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INVERSE LIMITS OF INTEGRAL DOMAINS ARISING FROM ITERATED NAGATA COMPOSITION

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Abstract

By iterating the type of pullback constructions in which $P^r VDs$ arise by Nagata composition, we are led to study a class of inverse limits $A = \varprojlim A_n$ of integral domains indexed by N. After identifying the prime spectrum, the localizations, and the integral closure of A, we then characterize when, i.a., such (typically infinite-dimensional) A is a Prüfer domain, Bézout domain, divided domain, or $P^r VD$.

1. Introduction

As the literature contained several interesting examples of directed unions of (commutative) integral domains, it was appropriate to study direct limits of integral domains in [10], [9]. The present paper initiates a similar study of inverse limits of integral domains, in part to expand upon examples such as the following (cf. Theorem 2.1 (c) and Corollary 2.7): if k is a field, then

$$k[[X_1]] + X_2k((X_1))[[X_2]] + X_3k((X_1, X_2))[[X_3]] + \cdots$$

is a valuation domain. For simplicity (and with an eye on the intended examples), we consider only inverse limits of directed systems indexed by N, the set of positive integers. As [10] was motivated largely by the result that the class of Prüfer domains is stable under direct limit, we begin by establishing the analogous result for inverse limit in the local case. Specifically, Theorem 2.1 (g) states that any inverse limit of valuation domains is a valuation domain. We finally establish a non-local analogue (concerning inverse limits of Prüfer domains) in Theorem 2.21. This result is developed in context of the special type of inverse limit to which this paper is devoted. This context is suggested by iterating the pullback construction of $P^r VDs$ in [12, Théorème 1.3], itself a generalization of the so-called Nagata composition of valuation domains [20, page 35].

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Applications of $P^r VDs$ are not new: see, for instance, their role in realizing arbitrary dimension sequences in [8] and their implicit role as iterated $D + M$ constructions in the work of Seidenberg [22] on constructing polynomial rings which realize specified Krull dimensions. However, "$P^\infty VDs$" such as the ring displayed above *are* new, inasmuch as they have infinite Krull dimension. To expand upon this observation, we identify Spec (A) in Theorem 2.5 (a), where A is the type of inverse limit under consideration here. One eventual upshot is Remark 2.9, which shows how to extend the factorization-theoretic work in [3] to produce an infinite-dimensional non-fragmented valuation domain with no atoms. To prepare for other applications (such as the above-mentioned result on Prüfer domains), we determine the integral closure of A (in Theorem 2.12) and the localizations of A (in Proposition 2.15 (d)). As additional applications, in the spirit of [10], we characterize when A is a divided domain (in the sense of [5]) in Corollary 2.17, a $P^r VD$ in Proposition 2.19, a pseudo-valuation domain (in the sense of [17]) in Corollary 2.20, and a Bézout domain in Corollary 2.23.

If D is an integral domain, we let $\dim(D)$ denote the Krull dimension of D, D' the integral closure of D, $qf(D)$ the quotient field of D, $\mathrm{Max}(D)$ the set of maximal ideals of D, and $\mathcal{U}(D)$ the set of units of D. As in [5], $P \in \mathrm{Spec}\,(D)$ is a *divided prime ideal of* D if $PD_P = P$; this is equivalent to saying that the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} D & \twoheadrightarrow & D/P \\ \cap\!\downarrow & & \cap\!\downarrow \\ D_P & \twoheadrightarrow & \mathrm{qf}\,(D/P) \end{array}$$

is a pullback. The integral domain D is a *divided domain* if each $P \in \mathrm{Spec}\,(D)$ is a divided prime ideal of D; and D is a *locally divided domain* if D_P is a divided domain for each $P \in \mathrm{Spec}\,(D)$. In this paper, the most important examples of divided domains are $P^r VDs$ (in particular, (pseudo-)valuation domains), and our most important examples of locally divided domains are Prüfer domains. Following [19, page 28], we denote the properties of lying-over, going-up, going-down and incomparability by LO, GU, GD and INC, respectively. Any unexplained material is standard, as in [15], [19].

After this paper had been drafted, the second-named author gave a talk on it at a meeting held in Fez, Morocco in October 1997. Following the talk, Professor M. Tabaâ mentioned to him that some of our results overlap those of Wiseman [24], a paper with which we had not been acquainted. Although Wiseman's motivations involving linearly compact modules in [24] are different from ours here, one of his contexts [24, page 1109] is a special case of ours. As a result, [24] contains special cases of our Theorem 2.1 (d) and Proposition 2.4 (d), as well as special cases of [11, Corollary 1.5 (5)]. Most significantly, in view of Proposition 2.4 (c), whose assertion is a hypothesis for the context of [24], we see that [24, Proposition 3.4] is essentially equivalent to our Theorem 2.12, concerning the commuting of inverse limit and integral closure.

2. Results

Recall that direct limit preserves integral domains [16, Proposition 6.1.6 (i)], reduced rings [16, Corollaire 6.1.3], and Prüfer domains [10, Proposition 2.5 (a)]. We begin by establishing some analogues for inverse limit.

THEOREM 2.1. *Let* $(A_n, \varphi_{n,m} : A_n \to A_m; n \geq m \geq 1)$ *be an inverse system of commutative rings, with* $\varphi_{n,n}$ *taken to be the identity map, and let* $A = \varprojlim A_n$ *be the inverse limit.*

Put $\Phi_n : A \to A_n$ *the canonical map, and* $Q_n := \ker(\Phi_n)$. *Then*

(a) $A = \{(a_n) \in \prod A_n : \varphi_{n+1,n}(a_{n+1}) = a_n$ *for each* $n \in \mathsf{N}\}$.

(b) *For each* $n \in \mathsf{N}$, Φ_n *is the composite of the inclusion map* $A \hookrightarrow \prod A_k$ *and the canonical projection* $\prod A_k \to A_n$.

(c) *For each* $n \in \mathsf{N}$, $Q_n = \{(a_k) \in A : a_k = 0$ *for each* $k \leq n\}$.

(d) $Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \ldots$ *and* $\bigcap Q_n = 0$.

(e) *If* A_n *is an integral domain for each n, then A is an integral domain.*

(f) *If* A_n *is reduced for each n, then A is reduced.*

(g) *If* A_n *is a valuation domain for each n, then A is a valuation domain.*

PROOF. According to the usual construction of inverse limit we may view $A = \{(a_n) \in \prod A_n : \varphi_{n,m}(a_n) = a_m$ whenever $n \geq m\}$. Then (a), (b) and (c) are immediate consequences of the definitions. Observe that the additive identity of A is $0 = (0_n) \in A$, where 0_n is the additive identity element of A_n.

Next, (d) follows from (c).

(e) Deny. Then there exist nonzero $\alpha = (\alpha_n)$, $\beta = (\beta_n) \in A$ such that $(\alpha_n \beta_n) = \alpha\beta = 0 = (0_n) \in A$. As $\alpha \neq 0 = (0_n)$, there exists an index i such that $\alpha_i \neq 0_i$. Similarly, $\beta_j \neq 0_j$ for some index j, since $\beta \neq 0$. Without loss of generality, $i \geq j$. As $\varphi_{i,j}(\beta_i) = \beta_j \neq 0_j$ and $\varphi_{i,j}$ is a homomorphism, $\beta_i \neq 0_i$. However, $\alpha_i \beta_i = 0_i$, contradicting the hypothesis that A_i is an integral domain.

(f) Suppose $\alpha = (\alpha_n) \in A$ is nilpotent. Then for some $\nu \in \mathsf{N}$, $(0_n) = 0 = \alpha^\nu = (\alpha_n^\nu)$. Hence, for each $n \in \mathsf{N}$, $\alpha_n^\nu = 0_n$, and so $\alpha_n = 0_n$ since A_n is reduced. It follows that $\alpha = (0_n) = 0$, whence A is reduced.

(g) Consider nonzero elements $\alpha = (\alpha_n)$, $\beta = (\beta_n) \in A$. By (a), it is enough to show that either $\alpha \in A\beta$ or $\beta \in A\alpha$.

Case 1: for each $n \in \mathbb{N}$, $\alpha_n = u_n\beta_n$ for some $u_n \in A_n$. If $n \geq m$, then

$$u_m\beta_m = \alpha_m = \varphi_{n,m}(\alpha_n) = \varphi_{n,m}(u_n\beta_n) = \varphi_{n,m}(u_n)\varphi_{n,m}(\beta_n) = \varphi_{n,m}(u_n)\beta_m.$$

Since A_m is an integral domain, either $\varphi_{n,m}(u_n) = u_m$ or $\beta_m = 0_m$. If $\beta_m \neq 0_m$ (regardless of the choice of n), then $u := (u_n) \in A$ and $\alpha = u\beta \in A\beta$. So, without loss of generality, $\beta_m = 0_m$ for some $m \in \mathbb{N}$. Let k be the maximal such m. (Note that k exists since $\beta \neq 0$ and $\varphi_{i,j}(0_i) = 0_j$ whenever $i \geq j$.) By the above reasoning, $\varphi_{n,m}(u_n) = u_m$ whenever $k < m \leq n$. Also, if $1 \leq i \leq k$, then $\beta_i = \varphi_{k,i}(\beta_k) = \varphi_{k,i}(0_k) = 0_i$, and so $\alpha_i = u_i\beta_i = 0_i$. Consider $v = (v_n) \in \prod A_n$ defined by

$$v_n = \begin{cases} u_n & \text{if } n > k \\ \varphi_{k+1,n}(u_{k+1}) & \text{if } 1 \leq n \leq k\ . \end{cases}$$

Since we have an inverse system, $\varphi_{n,m}(v_n) = v_m$ whenever $n \geq m$, whence $v \in A$. Also, $\alpha_n = v_n\beta_n$ for each n. (The assertion reduces to $0_n = 0_n$ if $n \leq k$ and to the choice of u_n if $n > k$.) Hence, $\alpha = v\beta \in A\beta$.

It remains to consider what happens if Case 1 does not apply. Since each A_i is a valuation domain, we are reduced, without loss of generality, to

Case 2: for some $n > m$, there exist $r_n \in A_n$ and $s_m \in A_m$ such that $\alpha_n = r_n\beta_n$, $\beta_m = s_m\alpha_m$, $\beta_n \notin A_n\alpha_n$, and $\alpha_m \notin A_m\beta_m$. In particular, $r_n \in A_n \setminus \mathcal{U}(A_n)$ and $s_m \in A_m \setminus \mathcal{U}(A_m)$. Then

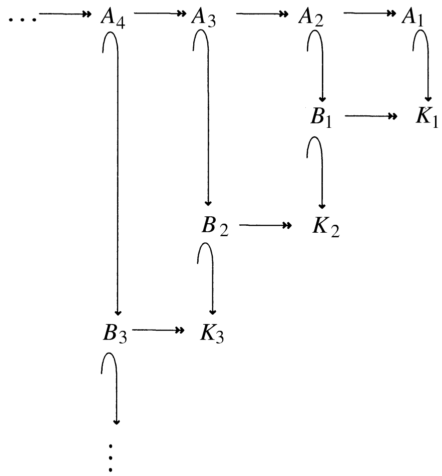
$$\alpha_m = \varphi_{n,m}(\alpha_n) = \varphi_{n,m}(r_n\beta_n) = \varphi_{n,m}(r_n)\varphi_{n,m}(\beta_n) = \varphi_{n,m}(r_n)\beta_m = \varphi_{n,m}(r_n)s_m\alpha_m\ .$$

Since s_m is a nonunit of A_m, $\varphi_{n,m}(r_n)s_m \neq 1$ and so, since A_m is an integral domain, $\alpha_m = 0_m$. This contradict $\alpha_m \notin A_m\beta_m$. Therefore, Case 2 does not occur.

We proceed to fix the *riding assumptions and notation* for the rest of the paper. We assume given $\{A_1; (K_n, B_n) : n \in \mathbb{N}\}$ such that for each n:

- B_n is a quasilocal integral domain with maximal ideal $M_n \neq 0$;
- $K_n = B_n/M_n$, $\varphi_n : B_n \to K_n$ is the canonical surjection and $qf(B_n) \subseteq K_{n+1}$; and
- A_1 is an integral domain but not a field, $qf(A_1) \subseteq K_1$.

For $n > 1$, let A_{n+1} be the pullback $A_{n+1} := B_n \times_{K_n} A_n$. Note that A_{n+1} is canonically contained in B_n and so also in K_{n+1}. Since $M_n \neq 0$, A_{n+1} and B_n share a common nonzero ideal (i.e. M_n), and so $qf(A_{n+1}) = qf(B_n)$: cf. [15, page 326]. Thus, the above data lead to the following set of pullback diagrams (reminiscent of the description of $P^r VDs$ in the diagram in [12, page 188]):



where $\varphi_{n,n-1} : A_n \to A_{n-1}$ is the canonical surjection arising from the fact that $A_n = \varphi_{n-1}^{-1}(A_{n-1})$ for all $n \geq 2$. If $n > m$, consider the surjection

$$\varphi_{n,m} = \varphi_{m+1,m} \circ \cdots \circ \varphi_{n,n-1} : A_n \to A_m,$$

with $\varphi_{n,n} : A_n \to A_n$ taken to be the identity map. The inverse system determined by the homomorphisms $\varphi_{n,m}$ is called *the inverse system generated by* $\{A_1; (K_n, B_n) : n \in \mathsf{N}\}$. Put, as above,

$$A := \varprojlim A_n, \quad \Phi_n : A \to A_n \quad \text{the canonical map, and} \quad Q_n := \ker(\Phi_n)\ .$$

We next collect some basic facts.

Lemma 2.2. *Under the riding assumptions, we have:*

(a) *For each* $n \in \mathsf{N}$, Φ_n *is surjective.*

(b) *For each* $n \in \mathsf{N}$, $Q_n \in \operatorname{Spec}(A)$ *and* $A/Q_n \cong A_n$, *so* A *is an integral domain.*

(c) *For each* $n \in \mathsf{N}$, $M_n = \ker(\varphi_{n+1,n})$ *and* $Q_n = \Phi_{n+1}^{-1}(M_n)$.

Proof. (a) follows from Theorem 2.1 (b), since each $\varphi_{n,n-1}$ is surjective. Then (b) follows from the First Isomorphism Theorem and Theorem 2.1 (e), since Φ_n is surjective and A_n is an integral domain. Finally, for (c), observe that the pullback description of A_{n+1} yields that $\ker(\varphi_{n+1,n}) = M_n$; then, since $\Phi_n = \varphi_{n+1,n} \circ \Phi_{n+1}$, we have that $Q_n = \ker(\Phi_n) = \Phi_{n+1}^{-1}(M_n)$.

The next result will often permit us to assume, without loss of generality, that $K_n = qf(A_n)$ for each $n \in \mathsf{N}$.

Proposition 2.3. *For each* $n \in \mathsf{N}$, *let* $K_n^* = qf(A_n)$ *and* $B_n^* = B_n \times_{K_n} K_n^*$. *Then:*

(a) *For each* $n \in \mathsf{N}$, *there is a canonical isomorphism* $A_{n+1} \cong B_n^* \times_{K_n^*} A_n$.

(b) *The inverse system generated by* $\{A_1; (K_n^*, B_n^*) : n \in \mathsf{N}\}$ *is the same as the inverse system generated by* $\{A_1; (K_n, B_n) : n \in \mathsf{N}\}$.

Proof. (a) follows from the canonical isomorphism

$$A_{n+1} = B_n \times_{K_n} A_n \cong (B_n \times_{K_n} K_n^*) \times_{K_n^*} A_n = B_n^* \times_{K_n^*} A_n.$$

We thus see that the two inverse systems have the same $\varphi_{n+1,n}$ maps, and (b) follows.

We pause to observe that in Proposition 2.3, B_n^* retains the property of B_n of being a quasilocal integral domain with maximal ideal M_n (cf. [11, Theorem 1.4, Proposition 2.1 (9)]). More precisely, by [2], Spec $(B_n) =$ Spec (B_n^*).

We observe also that, given our riding assumptions, A is never a Noetherian ring. As a matter of fact, if A is Noetherian then clearly A_n is also Noetherian for each $n \geq 1$ (Lemma 2.2 (b)); but A_2 is Noetherian if and only if B_1 is Noetherian, A_1 is a field and $A_1 \subset K_1$ is a finite (-dimensional) field extension [11, Theorem 2.3].

Finally, we observe that the ring

$$A = k[[X_1]] + X_2 k((X_1))[[X_2]] + X_3 k((X_1, X_2))[[X_3]] + \cdots$$

considered in the introduction, is the inverse limit of the inverse system generated by $\{k[[X_1]], (k((X_1, X_2, \ldots, X_n)), k((X_1, X_2, \ldots, X_n))[[X_{n+1}]]) : n \in \mathsf{N}\}$. Since, in this case, $A_1 = k[[X_1]]$ and $B_n = k((X_1, \ldots, X_n))[[X_{n+1}]]$ are (discrete) valuation domains, then, by induction on $n \geq 1$ and using Nagata composition, it is easy to see that

$$A_{n+1} = k[[X_1]] + X_2 k((X_1))[[X_2]] + \cdots + X_{n+1} k((X_1, X_2, \ldots, X_n))[[X_{n+1}]]$$

is also a valuation domain, for each $n \in \mathsf{N}$. Therefore, by Theorem 2.1 (g), A is a valuation domain.

We proceed to analyze the prime spectrum for the special type of inverse limit being studied here, from which we will deduce, in particular, further properties of our motivating example. For $r \geq n$ in N, put $Q_{r,n} := \ker(\varphi_{r,n})$.

Proposition 2.4.

(a) $Q_{r,n} = \Phi_r(Q_n)$ *if* $r \geq n$ *and* $Q_{r,n} = \varphi_{r,n+1}^{-1}(M_n)$ *if* $r \geq n+1$.

(b) *For each* $n \in \mathsf{N}$, $\varprojlim\{Q_{r,n} : r \geq n\} = Q_n$.

(c) *If* $r \geq n$, *then* $Q_{r,n}$ *is a divided prime ideal of* A_r.

(d) *For each* $n \in \mathsf{N}$, Q_n *is a divided prime ideal of* A.

Proof. (a) As $\varphi_{n,n}$ is an identity map, its kernel, $Q_{n,n}$, is 0. Of course, $\Phi_n(Q_n)$ is also 0, since $Q_n = \ker(\Phi_n)$. Thus, without loss of generality, $r \geq n+1$. As in the proof of Lemma 2.2 (c), $\Phi_n = \varphi_{r,n} \circ \Phi_r$, whence

$Q_n = \Phi_n^{-1}(0) = \Phi_r^{-1}(\ker(\varphi_{r,n})) = \Phi_r^{-1}(Q_{r,n})$. Since Φ_r is surjective, it follows that $\Phi_r(Q_n) = Q_{r,n}$. Since $\varphi_{r,n} = \varphi_{n+1,n} \circ \varphi_{r,n+1}$ and $M_n = \ker(\varphi_{n+1,n})$, we see similarly that $\varphi_{r,n+1}^{-1}(M_n) = \ker(\varphi_{r,n}) = Q_{r,n}$.

(b) If $r \geq n$, it follows from the surjectivity of $\varphi_{r+1,r}$ and the second assertion in (a) that $\varphi_{r+1,r}$ restricts to a surjection $Q_{r+1,n} \to Q_{r,n}$. Since $\varprojlim$ preserves monomorphisms, we can view $\varprojlim\{Q_{r,n} : r \geq n\}$ inside $\varprojlim\{A_r : r \geq n\}$, which, by cofinality, is just A. It now follows easily from the above comments that $\varprojlim\{Q_{r,n} : r \geq n\} = Q_n$.

(c) By Proposition 2.3 (b), we may assume that $K_n = qf(A_n)$ for each $n \in \mathsf{N}$. Of course, $Q_{r,n}$ is a prime ideal of A_r, since $\varphi_{r,n}$ is a surjective homomorphism and A_n is an integral domain. It remains only to prove the "divided" assertion. This is evident if $r = n$, since $Q_{n,n} = 0$. Next, for $r = n+1$, observe that $Q_{n+1,n} = M_n$ by (a) and, by a calculation using $K_n = qf(A_n)$, we check that $B_n = (A_{n+1})_{M_n}$ (cf. also [14, Lemma 1.1.6]). Then the pullback description of A_{n+1} may be identified with the statement that $A_{n+1} = (A_{n+1})_{M_n} \times_{qf(A_{n+1}/M_n)} (A_{n+1}/M_n)$; i.e., $Q_{n+1,n} = M_n$ is a divided prime ideal of A_{n+1}.

It follows easily from the second assertion in (a) that $\varphi_{r+1,r}^{-1}(Q_{r,n}) = Q_{r+1,n}$. Thus, in view of the pullback description of A_{n+i} for $i \geq 2$, a proof of (c) may be completed (by induction on r) by proving the following general result.

If B is a quasilocal integral domain with maximal ideal M and residue field K, $\varphi : B \to K$ the canonical surjection, D an integral domain with quotient field K, and P a divided prime ideal of D, then $P^* := \varphi^{-1}(P)$ is a divided prime ideal of $D^* := \varphi^{-1}(D) = D \times_K B$.

For a proof, observe as above that the condition $K = qf(D)$ ensures that $B = D^*_M$, whence $MD^*_M = M$ and, in particular, $MD^*_{P^*} = M$. As we may identify $D = D^*/M$ and $D_P = (D^*/M)_{P^*/M} = D^*_{P^*}/MD^*_{P^*} = D^*_{P^*}/M$, we have a pullback diagram

$$\begin{array}{ccc} D^* & \twoheadrightarrow & D \\ \downarrow & & \downarrow \\ D^*_{P^*} & \twoheadrightarrow & D_P \end{array}$$

In addition, since P is a divided prime ideal of D, there is a pullback diagram

$$\begin{array}{ccc} D & \twoheadrightarrow & D/P \\ \downarrow & & \downarrow \\ D_P & \twoheadrightarrow & \mathrm{qf}\,(D/P) \end{array}.$$

As we may identify $D^*/P^* = D/P$, juxtaposition of the above diagrams produces a pullback diagram

$$\begin{array}{ccc} D^* & \twoheadrightarrow & D^*/P^* \\ \cap\downarrow & & \cap\downarrow \\ D^*_{P^*} & \twoheadrightarrow & \mathrm{qf}\,(D^*/P^*) \end{array}$$

thus showing that P^* is divided in D^*.

(d) We show that if $a = (a_k) \in A \backslash Q_n$, then $Q_n \subseteq aQ_n$. Let $x = (x_k) \in Q_n$. Then $x_r \in Q_{r,n}$ whenever $r \geq n$, by (b). Also by (b), $a_r \in A_r \backslash Q_{r,n}$ whenever $r \geq n$. Hence, by (c), if $r \geq n$, there exists $y_r \in Q_{r,n}$ such that $x_r = a_r y_r$; moreover, y_r is uniquely determined since $a_r \neq 0$. Since $\varphi_{r+1,r}(x_{r+1}) = x_r$ and A_r is an integral domain, we see easily that $\varphi_{r+1,r}(y_{r+1}) = y_r$ for each $r \geq n$. Moreover, $x_i = 0$ whenever $i \leq n$, by Theorem 2.1 (c). Since $x_n = a_n y_n$ and A_n is an integral domain, $y_n = 0$. Put $y_i = 0$ if $1 \leq i < n$ and $y := (y_k) \in \prod A_k$. Observe that $y \in A$ and $x_k = a_k y_k$ for each $k \in \mathsf{N}$. In fact, $y \in Q_n$ by (b), and so $x = ay \in aQ_n$.

Since $A/Q_n \cong A_n$, $\mathrm{Spec}\,(A_n)$ embeds canonically in $\mathrm{Spec}\,(A)$. For convenience, we identify $\mathrm{Spec}\,(A_n)$ with its image in $\mathrm{Spec}\,(A)$. In particular, $\mathrm{Spec}\,(A_n) \subseteq \mathrm{Spec}\,(A_{n+1}) \subseteq \mathrm{Spec}\,(A)$. With this convention, the following useful result is easy to state.

THEOREM 2.5.
(a) $\mathrm{Spec}\,(A) \backslash \{0\} = \bigcup_{n=1}^{\infty} \mathrm{Spec}\,(A_n)$.
(b) *For each* $n \in \mathsf{N}$, $\mathrm{Max}(A) = \mathrm{Max}(A_n) = \mathrm{Max}(A_1)$.
(c) *Let* $a = (a_k) \in A$. *Then*

$$a \in \mathcal{U}(A) \Leftrightarrow a_k \in \mathcal{U}(A_k) \text{ for each (resp., some) } k \in \mathsf{N} \Leftrightarrow a_1 \in \mathcal{U}(A_1).$$

PROOF. (a) Since $M_n \neq 0$, it follows from Proposition 2.4 (a) that $Q_n \neq 0$, and so the zero prime ideal of A is not in (the canonical image of) $\mathrm{Spec}\,(A_n)$. Therefore, it suffices to show that if $0 \neq P \in \mathrm{Spec}\,(A)$, then $P \in \mathrm{Spec}\,(A_k)$ for some $k \in \mathsf{N}$. Now, for each k, since Q_k is divided (by Proposition 2.4 (d)), either $P \subset Q_k$ or $Q_k \subseteq P$. As $P \neq 0$, it follows from Theorem 2.1 (d) that P is not contained in $\bigcap Q_k$. Hence, there exists $k \in \mathsf{N}$ such that $Q_k \subseteq P$. In particular, $P \in \mathrm{Spec}\,(A_k)$.

(b) Let $n \in \mathsf{N}$. By Proposition 2.4 (d), Q_n is contained in each maximal ideal of A. Using $A/Q_n \cong A_n$ and the above convention, we infer $\mathrm{Max}(A) = \mathrm{Max}(A/Q_n) = \mathrm{Max}(A_n)$.

(c) If $M \in \mathrm{Max}(A)$, use the identifications in (b) to view M as

$M^{(k)} \in \mathrm{Max}(A_k)$ for each $k \in \mathsf{N}$. Since $\Phi_1(a) = a_1$, we see that $a \in M \Leftrightarrow a_1 \in M^{(1)}$. Therefore,

$$a \in \mathscr{U}(A) \Leftrightarrow a \notin M \quad \text{for each} \quad M \in \mathrm{Max}(A)$$
$$\Leftrightarrow a_1 \notin M^{(1)} \quad \text{for each} \quad M^{(1)} \in \mathrm{Max}(A_1) \Leftrightarrow a_1 \in \mathscr{U}(A_1).$$

Since ring-homomorphisms send units to units, the assertion follows.

Corollary 2.6. *A is an integral domain and* $\dim(A) = \infty$. *Moreover, A is quasilocal if and only if* A_1 *is quasilocal.*

Proof. The first assertion follows from Theorem 2.1 (e), the third from Theorem 2.5 (b). To show $\dim(A) = \infty$, note that $\mathrm{Spec}\,(A_n) \neq \mathrm{Spec}\,(A_{n+1})$ since $Q_{n+1,n} \neq 0$.

We next revisit the theme of Theorem 2.1 (g) for the special type of inverse limit under consideration.

Corollary 2.7. *The following conditions are equivalent:*

(1) *For each* $n \in \mathsf{N}$, A_n *is a valuation domain;*

(2) A_1 *is a valuation domain and, for each* $n \in \mathsf{N}$, B_n *is a valuation domain and* $K_n = qf(A_n)$*;*

(3) *A is a valuation domain.*

Proof. (1) $\Rightarrow$ (3) by Theorem 2.1 (g). Also, (3) $\Rightarrow$ (1) since $A/Q_n \cong A_n$ and factor domains of valuation domains are valuation domains [15, Proposition 13.2 (2)].

(1) $\Leftrightarrow$ (2): It follows from [15, Theorem 26.1 (1)] that if A_{n+1} is a valuation domain, then $K_n = qf(A_n)$. Therefore, applying [11, Theorem 2.4 (1)] to the pullback $A_{n+1} = B_n \times_{K_n} A_n$ leads to the following result: A_{n+1} is a valuation domain if and only if B_n and A_n are valuation domains and $K_n = qf(A_n)$. Using this result repeatedly, we obtain both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

Before addressing integral closure, we settle (in Remark 2.9) one of the motivating questions mentioned in the Introduction. To this end, we need only the following sortie into factorization theory.

Proposition 2.8. *If* A_n *has no atoms* (i.e., *irreducible elements*) *for each* $n \in \mathsf{N}$, *then A has no atoms.*

Proof. Deny. Choose an atom $\alpha = (a_n)$ of A. As $\alpha \neq 0$, we can assume that $a_n \neq 0$ for each n (by passing to some $(a_n)_{n \geq m}$ and using cofinality). By Theorem 2.5 (c), a_n is a nonunit of A_n for each n. By hypothesis, a_1 is not an atom of A_1, and so $a_1 = b_1 c_1$ for some $b_1, c_1 \in A_1 \backslash \mathscr{U}(A_1)$. Pick $b_2^* \in \varphi_{2,1}^{-1}(b_1)$ and $c_2 \in \varphi_{2,1}^{-1}(c_1)$. Then $a_2 - b_2^* c_2 \in \ker(\varphi_{2,1}) = Q_{2,1}$. Now, $c_2 \notin Q_{2,1}$ since $c_1 \neq 0$. Since Proposition 2.4 (c) ensures that $Q_{2,1}$ is a divided prime ideal of

A_2, we have $Q_{2,1} \subseteq A_2c_2$, and so $a_2 - b_2^*c_2 = c_2d_2$ for some $d_2 \in A_2$. It follows that $a_2 = b_2c_2$, with $b_2 := b_2^* + d_2 \in A_2$. Since

$$0 = \varphi_{2,1}(c_2d_2) = \varphi_{2,1}(c_2)\varphi_{2,1}(d_2) = c_1\varphi_{2,1}(d_2)$$

and A_1 is an integral domain, $\varphi_{2,1}(d_2) = 0$. Therefore, $\varphi_{2,1}(b_2) = \varphi_{2,1}(b_2^*)+ \varphi_{2,1}(d_2) = b_1 + 0 = b_1$. Iterating the above argument, we find for each $n \in \mathsf{N}$, nonunits b_n and c_n of A_n such that $a_n = b_nc_n$, $\varphi_{n+1,n}(b_{n+1}) = b_n$, and $\varphi_{n+1,n}(c_{n+1}) = c_n$. Evidently, $\beta := (b_n)$ and $\gamma := (c_n)$ are in A; in fact, they are nonunits of A, by Theorem 2.5 (c). However, $\alpha = \beta\gamma$, contradicting that α is an atom of A.

Remark 2.9. Following [3], an integral domain D is an *antimatter domain* if D has no atoms. As in [6], D is a *fragmented domain* if, for each $d \in D \backslash \mathcal{U}(D)$, there exists $e \in D \backslash \mathcal{U}(D)$ such that $d \in \bigcap_{n=1}^{\infty} De^n$. Any fragmented domain is an antimatter domain, but the converse is false. Indeed, for each $n \in \mathsf{N}$, [3, Corollary 3.11 (b)] uses repeated Nagata compositions to produce an n-dimensional antimatter valuation domain A_n (therein denoted V_n); and A_n is not fragmented since quasilocal fragmented domains are either fields or infinite-dimensional [6, Corollary 2.8]. What about $A := \varprojlim A_n$? We see quickly that A is a valuation domain (by Theorem 2.1 (g)), $\dim(A) = \infty$ (by Corollary 2.6), and A is an antimatter domain (by Proposition 2.8). To answer the question left open in [3], we now show that A is not fragmented. Since A is a valuation domain, [6, Corollary 2.6] translates the problem to showing that M, the maximal ideal of A, is not unbranched in A; that is, that M is not the union of the nonmaximal prime ideals of A. In view of Theorem 2.5 (a), (b), this conclusion follows from the analogous fact about A_2.

Theorems 2.10 and 2.12 address the integral closure of A.

Theorem 2.10. *Assume that $K_n = qf(A_n)$ for each $n \in \mathsf{N}$. Then the following conditions are equivalent:*

(1) *For each $n \in \mathsf{N}$, A_n is integrally closed;*
(2) *A_1 is integrally closed and, for each $n \in \mathsf{N}$, B_n is integrally closed;*
(3) *A is integrally closed.*

Proof. If P is a divided prime ideal of an integral domain D, we have the pullback description $D = D_P \times_{D_P/P} D/P$. It follows (cf. [11, Corollary 1.5 (5)]) that D is integrally closed if and only if both D_P and D/P are integrally closed.

(3) $\Rightarrow$ (1): Assume (3) and fix $n \in \mathsf{N}$. By Proposition 2.4 (d), Q_n is a divided prime ideal of A. Hence, by the above remark, A/Q_n is integrally closed. Then (1) follows since $A/Q_n \cong A_n$ by Lemma 2.2 (b).

(1) $\Rightarrow$ (3): It suffices to show that if $\xi \in A'$, then $\xi \in A$. Write $\xi = ab^{-1}$, with $a \in A$ and $0 \neq b \in A$. By Lemma 2.2 (c), there exists $m \in \mathsf{N}$ such that $b \notin Q_m$. Since ξ is integral over A, it is easy to see that $\eta := (a + Q_m)(b + Q_m)^{-1} \in qf(A/Q_m)$ is integral over $A/Q_m \cong A_m$. Since A_m is integrally closed by (1), $\eta = c + Q_m$ for some $c \in A$. It follows that $q := a - bc \in Q_m$. As Q_m is a divided prime ideal of A (by Proposition 2.4 (d)) and $b \notin Q_m$, we have $Q_m \subseteq Ab$. In particular, $q \in Ab$, and so $qb^{-1} \in A$. Hence, $\xi = c + qb^{-1} \in A + A \subseteq A$.

(1) $\Leftrightarrow$ (2): Since $K_n = qf(A_n)$, the pullback $A_{n+1} = B_n \times_{K_n} A_n$ leads canonically to $M_n = \ker(\varphi_n) = \ker(\varphi_{n+1,n})$, $B_n = (A_{n+1})_{M_n}$, and $A_n = A_{n+1}/M_n$. Hence, by the first comment of the proof, A_{n+1} is integrally closed if and only if both B_n and A_n are integrally closed. The assertion now follows by induction on n.

It is of some technical interest to note that the proof of the equivalence (1) $\Leftrightarrow$ (3) in Theorem 2.10 did not use the hypothesis that $K_n = qf(A_n)$ for each n. Also, the following condition can be added in those in Theorem 2.10:

(4) *there exists $m \in \mathsf{N}$ such that A_n is integrally closed for each $n \geq m$.*

Lemma 2.11. *Let C_1 be an overring of A_1 which is not a field. For $n \in \mathsf{N}$, define C_{n+1} inductively by $C_{n+1} := B_n \times_{K_n} C_n = \varphi_n^{-1}(C_n)$. If $m \geq n$ in N, let $\varphi_{m,n} : C_m \to C_n$ denote the natural extension of $\varphi_{m,n} : A_m \to A_n$. Consider the inverse system generated by $\{C_1; (K_n, B_n) : n \in \mathsf{N}\}$, and put $C := \varprojlim C_n$. Then:*

(a) *For each $n \in \mathsf{N}$, $M_n = (A_{n+1} : B_n) = (C_{n+1} : B_n)$.*

(b) *If $m > n$ in N, then $Q_{m,n} = \ker(\varphi_{m,n} : A_m \to A_n) = \varphi_{m,n+1}^{-1}(M_n) = \ker(\varphi_{m,n} : C_m \to C_n)$.*

(c) *For each $n \in \mathsf{N}$, Q_n is a divided prime ideal of C.*

(d) *A and C have the same quotient field and the same complete integral closure.*

Proof. The assertions in (a) and (b) follow easily from Proposition 2.4 (a) since $M_n \neq 0$. Moreover, replacing A_1 with C_1 in the earlier discussion, we see that (c) follows from (b) and Proposition 2.4 (b), (d). Finally, since Proposition 2.4 (a) ensures that $Q_1 \neq 0$, (d) follows since A and C share a common nonzero ideal (cf. [15, Theorem 13.1 (3)]).

Theorem 2.12. *Assume that for each $n \in \mathsf{N}$, B_n is integrally closed and $K_n = qf(A_n)$. Then A' is the inverse limit of the integral closures A_n' of A_n; that is, $(\varprojlim A_n)' = \varprojlim A_n'$.*

Proof. Since $B_n' = B_n$ by hypothesis, it follows that $A_{n+1}' \subseteq B_n$ and (cf. [11, Corollary 1.5 (5)]) $A_{n+1}' = B_n \times_{K_n} A_n'$. As in Lemma 2.11, consider the inverse system generated by $\{A_1'; (K_n, B_n) : n \in \mathsf{N}\}$. (The earlier theory applies since, by integrality, A_1' inherits from A_1 the property of not being a field.) Put $R := \varprojlim A_n'$. By Theorem 2.10, R is integrally closed; and by Lemma 2.11 (d), A and R have the same quotient field. As $\varprojlim$ preserves inclusions, $A = \varprojlim A_n \subseteq \varprojlim A_n' = R$, and so it suffices to prove that R is integral over A.

Let $x = (x_n) \in R \subseteq \prod A_n'$. Since $x_1 \in A_1'$, we have

$$x_1^r + a_{r-1}^{(1)} x_1^{r-1} + \cdots + a_1^{(1)} x_1 + a_0^{(1)} = 0$$

for some $r \in \mathsf{N}$ and elements $a_k^{(1)} \in A_1$. For each $n \in \mathsf{N}$, if $0 \leq k \leq r-1$, choose $b_k^{(n)} \in \varphi_{n,1}^{-1}(a_k^{(1)}) \in A_n$; of course, $b_k^{(1)} = a_k^{(1)}$. For $n = 2$, put

$$y_2 := x_2^r + b_{r-1}^{(2)} x_2^{r-1} + \cdots + b_1^{(2)} x_2 + b_0^{(2)} \in A_2'.$$

As $\varphi_{2,1}$ is a homomorphism and $\varphi_{2,1}(x_2) = x_1$, we have $\varphi_{2,1}(y_2) = 0$. Hence

$$c_0^{(2)} := -y_2 \in \ker(\varphi_{2,1}) \subseteq A_2$$

satisfies $x_2^r + b_{r-1}^{(2)} x_2^{r-1} + \cdots + b_1^{(2)} x_2 + (b_0^{(2)} + c_0^{(2)}) = 0$.

Since $c_0^{(2)} \in A_2$, we have $a_0^{(2)} := b_0^{(2)} + c_0^{(2)} \in A_2$. Also, if $1 \leq k \leq r-1$, then $a_k^{(2)} := b_k^{(2)} \in A_2$. Repeating the above argument, we find elements $a_k^{(3)} \in A_3$, $0 \leq k \leq r-1$, such that

$$x_3^r + a_{r-1}^{(3)} x_3^{r-1} + \cdots + a_1^{(3)} x_3 + a_0^{(3)} = 0$$

and $\varphi_{3,2}(a_k^{(3)}) = a_k^{(2)}$ for all k. Iterating the argument, we thus produce $a_k = (a_k^{(n)}) \in A$, $0 \leq k \leq r-1$, such that for each $n \in \mathsf{N}$,

$$x_n^r + a_{r-1}^{(n)} x_n^{r-1} + \cdots + a_1^{(n)} x_n + a_0^{(n)} = 0 \ .$$

Therefore, $x^r + a_{r-1} x^{r-1} + \cdots + a_1 x + a_0 = 0$; that is, x is integral over A.

In explaining that A_1' is not a field, the proof of Theorem 2.12 appealed to the lying-over theorem (cf. [19, Theorem 44]). It is natural to ask if LO (as well as GU, GD and INC) satisfies an analogue of Theorem 2.12; in particular, if (the above type of) $\varprojlim$ preserves LO. Our next result gives an affirmative answer. It is convenient to consider also the following property which was introduced in [21]. Recall that an inclusion $f : D \to E$ of integral domains is an *i-extension* if $\mathrm{Spec}\,(f) : \mathrm{Spec}\,(E) \to \mathrm{Spec}\,(D)$ is an injection.

PROPOSITION 2.13. *As above, consider the inverse system generated by* $\{A_1; (K_n, B_n) : n \in \mathsf{N}\}$, *with* $A = \varprojlim A_n$. *Consider a(nother) inverse system generated by* $\{A_{*1}; (K_{*n}, B_{*n}) : n \in \mathsf{N}\}$, *with* $A_* = \varprojlim A_{*n}$; *in the latter system, denote structures* $\varphi_{*m,n}$, Φ_{*n}, $Q_{*r,n}$, Q_{*n} *analogously to the corresponding "unstarred" structures in the former system. For each* $n \in \mathsf{N}$, *suppose given an injective ring-homomorphism* $h_n : A_n \to A_{*n}$; *and suppose that* $\varphi_{*(n+1),n} \circ h_{n+1} =$

$h_n \circ \varphi_{n+1,n}$ for each $n \in \mathsf{N}$. Let $h : A \to A_*$ be the induced injective ring-homomorphism. Then:

(a) *Let $Q \in \operatorname{Spec}(A_*)$ and $P := h^{-1}(Q) = Q \cap A \in \operatorname{Spec}(A)$. Then $Q = 0$ if and only if $P = 0$.*

(b) *Fix $n \in \mathsf{N}$. Consider $Q \in \operatorname{Spec}(A_{*n}) \subseteq \operatorname{Spec}(A_*)$ and $P \in \operatorname{Spec}(A_n) \subseteq \operatorname{Spec}(A)$. Then $h_n^{-1}(Q) = P$ if and only if $h^{-1}(Q) = P$.*

(c) *Let $\mathscr{P}$ be one of the following five properties: LO, GU, GD, INC, "(is an) i-extension". If h_n satisfies $\mathscr{P}$ for each $n \in \mathsf{N}$, then h satisfies $\mathscr{P}$.*

Proof. (a) It is useful to view h and the maps h_n as inclusions occasionally, and the notation "$Q \cap A$" in the statement of (a) is interpreted in this sense. (Of course, h is an injection since $\varprojlim$ preserves injections.) Now, of course, $0 \cap A = 0$. Thus, it suffices to show that if $Q \neq 0$, then $P \neq 0$. By Theorem 2.5 (a), we can view $Q \in \operatorname{Spec}(A_{*n})$ for some $n \in \mathsf{N}$. Since $\Phi_{*n} \circ h = h_n \circ \Phi_n : A \to A_{*n}$, functoriality of Spec gives a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(A_{*n}) & \xrightarrow{\operatorname{Spec}(\Phi_{*n})} & \operatorname{Spec}(A_*) \\ {\scriptstyle \operatorname{Spec}(h_n)}\downarrow & & \downarrow{\scriptstyle \operatorname{Spec}(h)} \\ \operatorname{Spec}(A_n) & \xrightarrow{\operatorname{Spec}(\Phi_n)} & \operatorname{Spec}(A) \end{array}$$

whose horizontal maps are viewed as inclusions. Chasing Q through this diagram, we find that $P = h^{-1}(Q)$ is (the canonical image of) $h_n^{-1}(Q) \in \operatorname{Spec}(A_n)$. By Theorem 2.5 (a), $h_n^{-1}(Q) \neq 0 \in \operatorname{Spec}(A)$, and so $P \neq 0$.

(b) This follows from the above commutative-diagram argument.

(c) All five proofs are similar. We give the proof for $\mathscr{P} = \text{GU}$ and leave the other proofs to the reader. Thus, we suppose given primes $P_1 \subseteq P_2$ in $\operatorname{Spec}(A)$ and $P_{*1} \in \operatorname{Spec}(A_*)$ such that $h^{-1}(P_{*1}) = P_1$, and we seek a prime $P_{*2} \in \operatorname{Spec}(A_*)$ such that $h^{-1}(P_{*2}) = P_2$ and $P_{*1} \subseteq P_{*2}$. Without loss of generality, $P_1 \neq P_2$.

Case 1: $P_1 = 0$. By (a), $P_{*1} = 0$. It suffices to find $P_{*2} \in \operatorname{Spec}(A_*)$ such that $h^{-1}(P_{*2}) = P_2$. Using Theorem 2.5 (a), choose $n \in \mathsf{N}$ such that $P_2 \in \operatorname{Spec}(A_n)$. Since h_n is an injection which satisfies GU, [19, Theorem 42] ensures that h_n satisfies LO, and so there exists $P_{*2} \in \operatorname{Spec}(A_{*n})$ such that $h_n^{-1}(P_{*2}) = P_2$. Then, by (b), $h^{-1}(P_{*2}) = P_2$.

Case 2: $P_1 \neq 0$. By (a), $P_{*1} \neq 0$. By applying Theorem 2.5 (a) three times and choosing the maximal of three subscripts, we find $n \in \mathsf{N}$ such that $P_1, P_2 \in \operatorname{Spec}(A_n)$ and $P_{*1} \in \operatorname{Spec}(A_{*n})$. Next, note that the injections $\operatorname{Spec}(\Phi_n)$ and $\operatorname{Spec}(\Phi_{*n})$ both preserve and reflect order (that is, inclusions of prime ideals). Thus, $P_1 \subseteq P_2$ when viewed in Spec (A_n), and so, in view of (b), it suffices to find P_{*2} in Spec (A_n) such that $P_{*1} \subseteq P_{*2}$ when viewed in Spec (A_{*n}) and $h_n^{-1}(P_{*2}) = P_2$. This, in turn, is accomplished since h_n satisfies GU.

Recall from [21] that an integral domain D is an *i-domain* in case the inclusion map $D \to E$ is an i-extension for each overring E of D. Evidently, being an i-domain is a local property of integral domains. It was shown in [21, Proposition 2.14 and Corollary 2.15] that an integral domain D is an i-domain if and only if $D \subseteq D'$ is an i-extension and D' is a Prüfer domain; and that an integral domain D is a quasilocal i-domain if and only if D' is a valuation domain.

Proposition 2.14. *Assume that for each $n \in \mathsf{N}$, B_n is integrally closed and $K_n = qf(A_n)$. Then the following conditions are equivalent:*

(1) *For each $n \in \mathsf{N}$, A_n is a quasilocal i-domain;*

(2) *A_1 is a quasilocal i-domain and, for each $n \in \mathsf{N}$, B_n is a valuation domain;*

(3) *A is a quasilocal i-domain.*

Proof. By Theorem 2.5 (b), A is quasilocal $\Leftrightarrow A_n$ is quasilocal for each $n \Leftrightarrow A_1$ is quasilocal. Thus, we may assume henceforth that each A_n is quasilocal. By Lemma 2.11 and Theorem 2.12, we have $A' = \varprojlim A'_n$, with $A'_n = A'/Q_n$ for each $n \in \mathsf{N}$ (cf. [11, Corollary 1.5 (5)]). In particular, if A' is a valuation domain, then so is each A'_n. Thus, in view of the material recalled from [21], (3) $\Rightarrow$ (1). On the other hand, (1) $\Rightarrow$ (3) by Theorem 2.12 and Theorem 2.1 (g), for (1) leads to A' being an inverse limit of valuation domains. By Corollary 2.7, if A'_n is a valuation domain for each n, then B_n is a valuation domain for each n. Consequently, (1) $\Rightarrow$ (2). Finally, to show that (2) $\Rightarrow$ (1), assume (2), observe that A'_1 is a valuation domain, and use Corollary 2.7 to conclude that A' is a valuation domain.

The above material leads one naturally to ask if (the ambient type of) $\varprojlim$ preserves Prüfer domains or (not necessarily quasilocal) i-domains. For this reason, we next address localizations of inverse limits. As usual, if D is an integral domain and $P \in \operatorname{Spec}(D)$, it is convenient to let $\mathsf{k}_D(P)$ denote $qf(D/P) = D_P/PD_P$.

Proposition 2.15. *Let $0 \neq P \in \operatorname{Spec}(A)$. Choose $m \in \mathsf{N}$ such that $P \in \operatorname{Spec}(A_m)$; thus, $P \supseteq Q_m$. (Such m exists by Theorem 2.5 (a).) Then for each $n \geq m$ in N, $P_n := \Phi_n(P)$ is a prime ideal of A_n. Then:*

(a) *$P \cong \varprojlim\{P_n : n \geq m\}$.*

(b) *$A/P \cong A_n/P_n$ for each $n \geq m$, and so $A/P \cong \varprojlim\{A_n/P_n : n \geq m\}$.*

(c) $\mathsf{k}_A(P) \cong \mathsf{k}_{A_n}(P_n)$ *for each* $n \geq m$*, and so* $\mathsf{k}_A(P) \cong \varprojlim\{\mathsf{k}_{A_n}(P_n) : n \geq m\}$.

(d) $A_P \cong \varprojlim\{(A_n)_{P_n} : n \geq m\}$*, the inverse limit of the inverse system generated by* $\{(A_m)_{P_m}; (K_n, B_n) : n \geq m\}$.

Proof. (a) Observe that $P_m = P$. Using Theorem 2.1 (d) and Lemma 2.2 (b), notice that $\varphi_{r,s}^{-1}(P_s) = P_r$ if $r \geq s \geq m$. It follows that $P \hookrightarrow A = \varprojlim\{A_n : n \geq m\}$ factors through $\varprojlim\{P_n : n \geq m\}$, and (a) follows.

(b), (c): Notice that $P_n \in \operatorname{Spec}(A)$ is identified with $P/Q_n \in \operatorname{Spec}(A/Q_n) = \operatorname{Spec}(A_n)$ for each $n \geq m$. Then $A/P \cong A_n/P_n$ by a standard isomorphism theorem. These isomorphisms are compatible with the isomorphisms $A_{n+1}/P_{n+1} \xrightarrow{\cong} A_n/P_n$ induced by $\varphi_{n+1,n}$, and (b), (c) follow easily.

(d) If $n \geq m$, we identify $\Phi_{n+2}(P) = P \supseteq Q_m \supseteq Q_{n+1}$, and so Proposition 2.4 (a) leads to

$$P \supseteq \Phi_{n+2}(Q_{n+1}) = Q_{n+2,n+1} = M_n.$$

Thus, by the universal mapping property of localizations, the inclusion map $A_{n+1} \to B_n$ extends to an inclusion $(A_{n+1})_{P_{n+1}} \to B_n$. Moreover, since $\varphi_{n+1,n}(A_{n+1} \backslash P_{n+1}) = A_n \backslash P_n$, the surjection $\varphi_{n+1,n}$ induces a surjection $(A_{n+1})_{P_{n+1}} \to (A_n)_{P_n}$. In particular, $\varphi_n^{-1}((A_n)_{P_n}) = (A_{n+1})_{P_{n+1}}$ and we have a pullback description $(A_{n+1})_{P_{n+1}} = B_n \times_{K_n} (A_n)_{P_n}$. Let m be increased, if necessary, so that P properly contains Q_m. Then we can let $R := \varprojlim (A_n)_{P_n}$, the inverse limit of the inverse system generated by $\{(A_m)_{P_m}; (K_n, B_n) : n \geq m\}$. By Lemma 2.11 (d), R has he same quotient field, say K, as A (and, hence, the same as A_P). Moreover, the universal mapping property of localization gives ring-homomorphisms $A_P \to (A_n)_{P_n}$ which, in view of the universal mapping property of $\varprojlim$, lead to a ring-homomorphism $A_P \to R$. As this map is evidently injective, we view it as an inclusion. It remains only to prove that $A_P = R$. To this end, observe via (a) that

$$\begin{aligned} A_P =& \{ab^{-1} \in K : a \in A, b \in A\backslash P\} = \{(a_n : n \geq 1)(b_n : n \geq 1)^{-1} \in K : \\ & a = (a_n) \in A, \quad b = (b_n) \in A, \quad b_n \in A_n \backslash P_n \text{ for each } n \geq m(b) \geq m \geq 1\} \\ & \supseteq \varprojlim\{(A_n)_{P_n} : n \geq m\} = R\ . \end{aligned}$$

Corollary 2.16. *Let* $0 \neq P \in \operatorname{Spec}(A)$*. As in Proposition 2.15, choose* $m \in \mathsf{N}$ *such that* $P \in \operatorname{Spec}(A_m)$ *properly contains* Q_m*, and put* $P_n := \Phi_n(P)$ *for each* $n \geq m$ *in* N*. Then* P *is a divided prime ideal of* A *if and only if* P_n *is a divided prime ideal of* A_n *for each (resp., some)* $n \geq m$*.*

Proof. If Q is a divided prime ideal of an integral domain D and Q contains $I \in \operatorname{Spec}(D)$, then Q/I is a divided prime ideal of D/I (cf. [5, Lemma 2.2 (c)]). Thus, if P is divided in A and $n \geq m$ in N, then $P_n \cong P/Q_n$ is divided in $A/Q_n \cong A_n$. For the converse, suppose that P_m is divided in A_m; that is, $A_m = (A_m)_{P_m} \times_{k_{A_m}(P_m)} A_m/P_m$. Now, by the proof of Proposition 2.15 (d), $(A_{n+1})_{P_{n+1}} = B_n \times_{K_n} (A_n)_{P_n}$ for each $n \geq m$. It follows from the pullback description of A_n that for each $n \geq m$, we have a canonical pullback diagram

$$\begin{array}{ccc} A_{n+1} & \twoheadrightarrow & A_n \\ \downarrow & & \downarrow \\ (A_{n+1})_{P_{n+1}} & \twoheadrightarrow & (A_n)_{P_n} \end{array} .$$

In cases like this, $\varprojlim$ commutes with pullbacks [18, Theorem 5.2, page 277]. Thus, by Proposition 2.15 (d), we have, with $n \geq m$, a pullback diagram

$$\begin{array}{ccc} A = \varprojlim A_n & \twoheadrightarrow & A_m \\ \downarrow & & \downarrow \\ A_P & \twoheadrightarrow & (A_m)_{P_m} \end{array} .$$

Since P_m is divided in A_m, Proposition 2.15 (b), (c) gives a pullback diagram

$$\begin{array}{ccc} A_m & \twoheadrightarrow & A_m/P_m \cong A/P \\ \downarrow & & \downarrow \\ (A_m)_{P_m} & \twoheadrightarrow & k_{A_m}(P_m) \cong k_A(P) \end{array}$$

Juxtaposing the last two pullback diagrams, we conclude that $A = A_P \times_{k_A(P)} A/P$; that is, P is divided in A.

In Corollary 2.16, it may be shown directly, using the dividedness of $Q_{n,m}$ (Proposition 2.4 (c)), that if P_m is divided, then so is P_n for each $n \geq m$.

COROLLARY 2.17. (a) *The following two conditions are equivalent:*

(1) *For each $n \in \mathbb{N}$, A_n is a divided domain;*

(2) *A is a divided domain.*

Moreover, if $K_n = qf(A_n)$ for each n, then (1) *and* (2) *are each equivalent to*

(3) *A_1 is a divided domain and, for each $n \in \mathbb{N}$, B_n is a divided domain.*

(b) *The assertions in* (a) *are valid if "divided" is replaced throughout by "locally divided".*

(c) *Suppose that for all $n \in \mathbb{N}$, B_n is integrally closed and $K_n = qf(A_n)$. Then the following conditions are equivalent:*

(1) *For each $n \in \mathsf{N}$, A_n is an i-domain;*
(2) *A is an i-domain;*
(3) *A_1 is an i-domain and, for each $n \in \mathsf{N}$, B_n is a valuation domain.*

Proof. (a) Since each A_n is (isomorphic to) a factor domain of A, (2) $\Rightarrow$ (1) by [5, Lemma 2.2 (c)]. Since the zero prime ideal is divided in any integral domain, (1) $\Rightarrow$ (2) follows from the "if" assertion in Corollary 2.16. Next, suppose that $K_n = qf(A_n)$ for each $n \in \mathsf{N}$. Since $A_{n+1} = B_n \times_{K_n} A_n$, it can be shown that A_{n+1} is a divided domain if and only if both B_n and A_n are divided domains. (This assertion is essentially a translation of the first assertion in [7, Proposition 2.12].) It is now evident that (3) $\Leftrightarrow$ (1).

(b) Any factor domain of a locally divided domain is locally divided [5]. Hence, (1) $\Leftrightarrow$ (2) by (a) and Proposition 2.15 (d). For a proof that (3) $\Leftrightarrow$ (1), assuming that $K_n = qf(A_n)$ for each n, use the following translation of the second assertion in [7, Proposition 2.12]: A_{n+1} is locally divided if and only if both B_n and A_n are locally divided.

(c) (2) $\Rightarrow$ (1) since D/P is an *i*-domain whenever P is a (divided) prime ideal of an *i*-domain D [21, p. 3]. Next, since being an *i*-domain is a local property, (1) $\Rightarrow$ (2) follows by combining Propositions 2.15 and 2.14. A similar combination yields (1) $\Leftrightarrow$ (3), once we notice that A has a nonzero prime ideal (cf. Theorem 2.5 (b)).

The next result will be used in determining when A is a $P^r VD$.

Corollary 2.18. Fix $n \in \mathsf{N}$, and suppose that $K_m = qf(A_m)$ for each $m \geq n$ in N. For $m \geq n+1$, inductively define $(B_n^\bullet)_m$ by

$$(B_n^\bullet)_{n+1} := \varphi_{n+1}^{-1}(B_n) \subseteq B_{n+1} \ ,$$
$$(B_n^\bullet)_{m+1} := \varphi_{m+1}^{-1}((B_n^\bullet)_m) \subseteq B_{m+1} \text{ if } m \geq n+1 \ .$$

Consider the inverse system generated by $\{(B_n^\bullet)_{n+1}; (K_m, B_m) : m \geq n+1\}$, and put $B_n^\bullet = \varprojlim\{(B_n^\bullet)_m : m \geq n+1\}$. Then $B_n^\bullet = A_{Q_n}$.

Proof. The inverse system in question satisfies our riding hypotheses. (The only issue may concern whether $(B_n^\bullet)_{n+1}$ is a field; if so, formally introduce $(B_n^\bullet)_n := B_n$, which is not a field.) Now, by Proposition 2.4 (a), $\Phi_{n+1}(Q_n) = M_n$; and, since $K_n = qf(A_n)$, $(A_{n+1})_{\Phi_{n+1}(Q_n)} = (B_n^\bullet)_{n+1}$ (cf. also [14, Lemma 1.1.6]). Similarly, one shows by induction on m that $(A_m)_{\Phi_m(Q_n)} = (B_n^\bullet)_m$ for each $m \geq n+1$. Now, since Proposition 2.4 (a) ensures that $Q_n \neq 0$, Proposition 2.15 (d) may be applied to $P := Q_n$ and the above inverse system, with the result that

$$B_n^\bullet = \varprojlim (B_n^\bullet)_m = \varprojlim (A_m)_{\Phi_m(Q_n)} = A_{Q_n}.$$

Recall from [17] that an integral domain D is a *pseudo-valuation domain* (PVD) if D has a valuation overring V such that $\mathrm{Spec}(D) = \mathrm{Spec}(V)$ as sets. Any PVD is quasilocal and, in fact, a divided domain. It is useful to recall from [17, Proposition 2.6] that if D is a PVD and $P \in \mathrm{Spec}(D)$ is non-maximal, then D_P is a valuation domain; and from [2, Proposition 2.6] that D is a PVD if and only if $D = V \times_k F$, where V is a valuation domain with residue field k and F is a subfield of k.

From [12, Définition 1.2 and Théorème 1.3] recall that a *pseudo–valuation domain of type* r ($P^r VD$) can be defined, by induction on $r \geq 0$, in the following way. A $P^0 VD$ is a PVD and, for $r \geq 1$, a $P^r VD$, D, is defined by a pullback diagram of the following type:

$$\begin{array}{ccc} D := \alpha^{-1}(C) & \longrightarrow & C \\ \downarrow & & \downarrow \\ R & \xrightarrow{\ \alpha\ } & F := R/N \end{array}$$

where R is a $P^{r-1} VD$, with maximal ideal N, F is its residue field, α is the canonical projection and C is a PVD with field of quotients isomorphic to F. For instance if k is a field and $r \geq 0$, then

$$k + X_1 k((Y_1))[[X_1]] + X_2 k((X_1, Y_1))[[X_2]] + \cdots + X_{r+1} k((X_1, Y_1, \ldots, X_r, Y_r))[[X_{r+1}]]$$

is a $P^r VD$.

PROPOSITION 2.19. *Let $0 \leq r \in \mathbb{Z}$. Then the following conditions are equivalent:*

(1) *There exists $m \in \mathbb{N}$ such that, for each $n \geq m$ in $\mathbb{N}$, A_n is a $P^r VD$;*

(2) *There exists $m \in \mathbb{N}$ such that A_m is a $P^r VD$ and, for each $n \geq m$ in $\mathbb{N}$, B_n is a valuation domain and $K_n = qf(A_n)$;*

(3) *A is a $P^r VD$.*

PROOF. (3) $\Leftrightarrow$ (2): By Proposition 2.4 (d), $\{Q_n : n \geq 1\}$ is a family of divided prime ideals of A. Using the definition of a $P^r VD$ (cf. also [12, Théorème 1.3]), we see that A is a $P^r VD$ if and only if there exists $m \in \mathbb{N}$ such that A/Q_m is a $P^r VD$ and A_{Q_m} is a valuation domain. Of course, $A/Q_m \cong A_m$. Now, if $K_n = qf(A_n)$ for each $n \geq m$, Corollary 2.18 gives that

$$A_{Q_m} = \varprojlim \{(B_m^\bullet)_k : k \geq m+1\} = B_m^\bullet .$$

Therefore, applying Corollary 2.7 to the construction in Corollary 2.18, we see that A_{Q_m} is a valuation domain if and only if, for each $k \geq m$, B_k is a valuation domain and $K_{k+1} = qf(B_k)$. Thus, (2) $\Rightarrow$ (3). Moreover, by [12, Théorème 1.3] (3) implies that there exists $m \in \mathsf{N}$ such that A_m is a $P^r VD$ and, for each $n \geq m$, that $K_n = qf(A_n)$ and so (3) $\Rightarrow$ (2).

(2) $\Leftrightarrow$ (1): Given $K_n = qf(A_n)$, we see from the pullback $A_{n+1} = B_n \times_{K_n} A_n$ that A_{n+1} is a $P^r VD$ if and only if A_n is a $P^r VD$ and B_n is a valuation domain (cf. [12, Théorème 1.3]). Thus, (2) $\Rightarrow$ (1). Moreover, in view of the above comments, (1) $\Rightarrow$ (2).

The next result is the PVD-theoretic analogue of Corollary 2.7 or of Proposition 2.19.

Corollary 2.20. *The following conditions are equivalent:*

(1) *For each $n \in \mathsf{N}$, A_n is a PVD;*

(2) *A_1 is a PVD and, for each $n \in \mathsf{N}$, B_n is a valuation domain and $K_n = qf(A_n)$;*

(3) *A is a PVD.*

Proof. It is well known that any factor domain of a PVD is a PVD. One may use this fact, together with the material recalled above, to fashion a proof of Corollary 2.20. Alternatively, the first sentence of this proof may be combined with the case $r = 0$ of the proof of Proposition 2.19; the upshot is another proof of Corollary 2.20, since a $P^0 VD$ is the same as a PVD.

We next present the promised "globalization" of Corollary 2.7. Its proof depends on Corollary 2.7 in much the same way that Corollary 2.17 (b) was proved using Corollary 2.17 (a).

Theorem 2.21. *The following conditions are equivalent:*

(1) *For each $n \in \mathsf{N}$, A_n is a Prüfer domain;*

(2) *A_1 is a Prüfer domain and, for each $n \in \mathsf{N}$, B_n is a valuation domain and $K_n = qf(A_n)$;*

(3) *A is a Prüfer domain.*

Proof. (3) $\Rightarrow$ (1) since $A_n \cong A/Q_n$ and factor domains of Prüfer domains are Prüfer domains [15, Proposition 22.5]. For (1) $\Rightarrow$ (3), assume (1); it suffices to show that A_P is a valuation domain for each $0 \neq P \in \mathrm{Spec}\,(A)$. By Proposition 2.15 (d), $A_P = \varprojlim (A_n)_{P_n}$ for some $P_n \in \mathrm{Spec}\,(A_n)$, considering all $n \geq m = m(P)$. By (1), each $(A_n)_{P_n}$ is a valuation domain and so, by Theorem 2.1 (c), A_P is also a valuation domain.

Since $A_{n+1} = B_n \times_{K_n} A_n$, it follows from well-known material on pullbacks (cf. [11, Theorem 2.4 (3)]) that A_{n+1} is a Prüfer domain if and only if A_n is a Prüfer domain, B_n is a (quasilocal Prüfer, that is) valuation domain, and $K_n = qf(A_n)$. Accordingly, (2) $\Leftrightarrow$ (1).

Prüfer domains are the integrally closed i-domains (cf. [15]). In view of Theorem 2.21, Corollary 2.17 (c) and Proposition 2.13 (c), we are led to consider integral domains D such that the inclusion map $D \to E$ satisfies INC for each overring E of D. According to [21, Proposition 2.26], these are the integral domains D for which D' is a Prüfer domain. These integral domains D were called *quasi-Prüfer domains* by Ayache, Cahen and Echi; they have been studied extensively in [14].

Corollary 2.22. *Suppose that for all $n \in \mathbb{N}$, B_n is integrally closed and $K_n = qf(A_n)$. Then the following conditions are equivalent:*

(1) *For each $n \in \mathbb{N}$, A_n is a quasi-Prüfer domain;*

(2) *A_1 is a quasi-Prüfer domain and, for each $n \in \mathbb{N}$, B_n is a valuation domain;*

(3) *A is a quasi-Prüfer domain.*

Proof. (3) $\Rightarrow$ (1) since $A_n \cong A/Q_n$ and D/P is a quasi-Prüfer domain whenever P is a (divided) prime ideal of a quasi-Prüfer domain D [14, Proposition 6.5.1]. For (1) $\Rightarrow$ (3), combine Theorems 2.12 and 2.21 [(1) $\Rightarrow$ (3)]. Finally, (2) $\Rightarrow$ (1) follows from the result that (if B_n is integrally closed and quasilocal, then) $A_{n+1} = B_n \times_{K_n} A_n$ is a quasi-Prüfer domain if and only if A_n is a quasi-Prüfer domain, B_n is a valuation domain, and $K_n = qf(A_n)$ [14, Corollary 1.1.9 (1)].

We next consider when A is a Bézout domain. Recall that an integral domain D is a *Bézout domain* if each nonzero finitely generated ideal of D is principal. Each Bézout domain is a Prüfer domain, but the converse is false (cf. [15]).

Corollary 2.23. *The following conditions are equivalent:*

(1) *For each $n \in \mathbb{N}$, A_n is a Bézout domain;*

(2) *A_1 is a Bézout domain and, for each $n \in \mathbb{N}$, B_n is a valuation domain and $K_n = qf(A_n)$;*

(3) *A is a Bézout domain.*

Proof. It follows from [13, Theorem 4.2 (c)] that if B^* is a quasilocal integral domain with residue field K^* and A^* is a (proper) subring of K^*, then $A^{**} := B^* \times_{K^*} A^*$ is a Bézout domain if and only if A^* is a Bézout domain, B^* is a valuation domain, and $K^* = qf(A^*)$. Thus, since $A_{n+1} = B_n \times_{K_n} A_n$, we see that A_{n+1} is a Bézout domain if and only if A_n is a Bézout domain, B_n is a valuation domain, and $K_n = qf(A_n)$. Therefore, (2) $\Rightarrow$ (1). Moreover, since any factor domain of a Bézout domain is a Bézout domain [15], we also have that (1) $\Rightarrow$ (2) and (in light of Theorem 2.21 [(3) $\Rightarrow$ (2)]) that (3) $\Rightarrow$ (2). It remains to show that [(1) and (2)] imply (3). Henceforth, assume (1) and (2); in particular, $K_n = qf(A_n)$ for each n.

Since $A_1 \cong A/Q_1$ and Q_1 is a divided prime ideal of A (see Lemma 2.2 (b) and Proposition 2.4 (d)), applying the above upshot of [13, Theorem 4.2 (c)] to the pullback $A = A_{Q_1} \times_{\mathsf{k}_A(Q_1)} A/Q_1$ yields the following conclusion: A is a Bézout domain if and only if A_1 is a Bézout domain and A_{Q_1} is a valuation domain. By (2), A_1 is a Bézout domain, and so we need only prove that A_{Q_1} is a valuation domain. This, in turn, follows since A is a Prüfer domain by Theorem 2.21 [(1) or (2) implies (3)].

Remark 2.24. (a) In view of the motivation provided by [10], Theorem 2.21 [(1) $\Rightarrow$ (3)] is an appealing result. It is instructive to consider a direct attempt at proving that $A = \varprojlim A_n$ is a Prüfer domain, assuming only that each A_n is a Prüfer domain and $\varphi_{m,n} : A_m \to A_n$ is surjective for each $m \geq n$ in N. Using the criterion for Prüfer domains that "each nonzero finitely generated ideal is invertible", one can construct a proof, somewhat in the spirit of that of Theorem 2.12, if one is free to use two facts: $\Phi_n : A \to A_n$ is surjective for each $n \in \mathsf{N}$; and $Q_{m,n} := \ker(\varphi_{m,n})$ is a divided prime ideal of A_m for each $m \geq n$ in N. The first of these "facts" is valid, but the second "fact" was shown in Proposition 2.4 (c) as a result of our riding hypotheses. Thus, we have sketched an alternate proof of Theorem 2.21 [(1) $\Rightarrow$ (3)]. More importantly, we have found a new way to motivate those riding hypotheses, for dividedness of the ideals $Q_{n+1,n}$ allows one to recover $\{(B_n, K_n) : n \in \mathsf{N}\}$ as in our riding hypotheses, assuming only that A_1 is not a field and $\varphi_{n+1,n}$ is not an isomorphism. Indeed, taking $B_n := (A_{n+1})_{Q_{n+1,n}}$ and $K_n = \mathsf{k}_{A_{n+1}}(Q_{n+1,n})$ leads to data satisfying our riding hypotheses. One moral is that both of our approaches to Theorem 2.21 [(1) $\Rightarrow$ (3)] naturally lead to consideration of divided prime ideals.

(b) Although we have determined when A (under our riding hypotheses) belongs to several important classes of integral domains, additional such results are possible. Analogously to Corollary 2.23, one could use part (b), rather than part (c), of [13, Theorem 4.2] to determine when A is a G-GCD domain (in the sense of [1]). Similarly, [13, Theorem 4.1] can be used to characterize when A is a Prüfer v-multiplication domain ($PVMD$). We leave the details to the interested reader.

In closing, we consider one of the motivations for much of [10] and the above work. Recall (cf. [4]) that an integral domain D is a *going-down domain* if the inclusion map $D \to E$ satisfies GD for each integral domain E containing D (equivalently, for each overring E of D). Many of the above-considered types of integral domains are examples of going-down domains. (For example, locally divided domains and i-domains are going-down domains; hence, so are valuation domains, divided domains, $P^r VDs$, $PVDs$, Prüfer domains, and Bézout domains: a redundant list which corresponds to the order in which the concepts appeared above. On the other hand, a quasi-Prüfer domain need not be a going-down domain; the same holds for a G-GCD domain and a $PVMD$.) Being a going-down domain is a local property of integral domains (cf. [5]). Any locally divided domain is a going-down domain [5, Remark 2.7 (b)]; the converse holds for integral domains which are seminormal (in the sense of [23]), essentially by the proof of [5, Theorem 2.5].

Proposition 2.25. *Consider the following three conditions:*

(1) *For each $n \in \mathsf{N}$, A_n is a going-down domain;*

(2) *A_1 is a going-down domain and, for each $n \in \mathsf{N}$, B_n is a going-down domain;*

(3) *A is a going-down domain.*

Then:

(a) (3) $\Rightarrow$ (1).

(b) *Assume that $K_n = qf(A_n)$ for each $n \in \mathsf{N}$. Then* (1) $\Leftrightarrow$ (2).

(c) *Assume that for each $n \in \mathsf{N}$, $K_n = qf(A_n)$, B_n is seminormal and A_1 is seminormal. Then* (1) $\Rightarrow$ (3).

(d) *Assume that for each $n \in \mathsf{N}$, $K_n = qf(A_n)$ and B_n is integrally closed. Then* (1) $\Rightarrow$ (3).

Proof. Since any factor domain of a going-down domain is a going-down domain [5, Remark 2.11 and Remark 3.2 (a), (b)], we have that (3) $\Rightarrow$ (1), giving (a). Assume henceforth that $K_n = qf(A_n)$ for each $n \in \mathsf{N}$. Then, by translating [7, Corollary 2.3] and applying it to the pullback $A_{n+1} = B_n \times_{K_n} A_n$, we see that A_{n+1} is a going-down domain if and only if both A_n and B_n are going-down domains. Then (b) follows easily.

(c) Notice that A_{n+1} is seminormal if and only if both A_n and B_n are seminormal. Then, given (1) and the assumptions in (c), we have that A_n is a seminormal going-down domain for each $n \in \mathsf{N}$. By the above remarks, A_n is locally divided for each $n \in \mathsf{N}$ and so, by Corollary 2.17 (b), A is a locally divided domain. In particular, A is a going-down domain, yielding (3) and completing the proof of (c).

(d) It is enough to show that A_P is a going-down domain for each maximal $P \in \operatorname{Spec}(A)$. By Proposition 2.15 (d), we may suppose A is quasilocal; by Corollary 2.6, A_1 is also quasilocal (and, by hypothesis, a going-down domain). Recall the following criterion [5, Theorem 2.5]: a quasilocal integral domain D is a going-down domain if and only if D has an integral overring E such that E is a divided domain and the inclusion map $D \to E$ is an *i*-extension. For $D := A_1$, take C_1 to be such an E. Notice, by integrality, that C_1 is not a field (since A_1 is not a field). For each $n \in \mathsf{N}$, put $C_{n+1} := B_n \times_{K_n} C_n$, and consider $C := \varprojlim C_n$. By the above criterion, it suffices to show that C is an integral overring of A, C is a divided domain, and the inclusion map $A \to C$ is an i-extension.

By Lemma 2.11 (d), C is an overring of A. (Of course, we may view $A \subseteq C$ since $\varprojlim$ preserves injections.) Moreover, by [11, Corollary 1.5 (5)], C_n is integral over A_n, for each $n \in \mathsf{N}$. It follows from the hypotheses that C_n is overring of A_n for each n, and so, by Theorem 2.12, $C \subseteq A'$. In particular, C is an integral overring of A. Next, notice that for each n, B_n is a quasilocal (semi)normal going-down domain and so, by the above comments, it follows from [5] that B_n is a divided domain. Hence, by applying [7, Proposition 2.12] to the pullback construction of C_{n+1}, we see by induction on n that C_n is a divided domain for each n. Therefore, by Corollary 2.17 (a), C is a divided domain. Finally, since $A_{n+1} = C_{n+1} \times_{C_n} A_n$, the topological description of prime spectra of pullbacks [11, Theorem 1.4] gives that $A_n \to C_n$ is an i-extension for each n. Therefore, by Proposition 2.13 (c), $A \to C$ is an i-extension, completing the proof.

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