MATH. SCAND. 87 (2000), 272–286

BOCHNER'S THEOREM FOR SEMIGROUPS: A COUNTEREXAMPLE

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1. Introduction

Weil [10], Povzner [5], and Raikov [6] proved (almost simultaneously) that if G is a locally compact abelian group, a function $\varphi\colon G \to \mathsf{C}$ admits an integral representation of the form

$$\varphi(s) = \int_{\widehat{G}} \sigma(s)\, d\mu(\sigma), \qquad s \in G$$

for some bounded measure μ on the dual group $\widehat{G}$ if and only if φ is continuous and positive definite in the sense that

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_j - s_k) \geq 0$$

for every choice of $s_1, \dots, s_n \in G$ and $c_1, \dots, c_n \in \mathsf{C}$.

So far, no generalization to semigroups is known which is of the same simplicity and generality. However, a comprehensive theory of integral representations of positive definite functions on semigroups exists. For this subject, see Berg, Christensen, and Ressel [2] and Berg [1].

Suppose $(S, +, *)$ is an abelian semigroup with zero and involution. A function $\varphi\colon S \to \mathsf{C}$ is *positive definite* if

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_j + s_k^*) \geq 0$$

for every choice of $s_1, \dots, s_n \in S$ and $c_1, \dots, c_n \in \mathsf{C}$. Denote by $\mathscr{P}(S)$ the set of all positive definite functions on S, and by $\mathscr{P}_1(S)$ the subset of those $\varphi \in \mathscr{P}(S)$ such that $\varphi(0) = 1$.

A function $\sigma\colon S \to \mathsf{C}$ is a *character* if $\sigma(0) = 1$, $\sigma(s^*) = \overline{\sigma(s)}$, and $\sigma(s + t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. The set $\widehat{S}$ of all bounded characters on S is a compact

Received April 1, 1997; in revised form March 30, 1998.

involution semigroup when considered with pointwise multiplication and complex conjugation and the topology of pointwise convergence. The semigroup $\widehat{S}$ has the unit 1 (the constant character), and $|\sigma(s)| \leq 1$ for all $\sigma \in \widehat{S}$ and $s \in S$.

Denote by $\mathscr{K}(\widehat{S})$ the set of all compact subsets of $\widehat{S}$. A measure μ defined on the Borel σ-field $\mathscr{B}(\widehat{S})$ is a *Radon measure* if $\mu(B) = \sup\{\, \mu(C) \mid \mathscr{K}(\widehat{S}) \ni C \subset B \}$ for each $B \in \mathscr{B}(\widehat{S})$. Denote by $M_+(\widehat{S})$ the set of all Radon measures on $\widehat{S}$, and by $M_+^1(\widehat{S})$ the subset of probability measures. For $\mu \in M_+(\widehat{S})$, define $\widehat{\mu}\colon S \to \mathsf{C}$ by

$$\widehat{\mu}(s) = \int_{\widehat{S}} \sigma(s)\, d\mu(\sigma), \qquad s \in S.$$

By the Lindahl-Maserick Theorem [4], a function $\varphi\colon S \to \mathsf{C}$ is bounded and positive definite if and only if $\varphi = \widehat{\mu}$ for some $\mu \in M_+(\widehat{S})$, and when the condition holds, μ is unique.

Now suppose S is equipped with a topology rendering addition and involution continuous. It might be hoped that if $\mu \in M_+(\widehat{S})$ were such that $\widehat{\mu}$ were continuous then μ-almost all characters would be continuous.

Certainly there are some topological $*$-semigroups, which are not groups, but for which the analogue of Bochner's theorem holds in the "continuity everywhere" version. Thus Ross [9] obtained a Bochner-type theorem for the duals of totally ordered semilattices which are locally compact in the order topology. Also, for $S = (\mathsf{R}_+^k, +, s^* = s)$, a Bochner-type theorem holds ([2], 4.4.7).

Suppose X is a locally compact Hausdorff space and $(\mathscr{K}, \cup)$ is the semilattice of compact subsets of X. A function $\varphi\colon \mathscr{K} \to \mathsf{R}$ is *continuous on the right* if for each $K \in \mathscr{K}$ and each $\varepsilon > 0$ there is an open neighbourhood G of K such that $|\varphi(K) - \varphi(L)| < \varepsilon$ for each $L \in \mathscr{K}$ satisfying $K \subset L \subset G$. The functions on $\mathscr{K}$ continuous on the right are the functions continuous for a certain topology on $\mathscr{K}$. As shown by Choquet [3] (cf. [2], 4.6.15 ff.), if $\mu \in M_+(\widehat{\mathscr{K}})$ is such that $\widehat{\mu}$ is continuous on the right then μ-almost all characters are continuous on the right.

Suppose S is a subsemigroup of R^k, containing 0, which is *conelike* in the sense that for each $s \in S$ there is some $a \in \mathsf{R}_+$ such that $\alpha s \in S$ for all $\alpha \geq a$. Ressel [7] proved that if either S carries the identical involution and 0 is in the closure of the set of those $s \in S$ such that $\{\, \alpha s \mid \alpha \in \mathsf{R}_+ \}$ intersects the interior of S, or S carries an involution inherited from R^k and 0 is in the closure of the interior of S, then Bochner's theorem holds in the "continuity everywhere" version. Moreover, in each case, continuity at 0 of a bounded positive definite function on S implies continuity everywhere.

Berg, Christensen, and Ressel ([2], p. 143) pointed out that on the compact semilattice $([0,1],\vee)$ with the usual topology, the only continuous character is the constant 1. On this semigroup, the function $s \mapsto 1-s$ is a bounded continuous positive definite function, the representing measure of which is concentrated on the set of discontinuous characters.

Call a function $\varphi: S \to \mathsf{C}$ 0-*continuous* if φ is continuous at 0. Berg, Christensen, and Ressel ([2], p. 143) suggested that the right dual object of a locally compact Hausdorff $*$-semigroup might be the set S' of bounded 0-continuous characters. By "Bochner's Theorem" for S we shall understand the statement (which may be true or false) that if $\mu \in M_+(\widehat{S})$ is such that $\widehat{\mu}$ is 0-continuous then μ-almost all characters are 0-continuous.

We note that Bochner's Theorem holds for S if S is one of the conelike subsemigroups of R^k for which Ressel proved a "continuity everywhere" version of Bochner's theorem (see above).

Suppose S is a topological abelian semigroup with zero and the identical involution. Call a subset U of S *cornered* if for all $u, v \in U$ there is some $w \in U$ such that $\sigma(w) \leq \min\{\sigma(u), \sigma(v)\}$ for all $\sigma \in \widehat{S}$. Say that S has *property C* if S has a neighbourhood base at 0 consisting of cornered sets. Ressel [8] proved that Bochner's Theorem holds for every semigroup having property C. In particular, this is so for every topological semilattice having a neighbourhood base at 0 consisting of subsemigroups (hence for every totally ordered semilattice considered with the order topology). Property C is stable under finite products and under open continuous homomorphic images (openness and continuity only to be required locally at 0).

Although it might be hoped that Bochner's Theorem, as we have defined it, would hold for every locally compact Hausdorff $*$-semigroup, or at least in the compact metrizable case, we shall destroy this hope by establishing the following:

THEOREM 1. *There is a compact metrizable semilattice which admits a continuous positive definite function represented by a measure supported by the set of* 0-*discontinuous characters.*

The semilattice S in our counterexample will be constructed in two steps: First, we define a compact semimetrizable but non-Hausdorff semilattice T and an appropriate function on it, then we proceed to the metric quotient semilattice S.

As a semilattice, T is simply the product of $(2^{\mathsf{N}},\cup)$ (the semilattice of subsets of the natural numbers) and $([0,1],\vee)$ where $\vee$ is the maximum operation on $[0,1]$. The topology on T is defined by choosing $\varphi \in \mathscr{P}(T)$ in a very special way and then referring to the following result:

THEOREM 2. *Suppose* $(T,\cup)$ *is a semilattice and* $\varphi \in \mathscr{P}(T)$. *The equation*

$$d(s,t) = \varphi(s) + \varphi(t) - 2\varphi(s \cup t) \tag{1}$$

defines a semimetric on T which renders $\cup$ and φ continuous.

2. Preparations

Suppose $(S, \cup)$ is a *semilattice*, that is, an abelian semigroup with zero satisfying $s \cup s = s$ for all $s \in S$ and considered with the identical involution. We consider S with the partial ordering $\leq$ defined by the condition that $s \leq t$ if and only if $s \cup t = t$. If $\sigma \in \widehat{S}$ and $s \in S$ then $\sigma(s)^2 = \sigma(s \cup s) = \sigma(s)$, so σ is $\{0,1\}$-valued.

Suppose $\varphi \in \mathscr{P}(S)$. Then φ is nonnegative since the definition of positive definiteness, with $n = 1$, $s_1 = s$, and $c_1 = 1$ yields $0 \leq \varphi(s \cup s) = \varphi(s)$. Moreover, φ is nonincreasing since if $s \leq t$ then the definition of positive definiteness, with $n = 2$, $s_1 = s$, $s_2 = t$, $c_1 = 1$, and $c_2 = -1$, yields

$$0 \leq \varphi(s \cup s) + \varphi(t \cup t) - 2\varphi(s \cup t) = \varphi(s) - \varphi(t).$$

In particular, $\varphi(s) \leq \varphi(0)$ for all $s \in S$, so φ is bounded and therefore $\varphi = \widehat{\mu}$ for some $\mu \in M_+(\widehat{S})$. (If S is totally ordered then conversely, every nonincreasing nonnegative function on S is positive definite, cf. [2], 4.4.18.) Since every character on S is $\{0,1\}$-valued,

$$0 \leq \int_{\widehat{S}} \big(1 - \sigma(s)\big)\big(1 - \sigma(t)\big)\, d\mu(\sigma) = \varphi(0) - \varphi(s) - \varphi(t) + \varphi(s \cup t) \tag{2}$$

for $s, t \in S$. This inequality can be written in the form

$$\varphi(0) - \varphi(s \cup t) \leq \big(\varphi(0) - \varphi(s)\big) + \big(\varphi(0) - \varphi(t)\big)$$

from which, by induction, it follows that

$$\varphi(0) - \varphi(s_1 \cup \cdots \cup s_n) \leq \sum_{j=1}^{n} \big(\varphi(0) - \varphi(s_j)\big) \tag{3}$$

for $s_1, \ldots, s_n \in S$.

Suppose A is a set. Denote by 2^A (resp. $2^{(A)}$) the set of all subsets (resp. finite subsets) of A. Consider the semilattice $S = (2^{(A)}, \cup)$. If $Q \in 2^A$ then a character σ on S is defined by the condition that $\sigma(P) = 1$ if and only if $P \subset Q$. In this way, $\widehat{S}$ is identified with 2^A, and for $\mu \in M_+(2^A)$ and $P \in S$ we write

$$\widehat{\mu}(P) = \mu(\{\, Q \in 2^A \mid P \subset Q \,\}).$$

LEMMA 1. *Suppose S is a topological abelian semigroup with zero and involution and $\varphi, \omega \in \mathscr{P}(S)$. If $\varphi + \omega$ is continuous at 0, so are φ and ω.*

Proof. The fact that $\varphi \in \mathscr{P}(S)$ implies $|\varphi(s)|^2 \le \varphi(0)\varphi(s+s^*)$ for $s \in S$. Hence, if $\varphi(0) = 0$ then φ vanishes identically. Thus we may assume $\varphi(0), \omega(0) > 0$. Now

$$\frac{|\varphi(0)-\varphi(s)|^2}{\varphi(0)} + \frac{|\omega(0)-\omega(s)|^2}{\omega(0)} \le [\varphi(0)-\varphi(s)-\varphi(s^*)+\varphi(s+s^*)] + [\omega(0)-\omega(s)-\omega(s^*)+\omega(s+s^*)],$$

and the right-hand side tends to 0 as $s \to 0$.

Proposition 1. *For a topological abelian semigroup S with zero and involution, the following two conditions are equivalent:*

(i) *Bochner's Theorem holds for S;*

(ii) *if K is a compact subset of $\widehat{S} \setminus S'$ and if $\mu \in M_+^1(K)$ then $\widehat{\mu}$ is discontinuous at* 0.

Proof. Suppose (i) does not hold. Choose $\lambda \in M_+(\widehat{S})$ such that $\widehat{\lambda}$ is 0-continuous, yet the inner measure $\lambda_*(\widehat{S} \setminus S')$ is positive. Choose a Borel set $A \subset \widehat{S} \setminus S'$ such that $\lambda(A) > 0$, then choose a compact set $K \subset A$ such that $\lambda(K) > 0$. If $\varphi = (\lambda|\widehat{S} \setminus K)\widehat{\ }$ and $\omega = (\lambda|K)\widehat{\ }$ then $\varphi + \omega = \widehat{\lambda}$, so φ and ω are 0-continuous by Lemma 1. Now $\mu = (\lambda|K)/\lambda(K)$ defines a probability measure μ on K with $\widehat{\mu}$ 0-continuous, so (ii) does not hold. Thus (ii) implies (i). It is trivial that (i) implies (ii).

Proposition 2. *Suppose $(S, \cup)$ is a first countable topological semilattice. If K is a compact subset of $\widehat{S} \setminus S'$, there is a sequence $(a_n)_{n=1}^\infty$, converging to 0 in S, such that $\sigma(a_n) = 0$ for infinitely many n for each $\sigma \in K$.*

Proof. Let $(A_n)_{n=1}^\infty$ be a neighbourhood base at 0 with $A_1 \supset A_2 \supset \cdots$. For $\sigma \in K$, choose $b_\sigma \in A_1$ such that $\sigma(b_\sigma) = 0$. The set $G_\sigma = \{\, \rho \in \widehat{S} \mid \rho(b_\sigma) = 0 \,\}$ is a neighbourhood of σ. Since K is compact, there exist $\sigma_1, \dots, \sigma_{n_1} \in K$ such that $K \subset \bigcup_{i=1}^{n_1} G_{\sigma_i}$. With $a_i = b_{\sigma_i}$ we have $a_1, \dots, a_{n_1} \in A_1$, and for each $\sigma \in K$ there is some $i \in \{1, \dots, n_1\}$ such that $\sigma(a_i) = 0$. Similarly, choose $a_{n_1+1}, \dots, a_{n_1+n_2} \in A_2$ such that for each $\sigma \in K$ there is some $i \in \{1, \dots, n_2\}$ such that $\sigma(a_{n_1+i}) = 0$. Continuing in this way, we get a sequence (a_n) with the desired property.

Suppose $(S, \cup)$ is a metrizable semilattice for which Bochner's Theorem fails. By Proposition 1, there exist a compact subset K of $\widehat{S} \setminus S'$ and a probability measure μ on K such that $\widehat{\mu}$ is continuous at 0. By Proposition 2, there is a sequence $(a_n)_{n=1}^\infty$, converging to 0 in S, such that $\sigma(a_n) = 0$ for infinitely many n for each $\sigma \in K$. Denote by T the semilattice $(2^{(\mathsf{N})}, \cup)$. Define a homomorphism $a\colon T \to S$ by

$$a(P) = \sum_{p \in P} a_p, \qquad P \in T.$$

Consider T with the initial topology associated with the mapping a. Then $\{p\} \to \emptyset$ in T as $\mathsf{N} \ni p \to \infty$. Define a continuous mapping $\widehat{a}: \widehat{S} \to \widehat{T}$ by $\widehat{a}(\sigma) = \sigma \circ a$, $\sigma \in \widehat{S}$. Let ν be the probability measure on $\widehat{T}$ defined by $\nu = \mu^{\widehat{a}}$ (the image of μ under the mapping $\widehat{a}$). With $L = \widehat{a}(K)$, we have $L \subset \widehat{T} \setminus T'$, L is compact, ν is supported by L, and the function $\widehat{\nu} = \widehat{\mu} \circ a$ is continuous at 0. Thus, every example of a metrizable semilattice for which Bochner's Theorem fails gives rise to an example in which the semigroup is $(2^{(\mathsf{N})}, \cup)$ equipped with a semimetric such that $\{p\} \to \emptyset$ as $p \to \infty$.

Proof of Theorem 2. Let $\mu \in M_+(\widehat{S})$ be the unique measure such that $\varphi = \widehat{\mu}$. Writing $K_s = \{\, \sigma \in \widehat{T} \mid \sigma(s) = 1 \,\}$ for $s \in T$, we have $\varphi(s) = \mu(K_s)$ and $\varphi(s \cup t) = \mu(K_s \cap K_t)$ for $s, t \in T$. Now $\varphi(s) - \varphi(s \cup t) = \mu_s(K_s \setminus K_t)$ and similarly for $\varphi(t) - \varphi(s \cup t)$, so

$$d(s,t) = \mu(K_s \triangle K_t), \qquad s, t \in T$$

with the notation $L \triangle M = (L \setminus M) \cup (M \setminus L)$. For $s, t, u \in T$ we have $K_s \setminus K_u \subset (K_s \setminus K_t) \cup (K_t \setminus K_u)$ and similarly for $K_u \setminus K_s$, so $K_s \triangle K_u \subset (K_s \triangle K_t) \cup (K_t \triangle K_u)$ and therefore $d(s,u) \leq d(s,t) + d(t,u)$, which shows that d is a semimetric. For $s, t, u, v \in T$ we have

$$\begin{aligned}&(K_s \cap K_u) \setminus (K_t \cap K_v)\\ &\quad = \big((K_s \cap K_u) \setminus K_t\big) \cup \big((K_s \cap K_u) \setminus K_v\big) \subset (K_s \setminus K_t) \cup (K_u \setminus K_v)\end{aligned}$$

and similarly for $(K_t \cap K_v) \setminus (K_s \cap K_u)$, so $(K_s \cap K_u) \triangle (K_t \cap K_v) \subset (K_s \triangle K_t) \cup (K_u \triangle K_v)$ and therefore

$$d(s \cup u, t \cup v) \leq d(s,t) + d(u,v), \qquad s, t, u, v \in T \tag{4}$$

which shows that T is a topological semigroup. Finally, since φ is nonincreasing then

$$|\varphi(s) - \varphi(t)| \leq d(s,t), \qquad s, t \in T \tag{5}$$

which shows that φ is continuous.

Recall that the space $(2^{(\mathsf{N})}, \cup)^{\widehat{}}$ is identified with 2^{N}.

Proposition 3. *Suppose $\mu \in M_+^1(2^{\mathsf{N}})$, $n \notin Q$ for infinitely many n for each Q in the support of μ, and define $\varphi = \widehat{\mu}$. If there is some semimetric on $2^{(\mathsf{N})}$ which makes the completion of $2^{(\mathsf{N})}$ a compact topological semigroup admitting a continuous extension of φ, and which satisfies $\{n\} \to \emptyset$ as $n \to \infty$, then the function d defined by $d(P,Q) = \varphi(P) + \varphi(Q) - 2\varphi(P \cup Q)$ is such a semimetric.*

PROOF. Suppose e is a semimetric satisfying the assumptions, and define d as in the statement. Every sequence in $2^{(\mathsf{N})}$ has a subsequence (P_n) such that $P_n \to \Pi$ for some Π in the completion of $2^{(\mathsf{N})}$ with respect to e, so $P_m \cup P_n \to \Pi$ and therefore $\varphi(P_n) \to \varphi(\Pi)$ and $\varphi(P_m \cup P_n) \to \varphi(\Pi)$ as $m, n \to \infty$ (denoting by φ also the continuous extension of φ on the completion of $2^{(\mathsf{N})}$ with respect to e). Thus $d(P_m, P_n) \to 0$ as $m, n \to \infty$, which shows that the completion of $2^{(\mathsf{N})}$ with respect to d is compact. The inequality (5) shows that φ extends to a continuous function on the completion of $2^{(\mathsf{N})}$ with respect to d. Finally, since $\{n\} \to \emptyset$ with respect to e as $n \to \infty$, and since φ is continuous with respect to e, then $d(\emptyset, \{n\}) = \varphi(\emptyset) - \varphi(\{n\}) \to 0$.

LEMMA 2. *Suppose K is a compact subset of 2^{N} such that $p \notin Q$ for infinitely many p for all $Q \in K$. Suppose $\mu \in M_+^1(2^{\mathsf{N}})$ and define $\varphi = \widehat{\mu}$. Then there is a sequence $(N_n)_{n=1}^\infty$ of positive integers such that with*

$$A_n = \{N_1 + \cdots + N_{n-1} + 1, \ldots, N_1 + \cdots + N_n\}, \qquad n \in \mathsf{N} \tag{6}$$

we have $\varphi(A_n) = 0$ for all n. If $\varphi(\{p\}) \to 1$ as $p \to \infty$ then $N_n \to \infty$ as $n \to \infty$.

PROOF. The set $\big\{ \{ Q \in 2^{\mathsf{N}} \mid p \notin Q \} \,\big|\, p \in \mathsf{N} \big\}$ is an open covering of K, so we can choose N_1 such that $\varphi(A_1) = 0$. Choose $N_2, N_3, \ldots$ similarly. If $\varphi(\{p\}) \to 1$ as $p \to \infty$ then from

$$1 = 1 - \varphi(A_n) \le \sum_{p \in A_n} \big(1 - \varphi(\{p\})\big) \le N_n \sup\{ 1 - \varphi(\{p\}) \mid p \in A_n \}$$

(cf. (3)) we get $N_n \to \infty$.

In the following, suppose that $(N_n)_{n=1}^\infty$ is a sequence of positive integers, and define a partition $(A_n)_{n=1}^\infty$ of N by (6). We plan to choose $\varphi_n \in \mathscr{P}_1(2^{(A_n)})$ suitably and then extend the φ_n to a positive definite function φ on $2^{(\mathsf{N})}$, in order to equip $2^{(\mathsf{N})}$ with the semimetric d defined as in Theorem 2 and proceed to the completion of $2^{(\mathsf{N})}$. One method of extending the φ_n to a positive definite function on $2^{(\mathsf{N})}$ is to define

$$\varphi(P) = \prod_{n=1}^\infty \varphi_n(P \cap A_n), \qquad P \in 2^{(\mathsf{N})}.$$

In terms of the measures $\mu_n \in M_+^1(2^{A_n})$ and $\mu \in M_+^1(2^{\mathsf{N}})$ such that $\varphi_n = \widehat{\mu}_n$ and $\varphi = \widehat{\mu}$, this would correspond to taking μ to be the product measure $\mu = \bigotimes_{n=1}^\infty \mu_n$. This, however, would not do, as we shall see next.

Say that a function $\varphi \colon 2^{(\mathsf{N})} \to [0, 1]$ has the *compactness property* if every sequence in $2^{(\mathsf{N})}$ has a subsequence (P_i) such that for some $s \in [0, 1]$ we have $\varphi(P_i) \to s$ and $\varphi(P_i \cup P_j) \to s$ as $i, j \to \infty$. If $\varphi \in \mathscr{P}_1(2^{(\mathsf{N})})$ then φ has the compactness property if and only if the completion of $2^{(\mathsf{N})}$ with respect to the semimetric d defined as in Theorem 2 is compact. Indeed, by the definition of d, a sequence (P_i) is a Cauchy sequence if and only if $\varphi(P_i) - \varphi(P_i \cup P_j) \to 0$ as $i, j \to \infty$.

LEMMA 3. *For $n \in \mathsf{N}$, suppose $\mu_n \in M^1_+(2^{A_n})$. Define $\mu = \bigotimes_{n=1}^{\infty} \mu_n \in M^1_+(2^{\mathsf{N}})$ and $\varphi = \widehat{\mu} \in \mathscr{P}_1(2^{(\mathsf{N})})$. Suppose $\varphi(A_n) = 0$ for all $n \in \mathsf{N}$. If $\varphi(\{p\}) \to 1$ as $p \to \infty$ then φ does not have the compactness property.*

PROOF. Suppose $\varphi(\{p\}) \to 1$ as $p \to \infty$. Choose $n_0 \in \mathsf{N}$ such that $1 - \varphi(\{p\}) < 1/6$ for $p \in A_n$ with $n \geq n_0$. Define

$$B_n^m = \{N_1 + \cdots + N_{n-1} + 1, \ldots, N_1 + \cdots + N_{n-1} + m\}$$

for $n \geq n_0$ and $m = 0, \ldots, N_n$. Then $\varphi(B_n^0) = 1$, $\varphi(B_n^{N_n}) = 0$, and by (3),

$$\varphi(B_n^{m-1}) - \varphi(B_n^m) \leq 1 - \varphi(\{N_1 + \cdots + N_{n-1} + m\}) < \tfrac{1}{6},$$

so there is some $m_n \in \{0, \ldots, N_n\}$ such that $1/2 < \varphi(C_n) < 2/3$ where $C_n = B_n^{m_n}$. Then $\varphi(C_i \cup C_j) = \varphi(C_i)\varphi(C_j) < 4/9$ for $i \neq j$, so no subsequence of (C_n) has the property required in the definition of the compactness property.

We thus have to find a measure μ on 2^{N} with the marginals μ_n with respect to the natural identification of 2^{N} with $\prod_{n=1}^{\infty} 2^{A_n}$, and distinct from the product measure. For $n \in \mathsf{N}$ and $k = 0, \ldots, N_n$, define $\Omega_{n,k} = \{ Q \in 2^{A_n} \mid \cdot\ |Q| = k \}$. Then

$$2^{A_n} = \bigcup_{k=0}^{N_n} \Omega_{n,k},$$

so there exist $b_{n,0}, \ldots, b_{n,N_n} \geq 0$ and measures $\mu_{n,0} \in M^1_+(\Omega_{n,0}), \ldots, \mu_{n,N_n} \in M^1_+(\Omega_{n,N_n})$ such that

$$\mu_n = \sum_{k=0}^{N_n} b_{n,k} \mu_{n,k}.$$

For $0 \leq t \leq 1$, define $\mu_{t,n} \in M^1_+(2^{A_n})$ by the condition that

$$b_{n,0} + \cdots + b_{n,k-1} \leq t < b_{n,0} + \cdots + b_{n,k} \Rightarrow \mu_{t,n} = \mu_{n,k}$$

together with $\mu_{1,n} = \mu_{n,N_n}$. Then

$$\mu_n = \int_0^1 \mu_{t,n}\, dt$$

in the sense that $\int f\, d\mu_n = \int_0^1 \int f\, d\mu_{t,n}\, dt$ for every function f on 2^{A_n}. Now define

$$\mu_t = \bigotimes_{n=1}^{\infty} \mu_{t,n} \in M_+^1(2^{\mathsf{N}}), \qquad 0 \le t \le 1$$

and

$$\mu = \int_0^1 \mu_t \, dt$$

in the sense that $\int f \, d\mu = \int_0^1 \int f \, d\mu_t \, dt$ for every continuous function f on 2^{N}. Then μ has the marginals μ_n.

Instead of specifying the numbers $b_{n,0}, \ldots, b_{n,N_n}$, we can specify the function $k_n\colon [0,1] \to \{0, \ldots, N_n\}$ defined by

$$b_{n,0} + \cdots + b_{n,k-1} \le t < b_{n,0} + \cdots + b_{n,k} \Rightarrow k_n(t) = k$$

and $k_n(1) = N_n$. Then

$$\mu_{t,n} = \mu_{n,k_n(t)}, \qquad 0 \le t \le 1.$$

Suppose that $\varphi_n(P)$, for $n \in \mathsf{N}$ and $P \in 2^{(A_n)}$, is a function of the cardinality of P alone:

$$\varphi_n(P) = \Phi_n(|P|)$$

where $|P|$ is the cardinality of P. The condition that such a function Φ_n exist is equivalent to the condition that $\mu_n(\{Q\})$, for $Q \in 2^{A_n}$, depend on the cardinality of Q alone, or

$$\mu_{n,k} = \binom{N_n}{k}^{-1} \sum_{Q \in \Omega_{n,k}} \varepsilon_Q$$

where ε_Q denotes the Dirac measure at Q. Then

$$\varphi_n(P) = \sum_{k=0}^{N_n} b_{n,k} \binom{N_n}{k}^{-1} \binom{N_n - j}{k - j} = \sum_{k=0}^{N_n} b_{n,k} \frac{k^{(j)}}{N_n^{(j)}}$$

for $P \in 2^{(A_n)}$ with $|P| = j$, with the notation $p^{(q)} = p(p-1)\ldots(p-q+1)$ for $p, q \in \mathsf{N}_0$. If we define

$$\varphi_{t,n} = \widehat{\mu}_{t,n}, \qquad 0 \le t \le 1$$

then

$$\varphi_{t,n}(P) = \frac{k_n(t)^{(j)}}{N_n^{(j)}} \qquad \text{for } P \in 2^{(A_n)} \text{ with } |P| = j.$$

In the following, let $T = (2^{\mathsf{N}} \times [0,1], \cup)$ be the semilattice defined by

$(P, u) \cup (Q, v) = (P \cup Q, u \vee v)$ for all (P, u) and (Q, v), where $\vee$ denotes the maximum operation on $[0, 1]$.

PROPOSITION 4. *For $n \in \mathsf{N}$, suppose $k_n \colon [0,1] \to \{0, \ldots, N_n\}$ is a nondecreasing function. Define*

$$\Phi_{t,n}(j) = \frac{k_n(t)^{(j)}}{N_n^{(j)}}, \qquad \varphi_{t,n}(P) = \Phi_{t,n}(|P|)$$

for $n \in \mathsf{N}$, $0 \le t \le 1$, $j = 0, \ldots, N_n$, and $P \in 2^{(A_n)}$, and

$$\varphi_t(P) = \prod_{n=1}^{\infty} \varphi_{t,n}(P \cap A_n), \qquad \varphi(P, u) = \int_u^1 \varphi_t(P)\, dt$$

for $0 \le t \le 1$, $P \in 2^{\mathsf{N}}$, and $0 \le u \le 1$. Then $\varphi_{t,n} \in \mathscr{P}_1(2^{(A_n)})$, hence $\varphi_t \in \mathscr{P}_1(2^{\mathsf{N}})$, hence $\varphi \in \mathscr{P}_1(T)$.

PROOF. With $\mu_{t,n}$ as above, we have $\varphi_{t,n} = \widehat{\mu}_{t,n} \in \mathscr{P}_1(2^{(A_n)})$. For $n \in \mathsf{N}$ the pointwise product $P \mapsto \prod_{m=1}^{n} \varphi_{t,m}(P \cap A_m)$ is in $\mathscr{P}(2^{\mathsf{N}})$. Hence so is the pointwise limit $P \mapsto \prod_{n=1}^{\infty} \varphi_{t,n}(P \cap A_n)$. For $t \in [0,1]$ the function $1_{[0,t]}$ is a character on $([0,1], \vee)$, hence positive definite, so $(P, u) \mapsto \varphi_t(P) 1_{[0,t]}(u)$ is in $\mathscr{P}(T)$. Hence so is

$$(P, u) \mapsto \int_0^1 \varphi_t(P) 1_{[0,t]}(u)\, dt = \varphi(P, u).$$

(Since k_n is nondecreasing, so are $t \mapsto \Phi_{t,n}(j)$, $t \mapsto \varphi_{t,n}(P)$, and $t \mapsto \varphi_t(P)$, so there is no measurability problem.)

We observe that the following formulas hold:

$$\varphi_t\left(P \cap \bigcup_{n=1}^{k} A_n\right) = \prod_{n=1}^{k} \varphi_{t,n}(P \cap A_n), \tag{7}$$

$$\varphi_t\left(P \cap \bigcup_{n=k+1}^{\infty} A_n\right) = \prod_{n=k+1}^{\infty} \varphi_{t,n}(P \cap A_n), \tag{8}$$

$$\varphi_t(P) = \varphi_t\left(P \cap \bigcup_{n=1}^{k} A_n\right) \varphi_t\left(P \cap \bigcup_{n=k+1}^{\infty} A_n\right). \tag{9}$$

LEMMA 4. *In order that $\varphi(\{p\}, 0) \to 1$ as $p \to \infty$, it is necessary and sufficient that $k_n(t)/N_n \to 1$ as $n \to \infty$ for $0 < t \le 1$.*

PROOF. For $p \in A_n$ we have

$$\varphi(\{p\},0)=\int_0^1\frac{k_n(t)}{N_n}\,dt.$$

If $k_n(t)/N_n\to 1$ as $n\to\infty$ for $0<t\le 1$ then $\varphi(\{p\},0)\to 1$ by bounded convergence. Conversely, if $\varphi(\{p\},0)\to 1$ as $p\to\infty$ then for $0<u\le 1$, from

$$\varphi(\{p\},0)\le u\frac{k_n(u)}{N_n}+1-u=1-\left(1-\frac{k_n(u)}{N_n}\right)u$$

we get $k_n(u)/N_n\to 1$.

With a notation different from one previously used, let $\mu\in M_+^1(\widehat{T})$ be the unique measure such that $\varphi=\widehat{\mu}$. Denote by L the set of those $\rho\in\widehat{T}$ such that $\rho(\{p\},0)=0$ for infinitely many $p\in\mathsf{N}$.

Lemma 5. *In order that $\mu(L)=1$, it suffices that $k_n(t)<N_n$ for all $n\in\mathsf{N}$ and $t\in[0,1]$.*

Proof. Suppose the condition holds. Then

$$\varphi_t(A_n)=\frac{k_n(t)^{(N_n)}}{N_n!}=0$$

for all $n\in\mathsf{N}$ and $t\in[0,1]$, so

$$\mu(\{\,\rho\in\widehat{T}\mid\rho(A_n,0)=1\,\})=\varphi(A_n,0)=0,$$

which shows $\mu(L_n)=1$ where L_n is the set of those $\rho\in\widehat{T}$ such that $\rho(\{p\},0)=0$ for at least one $p\in A_n$. Since $\bigcap_{n=1}^\infty L_n\subset L$, it follows that $\mu(L)=1$.

Consider T with the topology induced by the semimetric d defined as in Theorem 2.

Theorem 3. *In order that T be compact, it suffices that*

$$\frac{\log[N_n/k_n(v)]}{\log[N_n/k_n(u)]}\to 0\qquad\text{as } n\to\infty\text{ for } 0\le u<v\le 1.$$

Proof. Suppose the condition holds. Let a sequence (P_i,z_i) in T be given. Choose an increasing sequence $(i_k)_{k=1}^\infty$ in N such that

$$P_{i_k}\cap A_k=P_{i_l}\cap A_k,\qquad l\ge k.$$

(This is possible because $P_i\cap A_k$ is always one of the finitely many subsets of A_k.) Define $P\in 2^{\mathsf{N}}$ by $P\cap A_k=P_{i_k}\cap A_k$ for $k\in\mathsf{N}$. Write

$$Q_k = P_{i_k} \cap \bigcup_{n=k+1}^{\infty} A_n, \qquad k \in \mathsf{N}.$$

Choose a countable dense set U in $[0, 1]$, then choose an infinite subset K of N such that $\varphi_t(Q_k) \to s(t)$ as $K \ni k \to \infty$ for some $s(t) \in [0, 1]$ for all $t \in U$. Since $t \mapsto \varphi_t(Q)$ is nondecreasing for each Q then the function s is nondecreasing. If $t \in [0, 1]$ is such that s extends to a continuous function on $U \cup \{t\}$ then $\varphi_t(Q_k) \to s(t)$ as $K \ni k \to \infty$ for the unique $s(t)$ that makes the extension continuous. The remaining points of $[0, 1]$ form a countable set, so we can choose an infinite subset M of K such that $\varphi_t(Q_k) \to s(t)$ as $M \ni k \to \infty$ for some $s(t) \in [0, 1]$ for all $t \in [0, 1]$.

We claim that there is some $w \in [0, 1]$ such that

$$s(t) = 0 \quad \text{if } t < w, \qquad s(t) = 1 \quad \text{if } t > w. \tag{10}$$

The function s being nondecreasing, this is trivial if $s(t) \in \{0, 1\}$ for all $t \in [0, 1]$. Thus we may assume that there is some $w \in [0, 1]$ such that $0 < s(w) < 1$. The claim will then follow if we can show that the conditions $0 \le u < v \le 1$ and $Q \subset \bigcup_{n=k+1}^{\infty} A_n$ imply $\log \varphi_v(Q) \ge \varepsilon_k \log \varphi_u(Q)$ with $\varepsilon_k \to 0$ as $k \to \infty$. Since $\log \varphi_t(Q) = \sum_{n=k+1}^{\infty} \log \varphi_{t,n}(Q \cap A_n)$, it suffices to show

$$\log \varphi_{v,n}(R) \ge \delta_n \log \varphi_{u,n}(R), \qquad R \subset A_n$$

with $\delta_n \to 0$ as $n \to \infty$, or $\log \Phi_{v,n}(j) \ge \delta_n \log \Phi_{u,n}(j)$ for $0 \le j \le N_n$. Since $\log \Phi_{t,n}(j) = \sum_{i=0}^{j-1} \log[(k_n(t) - i)/(N_n - i)]$, it suffices to show

$$\log \frac{k_n(v) - i}{N_n - i} \ge \delta_n \frac{k_n(u) - i}{N_n - i}, \qquad 0 \le i < k_n(u).$$

In other words, with

$$f_n(x) = \frac{\log[(N_n - x)/(k_n(v) - x)]}{\log[(N_n - x)/(k_n(u) - x)]}, \qquad 0 \le x < k_n(u),$$

it suffices to show that $f_n(x) \to 0$, uniformly in x, as $n \to \infty$. We have $f_n(0) \to 0$ by hypothesis, so it suffices to show that each f_n is nonincreasing. Computation shows

$$f_n'(x) \le 0 \Leftrightarrow g_{n,x}\big(k_n(u)\big) \le g_{n,x}\big(k_n(v)\big)$$

where

$$g_{n,x}(y) = \frac{\log \dfrac{N_n - x}{y - x}}{\dfrac{1}{y - x} - \dfrac{1}{N_n - x}}, \qquad x < y < N_n,$$

so it suffices to show that $g_{n,x}$ is nondecreasing for all n and x. Differentiation shows

$$g'_{n,x}(y) \geq 0 \Leftrightarrow \log \frac{N_n - x}{y - x} \geq 1 - \frac{y - x}{N_n - x}.$$

The latter inequality is automatically fulfilled. This proves (10).

Finally, choose an infinite subset N of M such that $z_{i_k} \to z$ as $N \ni k \to \infty$ for some $z \in [0, 1]$. We claim that the sequence $\big((P_{i_k}, z_{i_k})\big)_{k \in N}$ converges to $(P, w \vee z)$. We have to show $d\big((P_{i_k}, z_{i_k}), (P, w \vee z)\big) \to 0$ as $N \ni k \to \infty$, and we do this by showing $\varphi(P_{i_k}, z_{i_k}) \to \varphi(P, w \vee z)$ and $\varphi(P_{i_k} \cup P, z_{i_k} \vee w \vee z) \to \varphi(P, w \vee z)$. Firstly,

$$\varphi(P_{i_k}, z_{i_k}) = \int_{z_{i_k}}^{1} \varphi_t(P_{i_k})\, dt \to \int_{z}^{1} \lim_k \varphi_t(P_{i_k})\, dt,$$

provided that the limit exists for almost all t, so it suffices to show

$$\lim_k \varphi_t(P_{i_k}) = \begin{cases} 0, & \text{t<w} \\ \varphi_t(P), & \text{t>w.} \end{cases}$$

But this follows from $\varphi_t(P_{i_k}) = \varphi_t(P \cap \bigcup_{n=1}^{k} A_n)\varphi_t(Q_k)$ because of (10), cf. (9). Secondly,

$$\varphi(P_{i_k} \cup P, z_{i_k} \vee w \vee z) \to \int_{w \vee z}^{1} \lim_k \varphi_t(P_{i_k} \cup P)\, dt,$$

provided that the limit exists for almost all t, so it suffices to show $\lim_k \varphi_t(P_{i_k} \cup P) = \varphi_t(P)$ for $t > w$. Now by (9),

$$\begin{aligned} 0 \leq \varphi_t(P) - \varphi_t(P_{i_k} \cup P) &= \varphi_t\left(P \cap \bigcup_{n=1}^{k} A_n\right)\left[\varphi_t\left(P \cap \bigcup_{n=k+1}^{\infty} A_n\right)\right. \\ &\left.- \varphi_t\left(Q_k \cup \left(P \cap \bigcup_{n=k+1}^{\infty} A_n\right)\right)\right] \leq \varphi_t\left(P \cap \bigcup_{n=k+1}^{\infty} A_n\right) \\ &- \varphi_t\left(Q_k \cup \left(P \cap \bigcup_{n=k+1}^{\infty} A_n\right)\right) \leq 1 - \varphi_t(Q_k) \end{aligned}$$

because of (2), and the right-hand side tends to 0 because of (10).

3. The counterexample

PROOF OF THEOREM 1. Define $N_n = 3^n$ and

$$k_n(t) = \lfloor N_n e^{-e^{-nt}} \rfloor, \qquad n \in \mathsf{N}, \quad t \in [0,1]$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Now define everything else as in Proposition 4. Consider T with the semimetric d defined as in Theorem 2. We have $k_n(t)/N_n \to 1$ as $n \to \infty$ for $0 < t \le 1$, so by Lemma 4, $\varphi(\{p\},0) \to 1 = \varphi(\emptyset,0)$ as $p \to \infty$, that is, $(\{p\},0) \to (\emptyset,0)$ in T. However, since $k_n(t) < N_n$ for all $n \in \mathsf{N}$ and $t \in [0,1]$ then $\mu(L) = 1$ by Lemma 5, so $\mu_*(\widehat{T} \setminus T') = 1$. For $0 \le u < v \le 1$ we have $k_n(v) > N_n e^{-e^{-nv}} - 1$ and $k_n(u) \le N_n e^{-e^{-nu}}$, so

$$\frac{\log[N_n/k_n(v)]}{\log[N_n/k_n(u)]} < e^{nu} \log \frac{e^{e^{-nv}}}{1 - e^{e^{-nv}}/N_n} < e^{-n(v-u)} - e^{nu} \log\left(1 - \frac{e}{N_n}\right) \to 0.$$

By Theorem 3 it follows that T is compact.

Define an equivalence relation $\sim$ in T by the condition that $s \sim t$ if and only if $d(s,t) = 0$, denote by $h(s)$ the equivalence class containing $s \in T$, and define $S = h(T)$. Then S is a metric space under the quotient metric, also denoted by d. If $s \sim t$ and $u \sim v$ then $s \cup u \sim t \cup v$ by (4), so we can make S a semilattice by defining $h(s) \cup h(u) = h(s \cup u)$ for $s, u \in T$. The inequality (4) carries over to S, which is therefore a topological semigroup. Since T is compact, so is S. The inequality (5) shows that there is a unique function Φ on S such that $\varphi = \Phi \circ h$. Clearly Φ is positive definite. The inequality (5) carries over to S and shows that Φ is continuous. If λ is the unique measure on $\widehat{S}$ such that $\Phi = \widehat{\lambda}$ then $\mu(A) = \lambda(\{\sigma \in \widehat{S} \mid \sigma \circ h \in A\})$ for $A \in \mathscr{B}(\widehat{T})$. In particular, $1 = \mu(L) = \lambda(\{\sigma \in \widehat{S} \mid \sigma \circ h \in L\})$. Since the conditions $\sigma \in \widehat{S}$ and $\sigma \circ h \in L$ imply that σ is discontinuous at 0 then $\lambda_*(\widehat{S} \setminus S') = 1$.

ACKNOWLEDGMENTS. Correspondence with Paul Ressel has been very helpful.

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