, A_1)$ of the form $F(z) = \sum_{k=1}^n F_k(z)a_k$, with F_k in $\mathscr{F}(\mathsf{C}, \mathsf{C})$ and a_k in $A_0 \cap A_1$. Calderón showed that $\mathscr{F}_0(A_0, A_1)$ is dense in $\mathscr{F}(A_0, A_1)$, which implies of course that $A_0 \cap A_1$ is dense in every $(A_0, A_1)_\theta$. Moreover, if X^0 denotes the closure of $A_0 \cap A_1$ in X then

$$(A_0, A_1)_\theta = (A_0^0, A_1)_\theta = (A_0, A_1^0)_\theta = (A_0^0, A_1^0)_\theta$$

with equality of norms. We shall also need the second Calderón interpolation method: let us denote by $\mathscr{G}(A_0, A_1)$ the family of functions g continuous on S with values in $A_0 + A_1$, holomorphic on S_0, such that $\|g(z)\|_{A_0+A_1} \leq c(1 + |z|)$, $g(j + it_1) - g(j + it_2) \in A_j$ for $t_1, t_2 \in \mathsf{R}$, $j = 0, 1$, and

$$\|g\|_{\mathscr{G}(A_0,A_1)} = \max_{j=0,1} \sup_{t_1,t_2 \in \mathsf{R}, t_1 \neq t_2} \left\| \frac{g(j + it_1) - g(j + it_2)}{t_1 - t_2} \right\|_{A_j} < \infty.$$

The space $\mathscr{G}(A_0, A_1)$ reduced modulo the constant functions and equipped with the norm above is a Banach space and the second complex interpolation spaces are defined by

$$(A_0, A_1)^\theta = \{g'(\theta), g \in \mathscr{G}\},$$

which are Banach spaces under the norm

$$\|a\|^{[\theta]} = \inf\{\|g\|_{\mathscr{G}(A_0,A_1)}, g \in \mathscr{G}, g'(\theta) = a\}.$$

The second method of interpolation is needed to identify the dual of an interpolation space: indeed the duality theorem asserts that if $A_0 \cap A_1$ is dense in both A_0 and A_1 then $(A_0, A_1)_\theta^\star = (A_0^\star, A_1^\star)^\theta$ for every $\theta \in]0, 1[$ with equality of norms. Calderón showed the inclusion $(A_0, A_1)_\theta \subset (A_0, A_1)^\theta$ and Bergh ([1]) proved that $\|a\|_{[\theta]} = \|a\|^{[\theta]}$ for every $a \in (A_0, A_1)_\theta$. It is well known that equality holds if one of the spaces A_0, A_1 is reflexive, but there is still no satisfactory characterization of spaces for which equality holds (see [2] for a survey).

Here we investigate another well known fact: the space $(L_p, L_q)_{\frac{1}{2}}$ is isometric to L_2 for every $p \in]1, +\infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$. More generally, if X is a reflexive Banach space compatible with its antidual $\overline{X^\star}$, that is the vector space $X^\star$ where the multiplication λx, $\lambda \in \mathsf{C}$, $x \in X^\star$, is replaced by the conjugate multiplication $\overline{\lambda} x$, then $(X, \overline{X^\star})_{\frac{1}{2}}$ is isometric with a Hilbert space provided $X \cap \overline{X^\star}$ is dense in X and in $\overline{X^\star}$. Pisier has showned (in [6] with Haagerup with a supplementary hypothesis, and in [9] in full generality) that if there is a continuous injection v of a Hilbert space H into X with dense range, and if we identify $\overline{X^\star}$ with the subspace $vv^\star(\overline{X^\star})$ of X, then the equality $(X, \overline{X^\star})_{\frac{1}{2}} = H$ holds again. In my thesis ([12]) I proved this equality in several other cases, in particular when X embeds in H (Cf. also [11]), or when X is a space with a 1-unconditional basis, or when X is a σ-order continuous rearrangement invariant Köthe function space. I also proved the equality when $X \cap \overline{X^\star}$ is dense in X and $\overline{X^\star}$ and with a supplementary hypothesis, but a simpler proof was given afterwards independently by Cobos and Schonbek ([5]). The main result of this paper is that equality holds if $X \cap \overline{X^\star}$ is dense in X and $|\langle K^{-1}a, K^{-1}b\rangle_H| \leq \|a\|_{X'} \|b\|_{F'}$ as soon as a and b are both in $X' \cap F'$ (Theorem 1), where F stands for the closure of $X \cap \overline{X^\star}$ in $\overline{X^\star}$, X', F', are the realizations of the duals of X and F inside the dual of $X \cap F$, and K is the natural isometry of H onto H'. This hypothesis holds in every case mentioned above and also in the case of a general Köthe space X such that $X \cap \overline{X^\star}$ is dense in X.

II. Complex interpolation of a Banach space with its dual

In all this section we shall assume that the Banach space X is compatible with its antidual $\overline{X^\star}$, and that there exists a Hilbert space H intermediate between X and $\overline{X^\star}$. Thus, as explained above, for some Hausdorff topological vector space $\mathcal{U}$, there exist continuous linear injections $i_0 : X \to \mathcal{U}$ and $i_1 : \overline{X^\star} \to \mathcal{U}$ such that $i_0(X) \cap i_1(\overline{X^\star}) \subset H \subset i_0(X) + i_1(\overline{X^\star}) \subset \mathcal{U}$. But in fact it is possible to simplify this notation and our presentation. We first observe that, without loss of generality, we can suppose, by redefining X, that $X \subset \mathcal{U}$ and that i_0 is the identity operator. The next step is to also make i_1 become the identity operator, by suitably adjusting the bilinear or sesquilinear mapping which is used to define the action of linear functionals on X. More specifically, let $Y = i_1(\overline{X^\star})$ and norm Y so that i_1 is an isometry. Define a map

$$\psi : X \times Y \to \mathsf{C}$$
$$(x, y) \mapsto i_1^{-1}(y)(x),$$

i.e. $\psi(x, y)$ is the value of the functional $i_1^{-1}(y) \in \overline{X^\star}$ when applied to the element $x \in X$. Since $i_1 : \overline{X^\star} \to Y$ is linear, ψ is sesquilinear with $\psi(\lambda x, y) = \lambda\psi(x, y) = \psi(x, \overline{\lambda} y)$. Thus, if we decide to define the action of all bounded linear functionals on X in terms of ψ, we can then in fact write $Y = \overline{X^\star}$, so that X and $\overline{X^\star}$ are subspaces of $\mathscr{U}$.

Now we can define what we mean by a *scalar product compatible with the duality*:

Definition. Let X be a Banach space compatible with its antidual $\overline{X^\star}$ and H be a Hilbert space intermediate between X and $\overline{X^\star}$. We say that the scalar product of H is compatible with the duality $(X, X^\star)$ if for every $h \in H$ such that $h = x + x^\star$ with $x \in X$ and $x^\star \in \overline{X^\star}$, we have

$$\langle h, a \rangle_H = \psi(x, a) + \overline{\psi(a, x^\star)} \text{ for every } a \in X \cap \overline{X^\star}.$$

Remark. The existence of an intermediate Hilbert space with a scalar product compatible with the duality $(X, \overline{X^\star})$ implies that $(X \cap \overline{X^\star}, \psi)$ is a prehilbertian space since

$$\langle h, a \rangle_H = \psi(h, a) = \overline{\psi(a, h)} \text{ for every } a, h \in X \cap \overline{X^\star}.$$

Conversely if $(X \cap \overline{X^\star}, \psi)$ is a prehilbertian space then its completion H is a Hilbert space, but there is no reason why this H should continuously embed into $X + \overline{X^\star}$.

From now on we assume that the scalar product of our intermediate Hilbert space H is compatible with the duality $(X, X^\star)$. We assume also without loss of generality that $X \cap \overline{X^\star}$ is dense in H. Then we can easily obtain the following:

Lemma. *In the above setting, we have* $(X, \overline{X^\star})_{\frac{1}{2}} \subset H$ *with norm less than or equal to one.*

Proof. As Pisier in [9], we shall use the bilinear interpolation theorem of Calderón. Let us first explain how to adapt it to the case of sesquilinear mappings. For any topological space B, let $\overline{B}$ denote the topological vector space which is B equipped with same topology (or norm) and with the operation $\lambda \odot b = \overline{\lambda} b$ for multiplication by scalars. Then (A_0, A_1) is a couple of Banach spaces contained in $\mathscr{U}$ if and only if $(\overline{A_0}, \overline{A_1})$ is such a couple contained in $\overline{\mathscr{U}}$. Next, consider an arbitrary element $F \in \mathscr{F}_0(A_0, A_1)$, i.e. $F(z) = \sum_{k=1} F_k(z) a_k$ where $F_k \in \mathscr{F}(\mathsf{C}, \mathsf{C})$ and $a_k \in A_0 \cap A_1$. Define $G : S \mapsto A_0 + A_1$ by setting $G(z) = \sum_{k=1}^n \overline{F_k(\overline{z})} \odot a_k$. Then clearly $G \in \mathscr{F}_0(\overline{A_0}, \overline{A_1})$ and

$G(\theta) = F(\theta)$. Furthermore $\|G\|_{\mathscr{F}_0(\overline{A_0},\overline{A_1})} = \|F\|_{\mathscr{F}_0(A_0,A_1)}$. By considering all such F and G it is easy to show that

$$\overline{(A_0, A_1)_\theta} = (\overline{A_0}, \overline{A_1})_\theta$$

with equality of norms. Hence one can deduce an interpolation theorem for sesquilinear mappings from Calderòn's theorem and the fact that for any Banach spaces A and B a map $\phi : A \times B \to \mathsf{C}$ is sesquilinear if and only if it is bilinear as a map from $A \times \overline{B}$ to C.

Now the sesquilinear form φ defined on $X \cap \overline{X^\star} \times \overline{X^\star} \cap X$ by $\varphi(a,b) = \langle a,b \rangle_H$ is bounded with norm less than or equal to one both on $X \times \overline{X^\star}$ and on $\overline{X^\star} \times X$ so that it extends by the bilinear interpolation theorem to a sesquilinear form of norm less than or equal to one on $(X, \overline{X^\star})_{\frac{1}{2}} \times (\overline{X^\star}, X)_{\frac{1}{2}} = ((X, \overline{X^\star})_{\frac{1}{2}})^2$. In particular we have for every x in $X \cap \overline{X^\star}$, $\varphi(x,x) = \|x\|_H^2 \leq \|x\|^2_{(X,\overline{X^\star})_{\frac{1}{2}}}$, hence $\|x\|_H \leq \|x\|_{(X,\overline{X^\star})_{\frac{1}{2}}}$. As $X \cap \overline{X^\star}$ is dense in $(X, \overline{X^\star})_{\frac{1}{2}}$ and as H and $(X, \overline{X^\star})_{\frac{1}{2}}$ are both continuously imbedded in $X + \overline{X^\star}$ we deduce that $(X, \overline{X^\star})_{\frac{1}{2}}$ is included in H with $\|x\|_H \leq \|x\|_{(X,\overline{X^\star})_{\frac{1}{2}}}$ for every x in $(X, \overline{X^\star})_{\frac{1}{2}}$.

In the sequel we shall make the supplementary assumption that $X \cap \overline{X^\star}$ is dense in X, and we shall let F denote the closure of $X \cap \overline{X^\star}$ in $\overline{X^\star}$. Now the couple we are really interested in is the couple (X, F), since we have $X \cap \overline{X^\star} = X \cap F$, and $(X, \overline{X^\star})_{\frac{1}{2}} = (X, F)_{\frac{1}{2}}$. The space H is continuously included in $X + \overline{X^\star}$, $X \cap F$ is dense in H, and $X + F$ is a closed subspace of $X + \overline{X^\star}$ (because the norm of $X + F$ is equal to the norm of $X + \overline{X^\star}$: indeed if $x + f = y + y^\star$ with x, $y \in X$, $f \in F$, $y^\star \in \overline{X^\star}$ then necessarily $y^\star \in F$ since $x - y = y^\star - f \in X \cap \overline{X^\star} = X \cap F$), therefore we obtain that $H \subset X + F$ (continuous inclusion). As $X \cap F$ is dense both in X and in F it is also dense in $X + F$, and H which contains $X \cap F$ is therefore dense in $X + F$. Let $\mathscr{V}$ be the dual space of $X \cap F$ and let us denote the action of $v \in \mathscr{V}$ on $x \in X \cap F$ by $\gamma(x, v)$, so that

$$\gamma : X \cap F \times \mathscr{V} \to \mathsf{C}$$

is a bilinear form. For each normed space A which contains $X \cap F$ densely, let A' denote the subspace of $\mathscr{V}$ consisting of those elements v for which the norm

$$\|v\|_{A'} = \sup\{|\gamma(x, v)| : x \in X \cap F, \|x\|_A \leq 1\}$$

is finite. Then A' is a realization of the dual space of A. In particular we will consider and use the space A' when A is any of the spaces X, F, $X + F$ and H. The two spaces X' and F' form a compatible couple with $\mathscr{V}$ as their containing space, and we have $(X + F)' = X' \cap F'$, $(X \cap F)' = \mathscr{V} = X' + F'$,

$((X,F)_{\frac{1}{2}})' = (X',F')^{\frac{1}{2}}$. Also the continuous inclusion $H \subset X+F$ implies the continuous inclusion $X' \cap F' \subset H'$. Now since $X \cap F$ is continuously included in H, each $h \in H$ defines an element $Kh \in \mathscr{V}$ such that

$$\gamma(x, Kh) = \langle x, h \rangle_H \text{ for all } x \in X \cap F.$$

This defines a one to one operator K which is an antilinear isometry of H onto H'. We are ready for theorem 1:

Theorem 1. *Let X be a Banach space compatible with $\overline{X^\star}$ such that $X \cap \overline{X^\star}$ is dense in X, and let F be the closure of $X \cap \overline{X^\star}$ in $\overline{X^\star}$. Let H be an intermediate Hilbert space between X and $\overline{X^\star}$ with a scalar product compatible with the duality $(X, X^\star)$ and $X \cap \overline{X^\star}$ dense in H. If*

$$|\langle K^{-1}a, K^{-1}b\rangle_H| \le \|a\|_{X'}\|b\|_{F'} \text{ for all } a, b \in X' \cap F'$$

then $(X, \overline{X^\star})_{\frac{1}{2}} = H$ with equality of norms.

Proof. The sesquilinear form φ defined on $X' \cap F' \times F' \cap X'$ by $\varphi(a,b) = \langle K^{-1}a, K^{-1}b\rangle_H$ is bounded with norm less than or equal to one both on $X' \times F'$ and on $F' \times X'$ by hypothesis so it extends by the bilinear interpolation theorem to a sesquilinear form of norm less than or equal to one on $(X',F')_{\frac{1}{2}} \times (F',X')_{\frac{1}{2}} = ((X',F')_{\frac{1}{2}})^2$. Using the same arguments as in the proof of the inclusion

$$(X,F)_{\frac{1}{2}} \subset H \text{ with norm } \le 1$$

we deduce the inclusion

$$(X',F')_{\frac{1}{2}} \subset H' \text{ with norm } \le 1.$$

On the other hand by dualizing the inclusion $(X,F)_{\frac{1}{2}} \subset H$ we obtain $H' \subset (X',F')^{\frac{1}{2}}$ with norm less than or equal to one. As $(X',F')_{\frac{1}{2}}$ is a subspace of $(X',F')^{\frac{1}{2}}$ with the same norm, this implies the equality

$$\|x\|_{H'} = \|x\|_{(X',F')_{\frac{1}{2}}} \text{ for every } x \in (X',F')_{\frac{1}{2}}.$$

Now $(X',F')_{\frac{1}{2}}$ is reflexive hence equal to $(X',F')^{\frac{1}{2}}$ thanks to the proposition below, so that eventually $((X,F)_{\frac{1}{2}})'$ is equal to H' with equality of norms, and so we obtain $(X,F)_{\frac{1}{2}} = H$.

For the sake of completeness let us state as a proposition the result we used in the previous proof (cf. also [12], Proposition II.1.3):

Proposition. *Let A_0, A_1 be two compatible Banach spaces with $A_0 \cap A_1$ dense in A_0 and A_1, let $\theta \in]0,1[$. If $(A_0^\star, A_1^\star)_\theta$ is reflexive then $(A_0^\star, A_1^\star)_\theta = (A_0^\star, A_1^\star)^\theta$.*

PROOF. We know that $(A_0^\star, A_1^\star)_\theta$ is a subspace of $(A_0^\star, A_1^\star)^\theta$ with the same norm, and we also know ([13], Lemma 2 or [11], [12]) that it is sequentially dense in $(A_0^\star, A_1^\star)^\theta$ for the weak star topology $\sigma((A_0^\star, A_1^\star)^\theta, (A_0, A_1)_\theta)$. Now if Y is a closed reflexive subspace of a dual Banach space $X^\star$ which is also sequentially weak star dense in $X^\star$ then Y is equal to $X^\star$.

Theorem 1 implies the result of Pisier mentioned in the introduction:

COROLLARY 1. *Let H be a Hilbert space, let $v : H \to X$ be an injection with dense range, and $H_1 = v(H)$. If we identify $\overline{X^\star}$ with the subspace of X defined by*

$$\overline{X^\star} = \{y \in H_1, |\langle x, y\rangle_{H_1}| \leq C\|x\|_X \ \forall x \in H_1\}$$

then $(X, \overline{X^\star})_{\frac{1}{2}} = H_1$ with equality of norms.

PROOF. Here we have $\overline{X^\star} \subset H_1 \subset X$ with continuous inclusions and H_1 dense in X, $X \cap \overline{X^\star} = \overline{X^\star}$, $X + \overline{X^\star} = X$, $F = \overline{X^\star}$. The scalar product of H_1 is compatible with the duality $(X, X^\star)$ by definition of $\overline{X^\star}$, and $\overline{X^\star}$ is dense in H_1 because every linear functional F bounded on H_1 which vanishes on $\overline{X^\star}$ is of the form $F(h) = \langle h, k\rangle_{H_1}$ with $k \in H_1$ hence if $\langle h, k\rangle_{H_1} = 0$ for every $h \in \overline{X^\star}$ then the value of the linear form h on $k \in X$ is zero for every $h \in \overline{X^\star}$ and therefore $k = 0$, i.e. $F = 0$. The space $\overline{X^\star}$ is also dense in X, and it is easy to check that K is an isometry from $\overline{X^\star}$ onto X', so that for every $a, b \in X' \cap \overline{X^\star}' = X'$,

$$|\langle K^{-1}a, K^{-1}b\rangle_{H_1}| = |\gamma(K^{-1}a, b)| \leq \|K^{-1}a\|_{\overline{X^\star}}\|b\|_{\overline{X^\star}'} = \|a\|_{X'}\|b\|_{\overline{X^\star}'}.$$

Therefore the theorem applies and we obtain $(X, \overline{X^\star})_{\frac{1}{2}} = H_1$ with equality of norms.

III. Applications

In this section we show how the special cases mentioned in the introduction become easy corollaries of Theorem 1.

COROLLARY 2. *Let H be a Hilbert space, let $v : X \to H$ be an injection with dense range, and let $Y = v(X)$ with norm $\|v(x)\|_Y = \|x\|_X$ for every $x \in X$. Let the duality between Y and $Y^\star$ be given by a bilinear functional which extends the bilinear functional $\beta : Y \times \overline{H} \to \mathsf{C}$, $\beta(y, h) = \langle y, h\rangle_H$, so that $H \subset \overline{Y^\star}$. Then $(Y, \overline{Y^\star})_{\frac{1}{2}} = H$ with equality of norms.*

PROOF. Here we have $Y \subset H \subset \overline{Y^\star}$ with $Y \cap \overline{Y^\star} = Y$ dense in H, $Y + \overline{Y^\star} = \overline{Y^\star}$, and F is the closure of Y in $\overline{Y^\star}$. Then $H \subset F$ densely so we can consider

$$\overline{F^\star} = \{y \in H, |\langle x, y\rangle_H| \leq C\|x\|_F \ \forall x \in H\}.$$

We have

$$Y \subset \overline{F^\star} \subset H \subset F \subset \overline{Y^\star},$$
$$F' \subset H' \subset \overline{F^\star}' \subset Y',$$

and for every a, b in $Y' \cap F' = F'$ we have $K^{-1}a$, $K^{-1}b$ in $\overline{F^\star}$ and

$$|\langle K^{-1}a, K^{-1}b\rangle_H| \leq \|K^{-1}a\|_F\|b\|_{F'} = \|K^{-1}a\|_{\overline{Y^\star}}\|b\|_{F'} = \|a\|_{Y'}\|b\|_{F'},$$

hence the theorem applies and we get the result.

Corollary 3. *Let X be a Banach space compatible with $\overline{X^\star}$ such that $X \cap \overline{X^\star}$ is dense in X and in $\overline{X^\star}$. Let H be an intermediate Hilbert space between X and $\overline{X^\star}$ with a scalar product compatible with the duality $(X, X^\star)$ and $X \cap \overline{X^\star}$ dense in H. If K^{-1} maps $X' \cap \overline{X^\star}'$ into $\overline{X^\star} \cap X$ then $(X, \overline{X^\star})_{\frac{1}{2}} = H$ with equality of norms.*

Proof. Here we have $F = \overline{X^\star}$, and it is easy to check that K maps $\overline{X^\star} \cap X$ into $X' \cap \overline{X^\star}'$ with $\|Kx\|_{X'} = \|x\|_{\overline{X^\star}}$ and $\|Kx\|_{\overline{X^\star}'} = \|x\|_X$ for every $x \in X \cap \overline{X^\star}$. Let a, $b \in X' \cap \overline{X^\star}'$. Then by hypothesis $K^{-1}a$, $K^{-1}b \in X \cap \overline{X^\star}$ so that

$$|\langle K^{-1}a, K^{-1}b\rangle_H| \leq \left\|K^{-1}a\right\|_{\overline{X^\star}}\left\|K^{-1}b\right\|_X = \|a\|_{X'}\|b\|_{\overline{X^\star}'}$$

and the theorem applies.

Before we state the next corollary let us explain the setting. A space X is called a space of sequences if X is a Banach space included in the space ω of all complex valued sequences such that the space c_{00} of finitely supported sequences is dense in X and the inclusion $X \to \omega$ is continuous with respect to the topology induced on ω by the family of semi-norms $p_n(x) = |x_n|$. We denote as usual by e_n the sequence whose all coordinates are 0 except the n^{th} which is equal to 1. If X is a Banach space with a basis $(b_n)_{n\geq 1}$, we identify X with the space of sequences which is the completion of c_{00} for the norm

$$\left\|\sum_{k=1}^n x_k e_k\right\| = \left\|\sum_{k=1}^n x_k b_k\right\|_X.$$

Then $i_1 : \overline{X^\star} \to \omega$ which maps a functional f to the sequence $(\overline{f(e_n)})_{n\geq 1}$ is a continuous linear injection. We set $Y = i_1(\overline{X^\star})$ and we norm Y so that i_1 is an isometry. Then we decide to define the action of bounded linear functionals on X in terms of

$$\psi : X \times Y \to \mathbf{C}$$
$$(x, y) \mapsto \sum_{k=1}^{\infty} x_k \overline{y_k}$$

so that we identify $\overline{X^\star}$ with Y. Now X and $\overline{X^\star}$ are both subspaces of ω and $X \cap \overline{X^\star}$ is continuously embedded into l^2. If we assume moreover that l^2 is continuously embedded in $X + \overline{X^\star}$ then l^2 is intermediate between X and $\overline{X^\star}$ and the scalar product of l^2 is clearly compatible with the duality $(X, X^\star)$. Also $X \cap \overline{X^\star}$ is automatically dense in l^2 and in X since it contains c_{00} which is dense in l^2 and in X. We consider as before the closure F of $X \cap \overline{X^\star}$ in $\overline{X^\star}$, and we dualize the inclusions

$$c_{00} \subset X \cap F \subset l^2 \subset X + F \subset X + \overline{X^\star} \subset \omega$$

into

$$X' \cap F' \subset l^{2'} \subset X' + F',$$

and we note that the duality $\gamma : X \cap F \times (X' + F')$ is given by $\gamma(x, Kh) = \sum_{k=1}^{\infty} x_k \overline{h_k}$ for every $h \in l^2$, so that we can write

$$K : l^2 \to l^{2'}$$
$$h \mapsto \overline{h}.$$

Now we are ready for Corollary 4:

Corollary 4. *Let X be a Banach space with a basis $(b_n)_{n\geq 1}$ and let $(b_n^\star)_{n\geq 1}$ be the sequence of coefficient functionals. We assume that the projections*

$$P_N : \overline{X^\star} \longrightarrow \overline{X^\star}$$
$$\sum_{k=1}^{+\infty} x_k b_k \star \longmapsto \sum_{k=1}^{N} x_k b_k^\star$$

are of norm less than one for every N and we interpolate X and $\overline{X^\star}$ in the setting of sequence spaces as explained above. If in this setting we have moreover that l^2 is continuously embedded in $X + \overline{X^\star}$ then $(X, \overline{X^\star})_{\frac{1}{2}} = l^2$ with equality of norms.

Proof. We only have to check the hypothesis of Theorem 1. Let a, $b \in X' \cap F'$. Then a and b are sequences and $\langle K^{-1}a, K^{-1}b\rangle_{l^2} = \sum_{k=1}^{\infty} \overline{a}_k b_k$. Now

$$\left|\sum_{k=1}^{N} \overline{a}_k b_k\right| = |\gamma(\overline{a}, b)| \le \|P_N(\overline{a})\|_F \|b\|_{F'} = \|P_N(\overline{a})\|_{\overline{X^\star}} \|b\|_{F'} \le \|\overline{a}\|_{\overline{X^\star}} \|b\|_{F'}$$
$$= \|a\|_{X'} \|b\|_{F'},$$

and letting N tend to $+\infty$ we obtain the desired inequality.

Before stating the last corollary let us recall (cf. [8]) that a Köthe function space on a complete σ-finite measure space (Ω, Σ, μ) is a Banach space X consisting of equivalence classes, modulo equality almost everywhere, of locally integrable functions such that:

1) if g belongs to X and if f is a measurable function such that $|f(\omega)| \le |g(\omega)|$ a.e. on Ω then f belongs to X and $\|f\| \le \|g\|$;

2) for every $\sigma \in \Sigma$ of finite measure the characteristic function χ_σ belongs to X.

Corollary 5. *Let X be a Köthe function space on a complete σ-finite measure space (Ω, Σ, μ) such that $X \cap \overline{X^\star}$ is dense in X, X and $\overline{X^\star}$ are intermediate between $L^1(\Omega)$ and $L^\infty(\Omega)$, and $L^2(\Omega)$ is intermediate between X and $\overline{X^\star}$. Then $(X, \overline{X^\star})_{\frac{1}{2}} = L^2(\Omega)$.*

Proof. Here X is a subspace of $L^1 + L^\infty$, the map $i_1 : \overline{X^\star} \to L^1 + L^\infty$ is defined by $i_1(f) = \overline{f}$, and the map ψ by $\psi(f, g) = \int_\Omega f\overline{g}$. The scalar product of L^2 is clearly compatible with the duality $(X, X^\star)$, and the map K is given by $K(h) = \overline{h}$. Then we only have to check the inequality mentioned in Theorem 1. Let a, $b \in X' \cap F'$. Write $\Omega = \cup_{n=1}^{\infty} \Omega_n$ with $\mu(\Omega_n) < \infty$ for every n. Then $a_n = \chi_{\Omega_n} \chi_{\{|a| \le n\}}$ is in $L^1 \cap L^\infty$ hence in $X \cap \overline{X^\star} \subset F$ and $a_n \to a$ a.e. as n tends to infinity with $|a_n| \le |a|$. We have

$$\langle K^{-1} a_n, K^{-1} b \rangle_{L^2} = \int_\Omega \overline{a_n} b \, d\mu$$

and

$$\left| \int_\Omega \overline{a_n} b \, d\mu \right| \le \|\overline{a_n}\|_F \|b\|_{F'} = \|\overline{a_n}\|_{\overline{X^\star}} \|b\|_{F'} \le \|\overline{a}\|_{\overline{X^\star}} \|b\|_{F'} = \|a\|_{X'} \|b\|_{F'}.$$

Now $\int_\Omega \overline{a_n} b \, d\mu \to \int_\Omega \overline{a} b \, d\mu = \langle K^{-1} a, K^{-1} b \rangle_{L^2}$ therefore $\left|\langle K^{-1} a, K^{-1} b \rangle_{L^2}\right| \le \|a\|_{X'} \|b\|_{F'}$ and the theorem applies.

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LMAM, UNIVERSITÉ BRETAGNE-SUD
1, RUE DE LA LOI
56 000 VANNES
FRANCE
Email: Frederique.Watbled@univ-ubs.fr