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SOME CHARACTERIZATIONS OF THE PROPERTIES (DN) AND $(\widetilde{\Omega})$

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Abstract

The aim of this paper is to show that

$$H_w(B,F) = H(B,F)$$

for every LB^∞ - regular compact set B in a Frechet space E if and only if F is a Frechet space having property (DN). At the same time, the equivalence between the existence of a LB^∞ - regular compact set B in a Schwartz - Frechet space E with an absolute basis and the property $(\widetilde{\Omega})$ of E is also established here.

1. Introduction

Let E be a Frechet space with the topology defined by an increasing system of semi-norms $\{\| \ \|_k\}$. For each subset B of E we define $\| \ \|_B^* : E^* \to [0, +\infty]$ given by

$$\|u\|_B^* = \sup\Big\{|u(x)| : x \in B\Big\}$$

where $u \in E^*$, the topological dual space of E.

Instead of $\| \ \|_{U_q}^*$ we write $\| \ \|_q^*$, where

$$U_q = \Big\{x \in E : \|x\|_q \leq 1\Big\}$$

Now we say that E has the property

(DN) $\exists p \ \exists d > 0 \ \forall q \ \exists k, C > 0 : \| \ \|_q^{1+d} \leq C\| \ \|_k \, \| \ \|_p^d$

$(\widetilde{\Omega})$ $\forall p \ \exists q, d > 0 \ \forall k \ \exists C > 0 : \| \ \|_q^{*1+d} \leq C\| \ \|_k^* \, \| \ \|_p^{*d}$

(LB^∞) $\forall \{\rho_n\}_{n\in\mathbb{N}} \uparrow \infty \ \forall p \ \exists q \ \forall n_0 \ \exists N_0 \geq n_0, \ C > 0 \ \forall u \in E^*$

$\exists n \in \mathbb{N} : n_0 \leq n \leq N_0 \quad \|u\|_q^{*1+\rho_n} \leq C\| \ \|_n^* \, \| \ \|_p^{*\rho_n}$

The above properties have been introduced and investigated by Vogt (see [9], [10]).

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Note that the following equivalent form of the property (DN) has been formulated by Zahariuta in [12]

$$((\mathrm{DN}))_{\mathrm{Z}} \quad \forall p\ \forall q, d > 0\ \exists k, C > 0 : \|\ \|_q^{1+d} \leq C\|\ \|_k \|\ \|_p^d$$

Let E and F be locally convex spaces and let $\Omega \subset E$ be open, $\Omega \neq \emptyset$. $f : \Omega \to F$ is called Gâteaux - holomorphic if for every $y \in F^*$, the topological dual space of F, the function $yf : \Omega \to \mathbf{C}$ is holomorphic. This means that its restriction to each finite dimensional section of Ω is holomorphic as a function of several complex variables.

A function $f : \Omega \to F$ is called holomorphic if f is continuous and Gâteaux - holomorphic on Ω.

Now let B be a compact subset in a locally convex space E and F a locally convex space. By the standard notation $H(B, F)$ denotes the space of germs of holomorphic functions on B with values in F with the inductive limit topology.

Recall that $f \in H(B, F)$ if there exists a neighbourhood V of B in E and a holomorphic function $\hat{f} : V \to F$ whose germ on B is f. A F-valued function f on B is called weakly holomorphic on B if for every $x^* \in F^*_\beta$, the topological dual space of F equipped with the strong topology $\beta(F^*, F)$, x^*f can be extended holomorphically to a neighbourhood of B. By $H_w(B, F)$ we denote the space of F-valued weakly holomorphic functions on B.

For details concerning holomorphic functions and germs of holomorphic functions on compact subsets of a locally convex space we refer to the books of Dineen [1] and Noverraz [6].

One of aims of this paper is to find some necessary and sufficient conditions for which

$$H_w(B, F) = H(B, F) \qquad (\omega)$$

The statement (ω) has been investigated by several authors. Siciak in [8] and Waelbroeck in [11] have considered this problem for the case, where $\dim E < \infty$ and F^*_β is a Baire space. After that, in [4] N. V. Khue and B. D. Tac have shown that (ω) holds in the case, where F^*_β is still Baire and either E is a nuclear metric space or F is nuclear. The Baireness of F^*_β plays a very important part in the works of the above authors. However, at present, when F^*_β is not Baire, in particular, F is a Frechet space which is not Banach (ω) has not been established by any authors.

In the second part of this paper we give a characterization of the property (DN) by showing that (ω) holds if F is a Frechet space having the property (DN) and B is a LB^∞ - regular compact set in a Frechet space E, where a compact set B in a Frechet space E is said to be LB^∞ - regular if $[H(B)]^*_\beta$ has property (LB^∞). Next, from the obtained result of the second section the third section is devoted to establishing some characterizations of the property $(\widetilde{\Omega})$ of a Schwartz - Frechet space E with an absolute basis.

In through paper F^*_{bor} denotes the space F^* equipped with the bornological topology associated with the topology of F^*_β. This is the most strong locally convex topology on F^* having the same bounded subsets as the $\beta(F^*, F)$ - topology. $[F^*_{\mathrm{bor}}]^*_\beta$ is equipped with the $\beta(F^{**}, F^*_\beta)$ - topology.

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2. Characterization of (DN)

The main result of this section is the following

2.1. Theorem. *Let F be a Frechet space. Then*

$$H_w(B, F) = H(B, F) \qquad (\omega)$$

holds for every LB^∞- regular compact set B in a Frechet space E if and only if F has property (DN).

In order to prove Theorem 2.1 we need some lemmas.

2.2. Lemma. *Every LB^∞ - regular compact set B in a Frechet space E is a set of uniqueness,* i.e. *if $f \in H(B)$ and $f\big|_B = 0$ then $f = 0$ on a neighbourhood of B in E.*

Proof. Let $\{V_n\}$ be a decreasing neighborhood basis of B in E. Given $f \in H(B)$ with $f\big|_B = 0$, choose $p \geq 1$ such that $f \in H^\infty(V_p)$. For each $n \geq p$, put

$$\varepsilon_n = \|f\|_n = \sup\Big\{|f(z)| : z \in V_n\Big\}$$

Then $\{\varepsilon_n\} \downarrow 0$. By the hypothesis $\big[H(B)\big]^*_\beta$ has property (LB^∞) and employing this with $\{\rho_n\} = \Big\{\sqrt{\log\frac{1}{\varepsilon_n}}\Big\} \uparrow \infty$ we have

$$\exists q \ \forall n_0 \ \exists N_0 \geq n_0, \ C_{n_0} > 0 \ \forall m > 0 \ \exists k_m : n_0 \leq k_m \leq N_0 :$$

$$\|f^m\|_q^{1+\rho_{k_m}} \leq C_{n_0} \|f^m\|_{k_m} \|f^m\|_p^{\rho_{k_m}}$$

which yields

$$\|f\|_q^{1+\rho_{k_m}} \leq C_{n_0}^{\frac{1}{m}} \|f\|_{k_m} \|f\|_p^{\rho_{k_m}}$$

Choose $n_0 \leq k \leq N_0$ such that

$$\text{Card}\left\{m : k_m = k\right\} = \infty$$

Then

$$\begin{aligned}\|f\|_q &\le \|f\|_k^{\frac{1}{1+\rho_k}} \|f\|_p^{\frac{\rho_k}{1+\rho_k}} \\ &\le (\varepsilon_k)^{\frac{1}{1+\rho_k}} (\varepsilon_p)^{\frac{\rho_k}{1+\rho_k}} \to 0\end{aligned}$$

as $k \to +\infty$.

Hence $f\big|_{V_q} = 0$. Lemma 2.2 is proved.

2.3. LEMMA. *Let F be a Frechet space having property* (DN). *Then* $\left[F^*_{\text{bor}}\right]^*_\beta$ *has property* (DN).

PROOF. Let $\{U_n\}$ be a decreasing neighbourhood basis of $0 \in F$. Since F has property (DN) we have

$$\exists p\ \forall q\ \exists k,\ C > 0 : \|\ \ \|_q \le r\|\ \ \|_p + \frac{C}{r}\|\ \ \|_k$$

for all $r > 0$, or in equivalent form [10]

$$\exists p\ \forall q\ \exists k,\ C > 0 : U_q^0 \subseteq rU_p^0 + \frac{C}{r} U_k^0 \quad \text{for all } r > 0$$

For $u \in \left[F^*_{\text{bor}}\right]^*_\beta$ and $r > 0$ we have

$$\begin{aligned}\|u\|_q^{**} &= \sup\left\{|u(x^*)| : x^* \in U_q^0\right\} \le \sup\left\{|u(x^*)| : x^* \in rU_p^0 + \frac{C}{r} U_k^0\right\} \\ &\le r\sup\left\{|u(x^*)| : x^* \in U_p^0\right\} + \frac{C}{r}\sup\left\{|u(x^*)| : x^* \in U_k^0\right\} = \\ &= r\|u\|_p^{**} + \frac{C}{r}\|u\|_k^{**}\end{aligned}$$

Hence $\left[F^*_{\text{bor}}\right]^*_\beta$ has property (DN).

Lemma 2.3 is proved.

PROOF OF THEOREM 2.1 *Sufficiency.* It suffices to prove that $H_w(B, F) \subset H(B, F)$. Let $f \in H_w(B, F)$ and F has property (DN), where B is a LB^∞ - regular compact set in a Frechet space E. By Lemma 2.2 B is a set of uniqueness and, hence, we can consider the linear map $\hat{f} : F^*_{\text{bor}} \to H(B)$ given by

$$\hat{f}(x^*) = \widehat{x^* f}$$

for $x^* \in F^*_{\text{bor}}$, where $\widehat{x^* f}$ is a holomorphic extension of $x^* f$ to some neighbourhood of B in E. Still by the uniqueness of B it follows that $\hat{f}$ has closed graph. On the other hand, F^*_{bor} is an inductive limit of Banach spaces, $H(B)$

is an (LF) - space so by closed graph theorem of Grothendieck [3] $\hat{f}$ is continuous. Since $\hat{f}$ maps bounded subsets of F^*_{bor} to bounded subsets of $H(B)$ then the dual map $\hat{f}' : [H(B)]^*_\beta \to [F^*_{\mathrm{bor}}]^*_\beta$ is also continuous. By the hypothesis $[H(B)]^*_\beta$ has property (LB^∞) and by Lemma 2.3 $\left[F^*_{\mathrm{bor}}\right]^*_\beta$ has property (DN). From a result of Vogt [9] it follows that there exists a bounded subset $L \subset H(B)$ such that $\hat{f}'(L^0)$ is a bounded subset of $[F^*_{\mathrm{bor}}]^*_\beta$, where L^0 denotes the polar of L in $[H(B)]^*_\beta$. Hence $\left(\hat{f}'(L^0)\right)^0$ is a neighbourhood of $0 \in \left[(F^*_{\mathrm{bor}})^*_\beta\right]^*_\beta$. Put $W = \left(\hat{f}'(L^0)\right)^0 \cap F^*_{\mathrm{bor}}$. Then W is a neibourhood of $0 \in F^*_{\mathrm{bor}}$. We have

$$\hat{f}(W) \subset L^{00} \cap H(B)$$

where L^{00} is the bi-polar of L. However $L^{00} \cap H(B)$ is the closure of the absolutely convex envelope of L and, hence, it is a bounded subset of $H(B)$. This shows that $\hat{f}(W)$ is bounded in $H(B)$. By the regularity of $H(B)$ there exists a neighbourhood U of B in E such that $\hat{f}(W)$ is contained and bounded in $H^\infty(U)$, the Banach space of bounded holomorphic functions on U. From the absorption of W it follows that $\hat{f}\left(F^*_{\mathrm{bor}}\right) \subset H^\infty(U)$. Now we can define a holomorphic function

$$g : U \longrightarrow \left[F^*_{\mathrm{bor}}\right]^*$$

given by

$$g(z)(x^*) = \hat{f}(x^*)(z)$$

for $z \in U$, $x^* \in F^*_{\mathrm{bor}}$.

We see that $g(z)(x^*) = \hat{f}(x^*)(z) = f(z)(x^*)$ for every $z \in B$, $x^* \in F^*$. This yields $g\big|_B = f$ and since B is a set of uniqueness, $g(U) \subset F$.

Necessity. By Vogt [9] it suffices to show that every continuous linear map T from $H(\Delta)$ to F is bounded on a neighbourhood of $0 \in H(\Delta)$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Consider $T^* : F^*_\beta \to [H(\Delta)]^*_\beta \cong H(\overline{\Delta})$. Since $T^*(x^*) \in H(\overline{\Delta})$ for all $x^* \in F^*_\beta$, we can define a map $f : \overline{\Delta} \to [F^*_\beta]^*_\beta$ given by

$$f(z)(x^*) = \delta_z(T^*(x^*))$$

for $x^* \in F^*_\beta$, $z \in \overline{\Delta}$, where δ_z is the Dirac functional defined by z

$$\delta_z(\varphi) = \varphi(z) \quad \text{for} \quad \varphi \in H(\overline{\Delta}).$$

From the weak continuity of T^* and δ_z we infer that $f(z)$ is $\sigma(F^*, F)$-continuous and, hence, $f(z) \in F$. Moreover, $f \in H_w(\overline{\Delta}, F)$. Since $\overline{\Delta}$ is LB^∞-regular it follows that $f \in H(\overline{\Delta}, F)$. Thus there exists a neighbourhood V of $\overline{\Delta}$ such that $f \in H^\infty(V, F)$. Hence, $B = f(V)$ is bounded in F. It is easy to see that T^* is bounded on B^0. Put $C = T^*(B^0) \subset [H(\Delta)]^*_\beta$ and $U = C^0$. Then U is a neighbourhood of $0 \in H(\Delta)$ and $T(U) \subset B^{00}$ is bounded in F.

Theorem 2.1 is proved.

3. Some characterizations of $(\widetilde{\Omega})$

This section is devoted to give some characterizations of the property $(\widetilde{\Omega})$ on a Schwartz - Frechet space E with an absolute basis.

The following theorem is the main result of this section.

3.1. THEOREM. *Let E be a Schwartz - Frechet space with an absolute basis. Then the following are equivalent*

(i) *There exists a compact set B of uniqueness in E such that $H_w(B, F) = H(B, F)$ for all Frechet spaces F having property* (DN).

(ii) *There exists a compact set B in E such that $[H(B)]^*_\beta$ has property* (LB^∞).

(iii) *There exists a compact set B in E which is not polar.*

(iv) *E has the property $(\widetilde{\Omega})$.*

PROOF.

(ii) $\Rightarrow$ (i) by Theorem 2.1.

Now we give the proof (i) $\Rightarrow$ (iii). The implication (i) $\Rightarrow$ (iii) is obtained from the following proposition

3.2. Proposition *Let B be a compact set of uniqueness in a Frechet space E having a Schauder basis and let*

$$H_w(B, F) = H(B, F)$$

for every Frechet space $F \in$ (DN). *Then B is not polar.*

PROOF. Otherwise, assume that B is polar. Choose a plurisubharmonic function φ on E such that $\varphi \neq -\infty$ and

$$\varphi\big|_B = -\infty$$

Consider the Hartogs domain Ω_φ given by

$$\Omega_\varphi = \left\{(z, \lambda) \in E \times \mathbf{C} : |\lambda| < e^{-\varphi(z)}\right\}$$

Since φ is plurisubharmonic, Ω_φ is pseudoconvex. Because E has a Schauder basis so Ω_φ is the domain of a holomorphic function f. Write the Hartogs expansion of f

$$f(z, \lambda) = \sum_{n=0}^{\infty} h_n(z)\lambda^n$$

where

$$h_n(z) = \frac{1}{2\pi i} \int\limits_{|\lambda| = e^{-\varphi(z)-\delta}} \frac{f(z,\lambda)}{\lambda^{n+1}} d\lambda, \quad \text{for } \delta > 0.$$

By the upper semi-continuity of φ it follows that h_n is holomorphic on E for all $n \geq 0$. Consider the function $g : B \to H(\mathsf{C})$, given by $g(z)(\lambda) = f(z,\lambda)$. Let $\mu \in [H(\mathsf{C})]^*_\beta$ be arbitrary. There exists $r > 0$ such that $\mu \in [H(r\overline{\Delta})]^*_\beta$, where $\overline{\Delta} = \{\lambda \in \mathsf{C} : |\lambda| \leq 1\}$. From the openness of Ω_φ it follows that there exists a neighbourhood V of B such that $V \times r\Delta \subset \Omega_\varphi$. By the absolute convergence of the series $\sum_{n=0}^{\infty} h_n(z)\lambda^n$ on $V \times r\Delta$ it follows that $\mu g \in H(V)$ and, hence, $g \in H_w(B, H(\mathsf{C}))$. Applying the hypothesis to $F = H(\mathsf{C})$ which has property (DN) we find a neighbourhood U of B in E and a bounded holomorphic function $\hat{g} \in H(U, H(\mathsf{C}))$ which is a holomorphic extension of g. We can write

$$\hat{g}(z,\lambda) = \sum_{n=0}^{+\infty} \hat{g}_n(z)\lambda^n$$

where $\hat{g}_n(z)$ is holomorphic on U for all $n \geq 0$. Choose a neighbourhood W of B such that $W \subset U$ and $W \times 2\Delta \subset \Omega_\varphi$. Define two holomorphic functions

$$\begin{aligned} H : W &\to H^\infty(\Delta) \\ z &\mapsto (h_0(z), h_1(z), \ldots) \\ G : W &\to H^\infty(\Delta) \\ z &\mapsto (\hat{g}_0(z), \hat{g}_1(z), \ldots) \end{aligned}$$

Since $H^\infty(\Delta)$ is a Banach space and $H\big|_B = G\big|_B$, it follows that there exists a neighbourhood W_1 of B in W such that $\widehat{g}\big|_{W_1 \times \Delta} = f\big|_{W_1 \times \Delta}$. Let X be a connected component of W_1. Since $X \times \mathsf{C}$ is connected, $\widehat{g}\big|_{X \times \Delta} = f\big|_{X \times \Delta}$, $X \times \Delta \subset \Omega_\varphi$ and Ω_φ is the domain of existence of f we have $X \times \mathsf{C} \subset \Omega_\varphi$. Hence $\varphi\big|_X = -\infty$. This is impossible.

Proposition 3.2 is proved.

The following proposition gives the implication (iii) $\Rightarrow$ (iv).

3.3. Proposition. *Let E be a Frechet space. If there exists a non polar compact set in E then E has property* $(\widetilde{\Omega})$.

Proof. By a result of Dineen - Meise - Vogt [2, Corollary 8 and Theorem 10].

Finally, the implication (iv) $\Rightarrow$ (ii) is given by the following proposition.

3.4. Proposition. *Let E be a Schwartz - Frechet space with an absolute ba-*

sis. If E has the property $(\widetilde{\Omega})$ *then there exists a balanced convex compact subset B of E such that* $[H(B)]^*_\beta$ *has property* (LB^∞).

PROOF. Let $\{e_j\}_{j\geq 1}$ be an absolute basis for E. From the hypothesis, by Vogt [9], there exists a balanced convex compact set B_1 in E such that

$$(1) \qquad (\widetilde{\Omega}_{B_1}) \quad \forall p\ \exists q,\ d>0,\ C>0 : \|\ \ \|_q^{*1+d} \leq C\|\ \ \|_{B_1}^*\ \|\ \ \|_p^{*d}$$

On the other hand, since $\{e_j\}_{j\geq 1}$ is an absolute basis it follows that $\|e_j^*\|_{B_1}^*\, e_j$ converges to $0 \in E$. Put

$$B = \overline{\mathrm{conv}}\,(B_1 \cup \bigcup_{j\geq 1} \|e_j^*\|_{B_1}^*\, e_j)$$

Now we prove that $[H(B)]^*_\beta$ has property (LB^∞).

In order to prove that $[H(B)]^*_\beta$ has property (LB^∞) by Vogt [9], it suffices to show that every continuous linear map $T : [H(B)]^*_\beta \longrightarrow H(\mathsf{C})$ is bounded in a neighbourhood of $0 \in [H(B)]^*_\beta$. Consider the function $f : B \longrightarrow H(\mathsf{C})$ given by

$$f(x)(\lambda) = T(\delta_x)(\lambda) \ \text{ for } x \in B,\ \lambda \in \mathsf{C}$$

where $\delta_x \in [H(B)]^*_\beta$ is the Dirac functional associated with x. We claim that f is weakly holomophic, i.e. $\mu f \in H(B)$ for all $\mu \in [H(\mathsf{C})]^*_\beta$. Indeed, since E is a Schwartz-Frechet space so $[H(B)]^*_\beta$ is also a Schwartz-Frechet space. Now let $\mu \in [H(\mathsf{C})]^*_\beta$ then $\mu T \in [[H(B)]^*_\beta]^*_\beta = H(B)$ which gives a holomorphic extension of μf. For each $s>0$ consider $h^s = R^s f$, where $R^s : H(\mathsf{C}) \longrightarrow H^\infty(2s\,\Delta)$ is the restriction map and $\Delta = \{\lambda \in \mathsf{C} : |\lambda| < 1\}$. Then h^s can be extended to a bounded holomorphic function $\widehat{h^s}$ on a neighbourhood V^s of B in E. Take $p \geq 1$ such that $B + U_p \subset V^1$ and $(\widetilde{\Omega}_{B_1})$ holds for E, where $U_p = \{x \in E : \|x\|_p \leq 1\}$. Let $V_1 = B + U_p$ and let $\overline{g} : (B \times \mathsf{C}) \cup (V_1 \times \overline{\Delta}) \longrightarrow \mathsf{C}$ be given by

$$\overline{g}(x,\lambda) = \begin{cases} f(x)(\lambda) & \text{for } x \in B,\ \lambda \in \mathsf{C} \\ \widehat{h}^1(x)(\lambda) & \text{for } x \in V_1,\ \lambda \in \overline{\Delta} \end{cases}$$

Obviously $\overline{g}$ is separately holomorphic in the sense of Siciak [7]. By $\mathscr{F}$ we denote the family of all finite dimensional subspaces P of $E(B)$, where $E(B)$ denotes the Banach space induced by B. Put

$$\overline{g}_P = \overline{g}\big|_{(B\cap P\times \mathsf{C})\cup(V_1\cap P\times\overline{\Delta})}$$

Since $B \cap P$ and $\overline{\Delta}$ are not pluri-polar in $V_1 \cap P$ and C, respectively, by Nguyen and Zeriahi [5] $\overline{g}_P$ is extended uniquely to a holomorphic function $\widehat{g}_P$ on $(V_1 \cap P) \times \mathsf{C}$. Since $V_1 \cap E(B) = \cup\{V_1 \cap P : P \in \mathscr{F}\}$ the family $\{\widehat{g}_P : P \in \mathscr{F}\}$ defines a Gâteaux holomorphic function $\widehat{g}$ on $(V_1 \cap E(B)) \times \mathsf{C}$.

On the other hand, $\overline{g}$ is holomorphic on $\{x \in B : \|x\|_B < 1\} \times \Delta$, by Zorn's theorem $\widehat{g}$ is holomorphic on $(V_1 \cap E(B)) \times \mathbb{C}$, where $V_1 \cap E(B)$ is equipped with the topology of $E(B)$.

Now we prove that $\widehat{g}$ is extended holomorphically to $\widehat{g}_1$ on $W \times \mathbb{C}$, a neighbourhood of $B \times \mathbb{C}$ in $E \times \mathbb{C}$ such that $\widehat{g}_1(W \times r\Delta)$ is bounded for $r > 0$. Let $q \geq p,\ d > 0,\ C > 0$ be chosen such that (1) holds.

Since $B = \overline{\text{conv}}\,(B_1 \cup \bigcup_{j \geq 1} \|e_j^*\|_{B_1}^* e_j)$ we have

$$\|e_j^*\|_{B_1}^* \, \|e_j\|_B \leq 1, \ \text{ for } j \geq 1$$

From the condition (1) we have

$$\left(\frac{1}{\|e_j\|_q}\right)^{1+d} \leq \frac{C}{\|e_j\|_B \, \|e_j\|_p^d} \tag{2}$$

Now let $\delta = \frac{1}{2}\left(C^{\frac{1}{1+d}} e\right)^{-1}$. Given $r > 0,\ d > 0$ we can find $s,\ D > 0$ such that

$$\|\sigma\|_r^{1+d} \leq D \, \|\sigma\|_s \, \|\sigma\|_1^d \tag{3}$$

for $\sigma \in H(\mathbb{C})$, where

$$\|\sigma\|_k = \sup\{|\sigma(z)| \ : \ |z| \leq k\}$$

Write the Taylor expansion of $g : V_1 \cap E(B) \longrightarrow H(\mathbb{C})$, the function associated to $\widehat{g} : (V_1 \cap E(B)) \times \mathbb{C} \longrightarrow \mathbb{C}$ at $0 \in E(B)$

$$g(x) = \sum_{n=0}^{\infty} P_n \, g(x)$$

where

$$P_n \, g(x)(\lambda) = \frac{1}{2\pi i} \int\limits_{|t|=1} \frac{\widehat{g}(tx, \lambda)}{t^{n+1}} \, dt$$

for $x \in V_1 \cap E(B),\ \lambda \in \mathbb{C}$.

Since $\widehat{h^s}$ is holomorphic at $0 \in E$ for every $s > 0$ we infer that $P_n \, g(\cdot)(\lambda)$ is continuous on E for every λ. Let $\widehat{P_n g}$ be the symmetric n-linear form associated with $P_n \, g$. We have

$$\sum_{n \geq 0} |P_n g(x)(\lambda)| \leq \sum_{n \geq 0} \sum_{j_1, \cdots, j_n \geq 1} \frac{|e_{j_1}^*(x)| \, \|e_{j_1}\|_q \cdots |e_{j_n}^*(x)| \, \|e_{j_n}\|_q}{\|e_{j_1}\|_q \cdots \|e_{j_n}\|_q} \times |\widehat{P_n g}(e_{j_1}, \cdots, e_{j_n})(\lambda)| \tag{4}$$

Using (2), (3) and (4) we get

$$\sum_{n\geq 0}|P_n g(x)(\lambda)| \leq \sum_{n\geq 0}\sum_{j_1,\cdots,j_n\geq 1}\frac{D^{\frac{1}{1+d}}C^{\frac{n}{1+d}}|e^*_{j_1}(x)|\,\|e_{j_1}\|_q\cdots|e^*_{j_n}(x)|\,\|e_{j_n}\|_q}{\|e_{j_1}\|_B^{\frac{1}{1+d}}\cdots\|e_{j_n}\|_B^{\frac{1}{1+d}}\|e_{j_1}\|_p^{\frac{d}{1+d}}\cdots\|e_{j_n}\|_p^{\frac{d}{1+d}}}$$

$$\times\|\widehat{P_n g}(e_{j_1},\cdots,e_{j_n})\|_s^{\frac{1}{1+d}}\|\widehat{P_n g}(e_{j_1},\cdots,e_{j_n})\|_1^{\frac{d}{1+d}}$$

$$\leq D^{\frac{1}{1+d}}\sum_{n\geq 0}C^{\frac{n}{1+d}}\frac{n^n}{n!}\|P_n g\|_{s,B}^{\frac{1}{1+d}}\|P_n g\|_{1,p}^{\frac{d}{1+d}}\|x\|_q^n$$

$$\leq D^{\frac{1}{1+d}}\|g\|_{B\times s\Delta}^{\frac{1}{1+d}}\|g\|_{U_p\times\Delta}^{\frac{d}{1+d}}\sum_{n=0}^{\infty}C^{\frac{n}{1+d}}\frac{n^n}{n!}\delta^n<+\infty$$

for $x\in\delta U_p$ and $|\lambda|<r$.

Thus g is extended holomorphically to $(\delta U_q\times \mathsf{C})\cup(V_1\times\overline{\Delta})$. By the same argument, as above, g is extended holomorphically to g_1 on $V_1\times\mathsf{C}$. Consider $\widehat{g}_1 : V_1\longrightarrow\dot{H}(\mathsf{C})$ associated with g_1. By the same above argument it follows that $\widehat{g}_1$ is locally bounded. Hence there exists a neighbourhood W of B in V_1 such that $\widehat{g}_1$ is bounded. Define a continuous linear map $S : [H^\infty(W)]^*\longrightarrow H(\mathsf{C})$ as

$$S(\mu)(\lambda)=\mu(\widehat{g}_1(\cdot,\lambda))$$

Since (1) holds for B_1 it holds for B. This shows that B is a set of uniqueness and we infer that span $\delta(B)$ is weakly dense in $[H(B)]^*_\beta$. Because $[H(B)]^*_\beta$ is reflexive span $\delta(B)$ is dense in $[H(B)]^*_\beta$, where $\delta : B\to[H(B)]^*_\beta$ is given by $\delta(x)(\varphi)=\varphi(x),\ x\in B,\ \varphi\in H(B)$. Now we have

$$T\Big(\sum_{j=1}^m\lambda_j\,\delta_{z_j}\Big)(\lambda)=\sum_{j=1}^m\lambda_j\,T(\delta_{z_j})(\lambda)=\sum_{j=1}^m\lambda_j f(z_j,\lambda)$$

$$=\sum_{j=1}^m\lambda_j\,\widehat{g}_1(z_j,\lambda)=\sum_{j=1}^m\lambda_j\,S(\delta_{z_j})(\lambda)=S\Big(\sum_{j=1}^m\lambda_j\,\delta_{z_j}\Big)(\lambda)$$

for $\lambda\in\mathsf{C}$.

Hence $S\big|_{[H(B)]^*_\beta}=T$ and $[H(B)]^*_\beta\in(\mathrm{LB}^\infty)$.

Proposition 3.4 is proved.

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