MATH. SCAND. 87 (2000), 251–271

APPROXIMATE FORMULAS FOR CANONICAL HOMOTOPY OPERATORS FOR THE $\bar\partial$ COMPLEX IN STRICTLY PSEUDOCONVEX DOMAINS

MATS ANDERSSON and JÖRGEN BOO

Abstract

Let $D = \{\rho < 0\}$ be a smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$ and ρ a strictly plurisubharmonic smooth defining function. We construct explicit homotopy operators for the $\bar\partial$ complex, which are approximately equal to the homotopy operators that are canonical with respect to the metric $\Omega = i\varphi(-\rho)\partial\bar\partial\log(1/-\rho)$ and weights $(-\rho)^\alpha$, where φ is a strictly positive smooth function. We also obtain an explicit operator which is approximately equal to the canonical homotopy operator for $\bar\partial_b$ on ∂D. From the explicit operators we obtain regularity results for these canonical operators, including C^∞ regularity and L^p-boundedness for the orthogonal projections onto Ker $\bar\partial$ and Ker $\bar\partial_b$. Previously it has been proved, in the ball case and $\varphi \equiv 1$, that the boundary values of the canonical operators coincide with the values of well-known explicit operators due to Henkin and Skoda et al. Previously Lieb and Range have constructed an explicit homotopy operator which is approximately equal to the canonical operator with respect to the metric $i\varphi\partial\bar\partial\rho$.

1. Introduction

Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary and let ρ be a strictly plurisubharmonic defining function. There are basically two ways to deal with the $\bar\partial$-equation in D; the L^2-methods due to Kohn, Hörmander et al from the 1960's, and the explicit methods developed by Grauert, Lieb, Henkin, Skoda and several others in the 1970's. For a long time these methods lived side by side without very much interaction.

Let K denote the Kohn operator; i.e., the L^2-bounded operator such that Kf is the L^2-minimal solution to $\bar\partial u = f$ if f is a $\bar\partial$-closed $(0,q)$-form and $Kf = 0$ if f is orthogonal to the kernel of $\bar\partial$ or if f is a function. In [12] Harvey and Polking found an explicit formula for K in the ball. In the case of a general strictly pseudoconvex domain D one cannot hope for explicit formulas for K, even in one variable. However, in [14] and [15] Lieb and Range constructed an explicit solution operator for the $\bar\partial$-equation which is approximately equal to the canonical operator K, provided that K is defined with respect to the metric $i\varphi\partial\bar\partial\rho$, where ρ is a smooth strictly plur-

Received December 15, 1997.

isubharmonic defining function and φ is a strictly positive smooth function on $\overline{D}$. By this explicit operator it is possible to obtain various regularity results for the abstractly defined operator K. However explicit, their formula was not as simple as the previously known explicit (weighted) solution operators.

It is well-known that in all natural estimates of solutions to $\bar{\partial}$ in D there is a certain difference in complex tangential and "complex normal" directions near the boundary. This phenomenon is reflected in the explicit expressions for the wellknown weighted solution operators. Therefore it is natural to expect that these operators should be related to a metric which takes this difference into account, such as the Bergman metric. In this paper we show that some special instances of the wellknown operators are indeed approximately equal to the abstractly defined operators that are canonical with respect to a metric Ω which is approximately the distance to the boundary times the Bergman metric.

More precisely, the metric in question is

$$\Omega = i(-\rho)\varphi\partial\bar{\partial}\log(1/-\rho),$$

where, as above, ρ is a smooth, strictly plurisubharmonic defining function and φ is a strictly positive smooth function on $\overline{D}$. In [2] we studied the spaces L^2_α, $\alpha > 0$, consisting of locally square integrable $(0,q)$-forms in D such that

$$\|f\|^2_\alpha = \frac{\Gamma(n+\alpha)}{2^n\pi^n\Gamma(\alpha)}\int_D(-\rho)^\alpha|f|^2dV$$

is finite, where the norm of f is taken with respect to the metric Ω and $dV = \Omega^n/n!$. To be precise, we only considered the case when $\varphi \equiv 1$; any choice of φ gives rise to the same space L^2_α but of course the norm will depend on φ. When f is a function, then $\|f\|_\alpha$ is equivalent to (and, for a suitable choice of φ, equal to) the usual L^2 norm with the weight $(-\rho)^{\alpha-1}$. Let K^{can}_α be the operator on L^2_α defined by letting $K^{\mathrm{can}}_\alpha f$ be the minimal solution in L^2_α to $\bar{\partial}u = f$ if $\bar{\partial}f = 0$ and 0 if f is orthogonal to $\mathcal{K}_\alpha = L^2_\alpha \cap \mathrm{Ker}\,\bar{\partial}$ or if f is a function.

We proved in [2] that when D is the ball in $\mathbf{C}^n$, $\varphi \equiv 1$ and $\rho = |z|^2 - 1$, then the boundary values of $K^{\mathrm{can}}_\alpha f$ coincide with the values given by the wellknown explicit operators found by Henkin, Skoda et al, thus giving a geometrical interpretation of these formulas. Moreover, we were able to compute the values of $K^{\mathrm{can}}_\alpha f$ even in the interior. As a corollary we proved that the orthogonal decomposition (1.2) below preserves regularity. The objective of this paper is to generalize to the strictly pseuodoconvex case. As was noted above, the best we should look for, are explicit operators that are approximately equal to the canonical ones. It is easy to see that

$$\bar{\partial} K_\alpha^{\mathrm{can}} f + K_\alpha^{\mathrm{can}} \bar{\partial} f = f - P_\alpha^{\mathrm{can}} f, \tag{1.1}$$

for any $f \in \mathrm{Dom}\ \bar{\partial}$, where P_α^{can} is the orthogonal projection of functions in L^2_α onto $A_\alpha = L^2_\alpha \cap \mathcal{O}(D)$. For any $f \in L^2_\alpha$, $K_\alpha^{\mathrm{can}} f$ is in Dom $\bar{\partial}$ and $\bar{\partial} K_\alpha^{\mathrm{can}}$ is the orthogonal projection $L^2_\alpha \to \mathscr{K}_\alpha$. Moreover, for any $f \in L^2_\alpha$, $K_\alpha^{\mathrm{can},*} f$ is in Dom $\bar{\partial}_\alpha^*$ and

$$\bar{\partial} K_\alpha^{\mathrm{can}} f + \bar{\partial}_\alpha^* K_\alpha^{\mathrm{can},*} f = f - P_\alpha^{\mathrm{can}} f, \tag{1.2}$$

see Proposition 3.7 in [2] (for the case $\varphi \equiv 1$, the general case follows in the same way).

Our main results are Theorem 1.1 and the corresponding result for the boundary complex, Theorem 1.2.

THEOREM 1.1. *For each $\alpha \geq 1$ there are explicit bounded operators operators K_α and P_α on L^2_α such that for any choice of φ, we have the following:*

(i) *K_α is even compact, and*

$$\bar{\partial} K_\alpha f + K_\alpha \bar{\partial} f = f - P_\alpha f, \quad f \in \mathrm{Dom}\ \bar{\partial}. \tag{1.3}$$

(ii) *There are compact operators H_α and R_α such that $\bar{\partial} R_\alpha$ is compact as well, H_α is self-adjoint and*

$$K_\alpha f = H_\alpha \bar{\partial}_\alpha^* f + R_\alpha f \quad \text{if} \quad f, \bar{\partial}_\alpha^* f \in L^2_\alpha. \tag{1.4}$$

(iii) *The operator $\bar{\partial} K_\alpha$ is a projection $L^2_\alpha \to \mathscr{K}_\alpha$.*

(iv) *The operators $P_\alpha - P_\alpha^*$ and $\bar{\partial} K_\alpha - (\bar{\partial} K_\alpha)^*$ are compact on L^2_α.*

(v) *All these operators, as well as their adjoints, map smooth forms onto smooth forms.*

Notice that the definition of adjoint operator depends on the choice of ρ and φ. The equation (1.3) in particular means that P_α is a projection $L^2_\alpha \to A_\alpha$, and that $K_\alpha f$ is a solution to $\bar{\partial} u = f$ if $\bar{\partial} f = 0$. The point of part (iii) is that $\bar{\partial} K$ extends to a bounded operator on all of L^2_α. Part (iv) means that P_α and $\bar{\partial} K_\alpha$ are approximately self-adjoint and therefore approximately the orthogonal projections. It also follows that K_α is the principal term of K_α^{can} in a certain way; for the precise statements, see Section 4.

If the defining function ρ is real-analytic and $v(\zeta, z)$ is the unique function near the diagonal that is holomorphic in z, anti-holomorphic in ζ, and such that $v(\zeta, \zeta) = -\rho(\zeta)$, then the principal term of the kernel for the boundary values of our operator K_α has the simple expression

$$\sum_{q=0}^{n-1} c_q \frac{\partial_\zeta \bar{v} \wedge (\bar{\partial}_z \partial_\zeta \bar{v})^q}{v^{\alpha-1+n-q} \bar{v}^{q+1}} \varphi(\zeta)^{q+1-n}.$$

Let us now assume that $n > 1$ and consider the boundary complex induced by $\bar\partial_b$. If f is a $(0,q)$-form on $\overline{D}$, we let $f|_b$ denote its complex tangential boundary values. On the boundary, $|f|$ is a norm of $f|_b$, cf. formula (2.5) below. Let L^2_b denote the space of complex tangential forms such that $\int_{\partial D} |f|^2 d\sigma$ is finite. Here $d\sigma$ is a certain surface measure that depends on Ω; for the precise definition, see Section 2. In particular, if φ is chosen appropriately, $d\sigma$ is the euclidean surface measure on ∂D.

A $(0,q)$-form $f \in L^2_b$ is in the image of $\bar\partial_b$ if and only if $f \in (\text{Ker } \bar\partial_b^*)^\perp$. If $q \le n-2$ this is equivalent to that $\bar\partial_b f = 0$. Therefore we can define the canonical operator K_b^{can} on L^2_b by requiring that $K_b^{\text{can}} f$ be the minimal solution to $\bar\partial_b u = f$ if $f \in (\text{Ker } \bar\partial_b^*)^\perp$, and $K_b^{\text{can}} f = 0$ if f is a function or a form that is orthogonal to Ker $\bar\partial_b$. Analogously to (1.1) we have

$$\bar\partial_b K_b^{\text{can}} f + K_b^{\text{can}} \bar\partial_b f = f - P_b^{\text{can}} f - S_b^{\text{can}} f, \tag{1.5}$$

for any $f \in \text{Dom } \bar\partial_b$, where P_b^{can} is the orthogonal projection of functions in L^2_b onto $A_b = L^2_b \cap \mathcal{O}(D)$ and S_b^{can} is the orthogonal projection $L^2_{b,n-1} \to$ Ker $\bar\partial_b^*$ ($L^2_{b,q}$ denotes the subspace of $(0,q)$-forms). For any $f \in L^2_b$, $K_b^{\text{can}} f$ is in Dom $\bar\partial_b$ and $\bar\partial_b K_b^{\text{can}}$ is the orthogonal projection $L^2_b \to \mathcal{K}_b = L^2_b \cap \text{Ker } \bar\partial_b$. Moreover, see Proposition 3.7 in [2], for any $f \in L^2_b$, $K_b^{\text{can},*} f$ is in Dom $\bar\partial_b^*$ and

$$\bar\partial_b K_b^{\text{can}} f + \bar\partial_b^* K_b^{\text{can},*} f = f - P_b^{\text{can}} f - S_b^{\text{can}} f. \tag{1.6}$$

We have the following analogous statement to Theorem 1.1.

THEOREM 1.2. *There are explicit bounded operators K_b, P_b and S_b on L^2_b such that the following hold (for any choice of φ):*

(i) *The operator K_b is compact and*

$$\bar\partial_b K_b f + K_b \bar\partial_b f = f - P_b f - S_b f, \quad f \in \text{Dom } \bar\partial_b. \tag{1.7}$$

(ii) *There are compact operators H_b and R_b such that $\bar\partial_b R_b$ is compact as well, H_b is self-adjoint and*

$$K_b f = H_b \bar\partial_b^* f + R_b f \quad \text{if} \quad f, \bar\partial_b^* f \in L^2_b. \tag{1.8}$$

(iii) *The operator $\bar\partial_b K_b$ is a projection $L^2_b \to (\text{Ker } \bar\partial_b^*)^\perp$.*

(iv) *The operators $P_b - P_b^*$ and $\bar\partial_b K_b - (\bar\partial_b K_b)^*$ are compact on L^2_b.*

(v) *All the operators, as well as their adjoints, map smooth forms onto smooth forms.*

Part (i) means that P_b is a projection $L^2_{b,0} \to A_b$, $I - S_b$ is a projection $L^2_{b,n-1} \to (\text{Ker } \bar\partial_b^*)^\perp$, and that $K_b f$ is a solution to $\bar\partial_b u = f$ if $\bar\partial_b f = 0$, ($f \perp$ Ker $\bar\partial_b^*$ if f is a $(0,n-1)$ form).

As for the operators in the interior, (iv) means that P_b and K_b are approximately P_b^{can} and K_b^{can}. However, the explicit operator S_b corresponding to S_b^{can} will approximate the latter one in the sense that $I - S_b$ is a projection onto $\left(\mathrm{Ker}\ \bar\partial_b^*\right)^{\perp}$ and $S_b^{\mathrm{can}} - S_b$ is compact. That is, the image of S_b will not be exactly Ker $\bar\partial_b^*$.

All occurrences of ‘‘compact’’ in Theorems 1.1 and 1.2 can be replaced by ‘‘somewhat regularizing’’, and then, following Kerzman and Stein [13], in principle any question of regularity, such as e.g. L^p and Hölder estimates, for the canonical operators K_α^{can} and P_α^{can} etc, can be reduced to the same question for the corresponding approximate explicit operator, see Section 4. For example we have

THEOREM 1.3. *Let $\alpha \geq 1$ or $\alpha = b$. The canonical operators P_α^{can}, $K_\alpha^{\mathrm{can},*}$ and K_α^{can} (and S_b^{can}) preserve C^∞-regularity, and in particular the orthogonal decompositions* (1.2) *and* (1.6) *preserve regularity. Moreover, they preserve L_α^p-boundedness, $1 < p < \infty$, as well.*

The C^∞ regularity result for the boundary complex is well-known, see [10], but the L^p-result has possibly not occurred before. Anyway, a variety of other regularity questions can be handled in this way.

An explicit operator that approximates the orthogonal projection on the boundary, the Szegö projection, in the same way, was found by Kerzman and Stein [13], and following the same lines Ligocka [16] found an explicit P_α as above (for $\alpha = 1$). The first step in [13] is to prove that the projection operator P_b, obtained from the Cauchy-Fantappie-Leray formula with a holomorphic support function, which a priori is just defined on say smooth functions, actually extends to a bounded operator on L_b^2. A nice proof of this fact can be found in [17]. The next step is to show that the support function can be chosen in such a way that $P_b - P_b^*$ is compact. Our construction follows the same lines. In Section 2 we construct explicit operators K_α^b, mapping forms in D onto complex tangential forms, and a holomorphic projection P_α such that $\bar\partial_b K_\alpha^b f + K_\alpha^b \bar\partial f = f|_b - (P_\alpha f)|_b$. Letting $\alpha \to 0$ we then obtain the desired operators in Theorem 1.2. To obtain Theorem 1.1 we extend these operators to the interior by a technique from [2] of representing forms in D by the boundary values of forms corresponding operators in a domain $\tilde D$ in $\mathbb{C}^{n+1}$. This is performed in Section 3. It is worth to point out that the kernel for operator $\bar\partial K_\alpha$ is truly singular so the L_α^2 boundedness is nontrivial and cannot be obtained by a brutal estimate. Finally in Section 4 we show how one can use our kernels to derive regularity results for the canonical operators; in particular we prove Theorem 1.3.

Throughout this paper D is assumed to be of class C^∞ but all results, with appropriate modifications of the formulations, hold if the boundary is just C^4.

ACKNOWLEDGEMENT. We are grateful to the referee for some important remarks and suggestions that have served to improve the presentation. We are also grateful to J. Bruna and M. Passare for several comments on the preprint [1], on which this paper is based.

2. Construction of the boundary values of the kernels

Define the function $v(\zeta, z)$ near the diagonal by

$$-v(\zeta, z) = \rho + \sum \rho_j(z_j - \zeta_j) + \tfrac{1}{2}\sum \rho_{jk}(z_j - \zeta_j)(z_k - \zeta_k), \tag{2.1}$$

where $\rho = \rho(\zeta), \rho_j = \partial\rho/\partial\zeta_j$ and so on. Then certainly $v(\zeta, z)$ is holomorphic in z near Δ and

$$v(z, \zeta) = \overline{v(\zeta, z)} + \mathcal{O}(|\zeta - z|^3) \quad \text{and} \quad \partial_\zeta v(\zeta, z) = \mathcal{O}(|\zeta - z|^2). \tag{2.2}$$

The relation (2.2) says that $v(\zeta, z)$ is approximately hermitean; it appeared first in in [13]. It can be verified by a direct computation but we prefer the more elucidatory argument that will be given below.

If ρ is real analytic we can take, and in fact we will take,

$$-v(\zeta, z) = \sum_{|\alpha|=0}^{\infty} \rho_\alpha(z - \zeta)^\alpha \tag{2.3}$$

near Δ. This is the polarization of $\rho(\zeta)$, i.e., the unique function $v(\zeta, z)$ that is holomorphic in z, satisfies $v(\zeta, \zeta) = -\rho(\zeta)$ and $v(z, \zeta) = \overline{v(\zeta, z)}$. If ρ is real analytic and v anyway is defined by (2.1) it is now clear that (2.2) holds. An arbitrary function ρ (of class C^3) can be approximated by real analytic functions in C^3-norm, and therefore (2.2) holds in general.

REMARK 1. By the same argument it follows that if we add terms up to order k in the definition (2.1), then (2.2) will hold with $k + 1$ and k instead of 3 and 2.

In what follows it is convenient to think of ρ as real analytic and $v(\zeta, z)$ as being its polarization, even though the property (2.2) is enough. Moreover, since ρ is strictly plurisubharmonic we have that

$$2 \operatorname{Re} v(\zeta, z) \geq -\rho(\zeta) - \rho(z) + \delta|\zeta - z|^2 \tag{2.4}$$

near the diagonal. We define $v(\zeta, z)$ globally by patching essentially with $|\zeta - z|^2$ (to be precise, with $|\zeta - z|^2 - \rho(\zeta)$, see the proof of Theorem 2.1), so that (2.4) holds globally. In particular, therefore $v(\zeta, z)^\alpha$ is well defined for all $\alpha > 0$.

For forms f and g, we let $\langle f, g\rangle$ denote the inner product generated by the metric form Ω and we let

$$(f, g)_\alpha = \frac{\Gamma(n+\alpha)}{2^n \pi^n \Gamma(\alpha)} \int_D (-\rho)^\alpha \langle f, g\rangle dV.$$

If $\langle f, g\rangle_\beta$ denotes the inner product with respect to $\beta = i\partial\bar{\partial}\rho$, for $(0, q)$-forms f and g we have that, cf., Lemma 2.1 in [2],

$$\text{(2.5)} \quad \langle f, g\rangle = \Big((-\rho)\langle f, g\rangle_\beta + \big\langle \bar{\partial}\rho \wedge f, \bar{\partial}\rho \wedge g\big\rangle_\beta\Big)\varphi^{-q}/B \ \text{ and } \ (-\rho)\Omega_n = \varphi^n B \beta_n,$$

where $B = -\rho + |\bar{\partial}\rho|_\beta^2$ and hence smooth up to the boundary and non-vanishing. Note that β is equivalent to the Euclidean metric since ρ is strictly plurisubharmonic. Hence dV is equivalent to the Lebesgue measure divided by the distance to the boundary. For a fixed $\alpha > 0$ we let an integral operator H_α and a kernel $h_\alpha(\zeta, z)$ be connected by the relation $H_\alpha f(z) = (f, \overline{h_\alpha(\cdot, z)})_\alpha$.

Let $\mathscr{E}_q$ denote the space of $(0, q)$-forms that are smooth up to the boundary and let $\mathscr{E}_q^b$ denote the space of smooth complex tangential $(0, q)$-forms. We also let $\mathscr{H}_q = \mathscr{E}_q \cap \operatorname{Ker} \bar{\partial}$ and $\mathscr{H}_q^b = \mathscr{E}_q^b \cap \operatorname{Ker} \bar{\partial}_b$. Thus $\mathscr{H}_0$ is the space of holomorphic functions that are smooth up to the boundary.

Theorem 2.1. *For each $\alpha > 0$ we have an operator $K_\alpha^b : \mathscr{E}_{*+1} \to \mathscr{E}_*^b$ and a projection $P_\alpha : \mathscr{E}_0 \to \mathscr{H}_0$ such that*

$$\text{(2.6)} \qquad \bar{\partial}_b K_\alpha^b + K_\alpha^b \bar{\partial} = I - P_\alpha|_b,$$

given by explicit kernels $k_\alpha^b(\zeta, z)$ and $p_\alpha(\zeta, z)$ that satisfy

$$\text{(2.7)} \quad k_\alpha^b(\zeta, z) = \sum_{q=0}^{n-1} c_{\alpha,n,q} \varphi(\zeta)^{q+1-n} \frac{\partial_\zeta \bar{v}(\zeta, z) \wedge (\bar{\partial}_z \partial_\zeta \bar{v}(\zeta, z))^q}{v(\zeta, z)^{n+\alpha-q-1} (\zeta, z)^{q+1}} + r_\alpha^b(\zeta, z),$$

$$c_{\alpha,n,q} = i^{1-q} \frac{\Gamma(\alpha + n - q - 1)}{\Gamma(n+\alpha)},$$

$$\text{(2.8)} \qquad |k_\alpha^b(\zeta, z)| \le C \frac{1}{|v|^{n+\alpha-1/2}}, \quad |r_\alpha^b(\zeta, z)| \le C \frac{1}{|v|^{n+\alpha-1}},$$

$$|\bar{\partial}_z r_\alpha^b(\zeta, z)| \le C \frac{1}{|v|^{n+\alpha-1/2}},$$

and

$$p_\alpha(\zeta,z)=\frac{\varphi(\zeta)^{-n}}{v(\zeta,z)^{n+\alpha}}+\varrho_\alpha(\zeta,z),\ \text{where} \tag{2.9}$$

$$|\varrho_\alpha(\zeta,z)|\le C\frac{1}{|v|^{n+\alpha-1/2}},\quad \zeta,z\in\overline{D}.$$

These kernels are essentially wellknown. The crucial point is the special form of the leading term of the kernel $k^b_\alpha(\zeta,z)$; as we will se later on it is (modulo lower order terms) ∂_ζ of a hermitean kernel.

We now turn our attention to the boundary complex. First notice that

$$\alpha(-\rho)^\alpha dV\to d\sigma:=\varphi^n\left|\bar\partial\rho\right|_\beta dS/2 \tag{2.10}$$

when $\alpha\to 0$, where dS is the surface measure induced by the metric β. Therefore it is natural to define the inner product

$$(f,g)_b=\frac{(n-1)!}{2^n\pi^n}\int_{\partial D}\langle f,g\rangle d\sigma \tag{2.11}$$

for complex tangential $(0,q)$-forms f and g. An operator H_b is then connected to the kernel $h_b(\zeta,z)$ by the relation $H_bf(z)=(f,\overline{h_b(\cdot,z)})_b$.

Remark 2. Unfortunately there is an error in the definition of $d\sigma$ on p. 250 in [2]. The definition should be as in (2.10) above.

Theorem 2.2. *There is an operator* $K_b\colon\mathscr{E}^b_{*+1}\to\mathscr{E}^b_*$ *and projections* $P_b\colon\mathscr{E}^b_0\to\mathscr{H}^b_0$ *and* $I-S_b\colon\mathscr{E}^b_{n-1}\to\left(\mathrm{Ker}\ \bar\partial^*_b\right)^\perp$ *such that*

$$\bar\partial_bK_b+K_b\bar\partial_b=I-P_b-S_b, \tag{2.12}$$

given by explicit kernels that satisfy

$$k_b(\zeta,z)=\sum_{q=0}^{n-2}i^{1-q}\frac{(n-q-2)!}{(n-1)!}\varphi(\zeta)^{q+1-n}\frac{\partial_\zeta\bar v(\zeta,z)\wedge(\bar\partial_z\partial_\zeta\bar v(\zeta,z))^q}{v(\zeta,z)^{n-q-1}\bar v(\zeta,z)^{q+1}}+r_b(\zeta,z),$$

where

$$|k_b(\zeta,z)|\le C\frac{1}{|v|^{n-1/2}},\quad |r_b(\zeta,z)|\le C\frac{1}{|v|^{n-1}},\quad |\bar\partial_{z,b}r_b(\zeta,z)|\le C\frac{1}{|v|^{n-1/2}} \tag{2.13}$$

and

$$p_b(\zeta,z)=\frac{\varphi(\zeta)^{-n}}{v(\zeta,z)^n}+\varrho_b(\zeta,z),\quad s_b(\zeta,z)=\frac{\left(\partial_\zeta\bar\partial_z\bar v\right)^{n-1}}{\bar v(\zeta,z)^n}+\varrho'_b(\zeta,z), \tag{2.14}$$

$$|\varrho'_b(\zeta,z)|\le C\frac{1}{|v|^{n-1/2}},\quad |\varrho_b(\zeta,z)|\le C\frac{1}{|v|^{n-1/2}},\quad \zeta\in\partial D,z\in D.$$

Both $P_b f$ and $S_b f$ are defined by first evaluating for $z \in D$ and then taking the boundary values.

PROOF OF THEOREM 2.1. Let $\alpha > 0$ be fixed. To begin with we let $\eta_j = z_j - \zeta_j$ and let $\chi = \chi(|\eta|)$ be a smooth function supported and identically 1 near Δ and set

$$q_j(\zeta, z) = \chi\left(\rho_j + \tfrac{1}{2}\sum_k \rho_{jk}\eta_k\right) - (1-\chi)\bar{\eta}_j$$

(or possibly with some more terms if ρ is real analytic). Then we define v globally by $-v(\zeta, z) = q(\zeta, z) \cdot \eta + \rho(\zeta)$, and if we let $s(\zeta, z) = -q(z, \zeta)$ we also get, cf. (2.2),

$$-s(\zeta, z) \cdot \eta - \rho(z) = v(z, \zeta) = \overline{v(\zeta, z)} + \mathscr{O}(|\eta|^3). \tag{2.15}$$

Using the notation $s \sim \sum s_j d\zeta_j$ and $q \sim \sum q_j d\zeta_j$, we define the operators

(2.16) $\hat{K}f(z) =$

$$\int_D \sum_{k=0}^{n-1} c'_{\alpha,n,k} \frac{(-\rho)^{\alpha-1} f \wedge s \wedge (\bar{\partial}s)^k \left(-\rho(\bar{\partial}q)^{\bar{n}-k-1} - (n-k-1)q \wedge \bar{\partial}\rho \wedge (\bar{\partial}q)^{n-k-2}\right)}{(-q \cdot \eta - \rho)^{\alpha+n-k-1}(-s \cdot \eta)^{k+1}}$$

for $z \in \partial D$, where $c'_{\alpha,n,k} = (i/2\pi)^n \Gamma(\alpha+n-k-1)/(n-k-1)!\Gamma(\alpha)$, and

$$\hat{P}f(z) = c_{\alpha,n,-1} \int_D \frac{(-\rho)^{\alpha-1} f(-\rho(\bar{\partial}q)^n - n\bar{\partial}\rho \wedge q \wedge (\bar{\partial}q)^{n-1})}{(-q \cdot \eta - \rho)^{n+\alpha}}, \quad z \in D.$$

If f is smooth, then $\hat{K}f$ and $\hat{P}f$ are smooth and, see [7], the relation

$$\bar{\partial}_b \hat{K}f + \hat{K}\bar{\partial}f = f|_b - \hat{P}f|_b. \tag{2.17}$$

holds. Let $\hat{K}_q$ and $\hat{P}_q$ denote the components that are $(0, q)$ in dz. Since $q(\zeta, z)$ is holomorphic in z near the diagonal, all components $\hat{P}_q$ but $\hat{P}_0$ are smooth since the singularities are cancelled. Notice that the leading term in $\hat{K}_q$ is the one corresponding to $k = q$ in the sum, since in all the other ones the singularity is cancelled; the leading term is in fact the desired one, but since $\hat{P}_q$ does not vanish identically for $q > 0$, the operator $\hat{K}_q f$ is just (the boundary values of) an approximate solution if $\bar{\partial}f = 0$. To get rid of this flaw, let $\mathscr{L}$ be a C^∞ homotopy for $\bar{\partial}$ and $\mathscr{Q}$ the corresponding holomorphic projection, i.e. both of them map $\mathscr{E}_*$ into itself ($\mathscr{L}$ decreasing the degree one unit) and

$$\bar{\partial}\mathscr{L} + \mathscr{L}\bar{\partial} = I - \mathscr{Q}. \tag{2.18}$$

Such a homotopy can e.g. be obtained by the formula above, choosing s and

q in an appropriate way in $\overline{D \times D}$ so that q is holomorphic in z, see e.g. [3] for details.

Remark 3. It is proved in [21] that one actually can choose a global support function $v(\zeta, z)$ that is holomorphic in z and satisfies (2.2).

So far we have only used $s(\zeta, z)$ for z on the boundary. One can make an extension inwards so that the relation (2.17) holds in D; see [7]. Applying $\bar\partial$ both from the left and from the right of the equality (2.17) we get the additional relation

$$\bar\partial \hat P = \hat P \bar\partial. \tag{2.19}$$

Let us now define $K^b = \hat K + \mathscr{L}\hat P$ and $P = \mathscr{Q}\hat P$. It is readily verified that $(\mathscr{L}\hat P)_q$ takes smooth forms to smooth forms for all $q \ge 0$, and therefore $K^b = \hat K +$ smooth operator. Moreover, $\mathscr{Q}\hat P = \hat P_0 - \mathscr{L}\bar\partial\hat P_0$, where $\hat P_0$ defined as above just is the component acting on $(0,0)$-forms. Since the kernel for $\hat P_0$ is holomorphic near the diagonal, $\bar\partial\hat P_0$ has no singularity, and hence $P = \hat P +$ smooth operator. From (2.17) and (2.19) we get

$$\bar\partial_b K^b + K^b\bar\partial = I - P|_b, \tag{2.20}$$

where P is holomorphic and only acts on $(0,0)$-forms, and so K^b is a true homotopy operator for $\bar\partial$ whose leading term is $\hat K$, and $\hat P_0$ is the leading term of the corresponding projection. We are now going to rewrite the leading terms and to this end we need

Proposition 2.3. *If ρ is C^3 and v, s, q are defined as above, then*

$$\partial_\zeta \bar v = -s + \mathcal{O}(|\eta|) = q + \mathcal{O}(|\eta|) = -\partial\rho + \mathcal{O}(|\eta|), \tag{2.21}$$

$$s \wedge q = \mathcal{O}(|\eta|), \quad \partial\rho \wedge \partial_\zeta \bar v = \mathcal{O}(|\eta|), \tag{2.22}$$

$$\bar\partial_\zeta q = \partial\bar\partial\rho + \mathcal{O}(|\eta|), \tag{2.23}$$

$$\partial\rho \wedge \partial_\zeta \bar v = s \wedge q + \mathcal{O}(|\eta|^2), \tag{2.24}$$

and

$$\bar\partial_z s - \bar\partial_z \partial_\zeta \bar v = \mathcal{O}(|\eta|). \tag{2.25}$$

The estimates in the lemma hold for the Euclidean metric and hence they hold for our metric as well, since $|f| \lesssim |f|_\beta$ for all forms, cf. (2.5); in fact we have strict inequality if and only if f contains a factor $\bar\partial\rho$ or $\partial\rho$. The crucial part is (2.24) which first occurred in [5]. The other ones are more or less direct consequences of the definitions. All of them but (2.25) can be found in [3] so let us restrict ourselves to this one.

PROOF OF (2.25). From (2.15) we get

$$\partial_\zeta \overline{v(\zeta,z)} + \mathcal{O}(|\eta|^2) = \sum s_j d\zeta_j - \sum (\partial_\zeta s_j)(z_j - \zeta_j) = s - \sum (\partial_\zeta s_j)(z_j - \zeta_j),$$

and so

$$\bar\partial_z \partial_\zeta \overline{v(\zeta,z)} + \mathcal{O}(|\eta|) = \bar\partial_z s - \sum (\bar\partial_z \partial_\zeta s_j)(z_j - \zeta_j) + \mathcal{O}(|\eta|) = \bar\partial_z s + \mathcal{O}(|\eta|).$$

We also need

LEMMA 2.4. *Suppose that the kernels $K(\zeta,z)$ and $k(\zeta,z)$ are connected by the relation $\int_D f \wedge K = \int_D \langle f, \bar k\rangle dV$ (f being a $(0,q)$ form and thus K being $(n, n-q)$ in ζ). Then*

$$K = c_q \bar k \wedge \Omega_{n-q} = c_q \bar k \wedge \varphi^{n-q}(-\rho\beta_{n-q} + \gamma \wedge \beta_{n-q-1})/(-\rho),$$

$$k = \pm * K \text{ and } |K| = |k|,$$

where $*$ is the Hodge star with respect to Ω, and $c_q = 1$ if q is even and $c_q = -i$ if q is odd.

PROOF. For any forms f and g we have that $\langle f, g\rangle dV = f \wedge *g$. Moreover, if they are $(0,q)$ forms, then $\langle f, g\rangle \Omega_n = c_q f \wedge \bar g \wedge \Omega_{n-q}$. Now, the lemma follows since $** = \pm 1$ and $*$ is an isometry.

Notice that (2.5) and (2.21) imply that $|\partial\rho| \sim \sqrt{-\rho}$ and $|\partial_\zeta \overline{v(\zeta,z)}| \sim \sqrt{-\rho} + |\zeta - z|$. If $\hat K$ is the kernel such that $\hat K f = \int (-\rho)^\alpha \hat K \wedge f$, then cf. (2.16), $|\hat K| \lesssim |v|^{-(n+\alpha-1/2)}$. By repeated use of Proposition 2.3 one can verify that

$$\hat K = \sum_{q=0}^{n-1} c_{\alpha,n,q} \frac{\partial_\zeta \bar v \wedge (\bar\partial_z \partial_\zeta \bar v)^q}{v^{\alpha+n-1-q}\bar v^{q+1}} \wedge (-\rho\beta + (n-q-1)\gamma) \wedge \beta^{n-q-2}/(-\rho) + R$$

where $|R| \lesssim |v|^{-(n+\alpha-1)}$. In view of Lemma 2.4 this proves (2.7) and the first two estimates in (2.8). The third estimate in (2.8) follows in the same way, just noting that the operator $\bar\partial$ at most increases the singularity half a unit. In the same way one can rewrite the expression for $\hat P f$ and obtain the stated properties of Pf. Thus Theorem 2.1 is proved.

PROOF OF THEOREM 2.2. The proof is performed along the same lines as the previous one. Let $\hat K_b f$ be the limit, when $\alpha \to 0$, of the $n-2$ first terms in the expression for $\hat K f$ above. Then