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ON THE CLOSURE IN $\overline{M}_g$ OF SMOOTH CURVES HAVING A SPECIAL WEIERSTRASS POINT

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Abstract

Let $\overline{wt(2)}$ be the closure in $\overline{M}_g$, the coarse moduli space of stable complex curves of genus $g \geq 3$, of the locus in M_g of curves possessing a Weierstrass point of weight at least 2. The class of $\overline{wt(2)}$ in the group $\mathrm{Pic}(\overline{M}_g) \otimes \mathbf{Q}$ is computed. The computation heavily relies on the notion of "derivative" of a relative Wronskian, introduced in [15] for families of smooth curves and here extended to suitable families of Deligne-Mumford stable curves. Such a computation provides, as a byproduct, a simpler proof of the main result proven in [6].

0. Introduction

0.1. Let M_g (resp. $\overline{M}_g$) be the coarse moduli space of complex smooth (resp. Deligne-Mumford stable) projective curves of genus $g \geq 3$ and let $wt(2)$ be the subset of M_g defined by the locus of (isomorphism classes of) curves possessing a Weierstrass point of weight at least 2. Such a set has been equipped with a scheme structure by Ponza in his doctoral thesis ([29]; see also [15]) by using the notion of *derivative* of the wronskian relative to a proper flat family of smooth curves. The locus $wt(2)$ turns out to be a divisor in M_g, and the purpose of this paper is to compute the class of $\overline{wt(2)}$, its closure in $\overline{M}_g$, in the group $\mathrm{Pic}(\overline{M}_g) \otimes \mathbf{Q}$. The result is gotten by extending the notion of derivative of the relative wronskian (see sect. 2 for details) to a family of stable curves whose general fiber is smooth and non-hyperelliptic, so providing a new application of the tools introduced in [15].

0.2. As one may reasonably expect, the divisor $wt(2)$ is strongly related to two other natural divisors, defined in terms of curves possessing some special Weierstrass points, which have been extensively studied in the literature. The first one is the locus D_{g-1} of the curves having a Weierstrass point whose first non gap is $g-1$. The second one is the locus $E(1)$ of the curves possessing a Weierstrass point of type $g+1$: a point P of a curve C of genus

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g is said to be of type $g+1$ if there exists a non zero canonical divisor containing nP, with $n \geq g+1$.

Let us denote by $\overline{D_{g-1}}$ and by $\overline{E(1)}$ the closures, respectively, of D_{g-1} and $E(1)$ in $\overline{M}_g$. The class of $\overline{D_{g-1}}$ and $\overline{E(1)}$ in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ have been computed respectively by Diaz ([7]) and Cukierman ([6]).

Both computations are based on the theory of the compactification of the Hurwitz scheme by means of the *admissible covers*, according to Harris and Mumford ([21]).

0.3. Roughly speaking, Diaz gets his results by an enumeration of all the possible admissible coverings which may occur as a degeneration of families of curves whose general fiber has a Weierstrass point whose first non gap is $g-1$. Conceptually, this amounts to consider "curves" in the boundary of $\overline{M}_g$ and to compute their intersections with the divisor $\overline{D_{g-1}}$.

Cukierman's computations involve, instead, a fine analysis of the singularities of the closure of the Weierstrass locus $\overline{\mathscr{W}}$, that sits in the "universal curve" over $\overline{M}_g$, along the locus N of nodes of irreducible curves. Concretely, this amounts to compute several intersection numbers among the various branches of the preimage of N in the compactified Hurwitz scheme. Such data are the needed inputs to apply a Hurwitz formula with singularities ([16], p. 500) to the morphism $\psi : \tilde{\mathscr{H}} \longrightarrow \overline{M}_g$, gotten from the composition of the natural maps occurring in the diagram:

$$\begin{tikzcd}
\tilde{\mathcal{H}} \arrow[r, "n"] \arrow[drr, "\psi"'] & \overline{\mathcal{H}} \arrow[r, "h"] & \overline{\mathcal{W}} \arrow[r, hook] \arrow[d, "\pi"] & \overline{M}_{g,1} \arrow[dl, "\pi"] \\
 & & \overline{M}_g &
\end{tikzcd}$$

where $\overline{\mathscr{H}}$ is the compactified Hurwitz scheme and $n : \tilde{\mathscr{H}} \longrightarrow \overline{\mathscr{H}}$ is its normalization. Combining the expression of the branch locus in $\overline{M}_g$ of the map ψ, with the expression found by Diaz in [7] for the class of $\overline{D_{g-1}}$, Cukierman finally gets the searched expression for the class $[\overline{E(1)}]$ in the group $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$.

0.4. As for $wt(2)$ the situation is as follows. Suppose that $\pi : \mathfrak{X} \longrightarrow S$ is a proper flat family of smooth curves of genus g parametrized by some smooth scheme of finite type over $\mathbb{C}$. Define $wt(2)(S)$ as the set $s \in S$ such that P is a Weierstrass point of weight at least 2 on $\mathfrak{X}_s$, the scheme theoretical fiber of π over $s \in S$.

Using a suitable notion of "derivative" of the wronskian relative to a proper flat family $\pi : \mathrm{X} \longrightarrow S$ of smooth curves, recalled in Sect. 4, Ponza is able to equip $wt(2)(S)$ with a structure of closed subscheme of S. Once such a scheme structure has been given, the class (see section 1) of $wt(2)$ in $\mathrm{Pic}(M_g) \otimes \mathbb{Q}$, $[wt(2)]$, may be easily computed, and by a direct calculation Ponza proves that the equality:

$$[wt(2)] = [E(1)] + [D_{g-1}], \tag{1}$$

holds in $\mathrm{Pic}(M_g) \otimes \mathbb{Q}$, for $g \geq 4$.

Relation (1) looks nice, because it says that, even at the level of divisors classes, the union of $E(1)$ and D_{g-1} is exactly the set of all the smooth curves having a non normal Weierstrass point, a very well known fact from a set theoretical point of view. Let $\overline{wt(2)}$ be the closure of $wt(2)$ in $\overline{M}_g$. The new goal is to compute the class $[\overline{wt(2)}]$ of $\overline{wt(2)}$ in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$. Now, using equality (1) and the fact that $E(1)$ and D_{g-1} are components of $wt(2)$ (in the scheme theoretic sense) we are able to prove, by an easy argument, that:

THEOREM 4.5. *In the moduli space $\overline{M}_g$ of the stable curves of genus $g \geq 4$ one has*

$$\overline{wt(2)} = \overline{E(1)} \cup \overline{D_{g-1}}, \tag{26}$$

in the scheme theoretic sense and the following equality holds in the group $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$:

$$[\overline{wt(2)}] = [\overline{E(1)}] + [\overline{D_{g-1}}]. \tag{27}$$

Notice that in the literature (e.g. [6], p. 339, remark 3.3.2) one can find several claims that the locus of curves having a WP of weight at least 2 is the set theoretic union of $E(1)$ and D_{g-1}. However, the author did not find any explicit similar statement in the scheme theoretic setting. Hence, theorem 4.5 is certainly a new result, although very natural and probably not surprising at all.

0.5. It turns out that, because of Theorem 4.5., the problem of computing the class of $\overline{wt(2)}$ may be considered essentially solved. Such a class is simply the sum of the class computed by Diaz and the class computed by Cukierman. However, we believe that it is quite remarkable that if on one hand the computation of $\overline{D_{g-1}}$ and $\overline{E(1)}$ is far from being a trivial matter (look, e.g., at the papers [7] and [6]), the class $[\overline{wt(2)}]$ may be very easily computed, as in a routine exercise, by means of absolutely standard techniques, once one has learned to *differentiate* a wronskian. As far as we know the computation of such a class would be the first example in the literature of computation of a class in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ which does not require the theory of *admissible covers*.

This turns out to be important at least from a "pedagogical" point of view. In fact, assuming (as done in [6]) the result by Diaz on the class of $\overline{D_{g-1}}$, the easy computation of $\overline{wt(2)}$ provides, by virtue of Theorem 4.5, a much simpler proof of the Cukierman's expression for the class of $\overline{E(1)}$ (the latter being in fact equal to the difference $[\overline{wt(2)}] - [\overline{D_{g-1}}]$). Instead of working directly on $\mathrm{Pic}(\overline{M}_g) \otimes \mathbf{Q}$, the computation will be performed, as it is customary to do (Cfr. [2], [21], [19]), in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbf{Q}$, the *Picard group of the moduli functor* $\overline{\mathcal{M}}_g$ of stable curves, whose definition is quickly reviewed in Sect. 1.7 (see however [21], p. 50, for more details). Hence, we are able to prove, by a straightforward computation, that:

5.1. Theorem. *In the Picard group of the moduli functor* $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbf{Q}$ ($g \geq 4$)*, the following equality holds*

$$[\overline{wt(2)}] = (3g^4 + 4g^3 + 9g^2 + 6g + 2)\lambda - \tfrac{1}{6}g(g+1)(2g^2 + g + 3)\delta_0 + \\ - (g^3 + 3g^2 + 2g + 2)\sum_{i=1}^{[\frac{g}{2}]} i(g-i)\delta_i. \qquad (28)$$

where λ is the first Chern class of the Hodge bundle $\mathbf{E}$ (see Sect. 1.6) and δ_i are the classes in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbf{Q}$ corresponding to the boundary components of $\overline{M}_g$ (Sect. 1.5-1.6).

0.6. A few words about the main ingredient of the proof of Theorem 5.1. If one writes:

$$[\overline{wt(2)}] = a\lambda - b_0\delta_0 - b_1\delta_1 - \ldots - b_{[g/2]}\delta_{[g/2]},$$

the first step consists in computing the coefficients $b_1, b_2, \ldots$ up to $b_{[g/2]}$.

To do this, one needs a result by Cukierman ([6], p. 326) about the inflectionary locus in $\mathfrak{X}$ of the relative dualizing sheaf of a family $\mathfrak{X} \longrightarrow \mathrm{Spec}(\mathbf{C}[[T]])$ of stable curves of genus $g \geq 3$ whose generic fiber is smooth and the special fiber is reducible (and hence singular). A proper use of such a result reduces all the matter to compute the top Chern class of a certain rank 2 vector bundle.

It is however worth to remark that such a bundle is the appropriate substitute of the sheaf of principal parts of order 1 of a relative line bundle $\mathscr{L}$ sitting over the total space of a family $\pi : \mathfrak{X} \longrightarrow S$, which may have singular (stable) fibers. The "ordinary" sheaf of the principal parts of such $\mathscr{L}$ is, in general, only a coherent sheaf (because if $\pi : \mathfrak{X} \longrightarrow S$ has singular fibers the sheaf of the differentials relative to π is no longer locally free). Such a substitute (see Sect. 2 for a sketchy description) is the relative version of the jets bundles on singular Gorenstein curves constructed in [12]. (see also [13] for a survey, and [25] for a characteristic free construction): it has the nice feature that, in the situation we are interested in, the k-th jet of a relative line bundle $\mathscr{L}$ on a family of stable curves, is still locally free (a rank $k+1$ vector bundle). Such a construction is included in the paper because, for its limited purposes, we do not know any easier reference.

0.7. The last important step is to use the relation:

$$a - 12b_0 + b_1 = 0, \qquad (33)$$

to compute b_0. In fact, the coefficient b_1 is known by the previous calculations, while the coefficient a has been computed by Ponza in ([29]). As far as the author knows, the only proof of relation (33) available in the literature is a consequence of a intersection theoretical computation based on Cukierman's result (see F. M. Cukierman, Ph.D. Thesis, Brown University 1987, page 56, remark (d)), which is exactly what we want to avoid! In fact, we show in Sect. 5 that such a relation may be inferred from a purely geometrical statement, proven in Sect. 3, which seems to be interesting in its own:

LEMMA 3.2. *Suppose that $\pi : \mathfrak{X} \longrightarrow S$ is a flat proper family of curves of arithmetic genus $g \geq 3$ parametrized by some smooth scheme of finite type over* Spec($\mathbf{C}$). *Suppose that the general curve of the family is smooth and that the special fiber $\mathfrak{X}_0$ is an integral curve having a cusp at a point P_0. Let $n : \tilde{\mathfrak{X}}_0 \longrightarrow \mathfrak{X}_0$ be the normalization of $\mathfrak{X}_0$ and let $Q = n^{-1}(P_0)$. If there is a section of WP's having weight at least* 2 *degenerating to the cusp P_0, then Q is a Weierstrass point for $\tilde{\mathfrak{X}}_0$.*

0.8. The computational strategy described in section 5, suggests one more question. In [15] it is proven that the locus $wt(3) \subset M_g$ of (isomorphisms classes of) curves of genus $g \geq 4$ possessing a Weierstrass point of weight at least 3 has codimension 2 in M_g. In the quoted reference, the class $[wt(3)]$ of $wt(3)$ is computed in the Chow group $A^2(M_g)$ ([15], Prop. 4.9). Let $\overline{wt(3)}$ be the closure of $wt(3)$ in $\overline{M}_g$. What is the expression of the class $[\overline{wt(3)}]$ in the Chow group $A^2(\overline{M}_g)$?

We are not able yet to provide an answer to such a question. However, for the time being and as a final remark, it seems worth of saying that we believe that imitating the same kind of arguments used to compute $[\overline{wt(2)}]$, one might be able to compute the class $[\overline{E(1)}]$ in a completely independent way - and with no use of the geometry of the Hurwitz scheme - from the knowledge of $[\overline{wt(2)}]$ and of $[\overline{D_{g-1}}]$. As a byproduct, this would provide a new simple proof of the expression found by Diaz in [7] for the class $[\overline{D_{g-1}}]$ and some new insight for attempting the solution of the above major question.

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1. Preliminaries and Notation

1.1. In this paper we shall deal with Deligne-Mumford stable projective algebraic curves of genus $g \geq 2$ ([9]) over the complex field C (Compact Riemann Surfaces if smooth) or with suitable families of them. We assume known the basic definitions of Weierstrass point on a smooth projective curve of genus g as well as the notion of Weierstrass Gaps Sequence (WGS) at a point P of a smooth curve (see [3], [10], [17] for references). The *Weierstrass weight* of a point $P \in C$, $wt(P)$, is defined to be the order of vanishing of the wronskian section $\mathsf{W} \in H^0(C, K^{\otimes \frac{g(g+1)}{2}})$ at P, where K is the canonical bundle of C. Let $\underline{\omega} = (\omega_1, \ldots, \omega_g)$ be a basis of $H^0(C, K)$, and let (U, z) be a local coordinate chart trivializing K, such that $z(P) = 0$ and $\omega_{i|_U} = u_i(z)dz$. The local expression $\mathsf{W}_{|_U}(z)$ of the wronskian shall be often written, for short, as:

$$\mathbf{u}(z) \wedge \mathbf{u}'(z) \wedge \ldots \wedge \mathbf{u}^{(g-1)}(z),$$

which is nothing but an abbreviated notation for:

$$\det\begin{pmatrix} \mathbf{u}(z) \\ \vdots \\ \mathbf{u}^{(g-1)}(z) \end{pmatrix}, \tag{2}$$

having set $\mathbf{u} = (\mathbf{u_1}, \ldots, \mathbf{u_g}) \in \mathbf{O_C(U)}^{\oplus \mathbf{g}}$. The order of vanishing at P of the wronskian section W is the order of vanishing of $\mathsf{W}_{|_U}(z)$ at $z = 0$. Such a definition does not depend neither on the particular chosen basis for the holomorphic differentials nor on the local coordinate around P.

1.2. In the following, as usual, $M_{g,n}$ and $\overline{M}_{g,n}$ will denote, respectively, the n-pointed moduli space of smooth curves of genus g, and the moduli space of stable n-pointed curves (a stable n-pointed curve $(C, P_1, \ldots, P_n)$ is a connected projective curve having only nodes as singularities and such that on each smooth rational component there are at least 3 *special points*, a special point being either a singular point or a marked point P_i). The spaces $M_{g,0}$ and $\overline{M}_{g,0}$ are simply denoted by M_g and $\overline{M}_g$ respectively. If $g \geq 2$, $\dim(M_{g,n}) = 3g - 3 + n$.

1.3. We recall, for the reader's convenience, notation from the paper [8]. In $M_{g,1}$, the coarse moduli space of stable pointed curves of genus g (often said the "universal curve" over M_g), one defines the locus $\mathscr{V}(1, n_2, \ldots, n_g)$ set theoretically described as:

$$\{(C, P) \in M_{g,1} | \quad WGS(P) = \{1, n_2, \ldots, n_g\}\}.$$

Similarly, one may define some loci in M_g, the coarse moduli space of smooth curves of genus g, set theoretically defined as:

$$W(1, n_2, \ldots, n_g) = \{[C] \in M_g | \text{C has a point P with WGS(P)} = \{1, \mathrm{n}_2, \ldots, \mathrm{n}_\mathrm{g}\}\},$$

In this paper we shall be concerned with the loci in M_g (and their closures in $\overline{M}_g$) set theoretically defined as:

$$D_{g-1} = \{[C] \in M_g | \text{C has a Weierstrass point of type g} - 1\},$$

and:

$$E(1) = \{[C] \in M_g | \text{C has a Weierstrass point of type g} + 1\}.$$

The loci D_{g-1} and $E(1)$ can be equipped with a scheme structure so that they become closed subschemes of M_g. Such a scheme structure is defined in [7] and [8], and it is described in Sect. 4 of this paper on the base of a stable curve $\pi : \mathfrak{X} \longrightarrow S$ over S (see below). The locus D_{g-1} turns out to be non empty, irreducible and of the "expected dimension" $3g - 4$ in M_g([31], [1]).

In the last section, instead, we shall deal with the closure in $\overline{M}_g$ of the locus $wt(2)$ introduced in [29], [15]. The locus $wt(2)$ is set theoretically defined as:

$$wt(2) = \{[C] \in M_g | \text{C has a point P such that wt(P)} \geq 2\}.$$

and it is given of a scheme structure in [15]. Such a scheme structure will be recalled in Sect. 4 on the base of a stable curve $\pi : \mathfrak{X} \longrightarrow S$ over S.

1.4. Let $\pi : \mathfrak{X} \longrightarrow S$ be a stable curve of genus g over a smooth scheme S of finite type over C (Cf. Sect. 2 for definitions), i.e. $\pi : \mathfrak{X} \longrightarrow S$ is a proper flat morphism such that the fibers are all Deligne-Mumford stable curves of genus g.

Let ω_π be the relative canonical sheaf of the family ([9], p. 76). One can then define some classes in the Chow ring, $A^*(S)$, of S:

$$\kappa_i = \pi_*\left[c_1(\omega_\pi)^{i+1}\right], \tag{3}$$

the so-called κ-*classes*, and

$$\lambda_1 = \lambda = c_1(\pi_*\omega_\pi) = c_1(\bigwedge^g \pi_*\omega_\pi) \quad \text{and} \quad \lambda_i = c_i(\pi_*\omega_\pi), \tag{6}$$

the so-called λ-*classes*. As for definitions (3), one has that (see [7]) $\kappa_0 = (2g-2)[S]$, where $[S]$ is the fundamental class of S (the identity of the ring $A^*(S)$). As for definitions (4), instead, we remark that $\pi_*\omega_\pi$ is a locally free rank g sheaf of O_S-modules, often denoted in the literature as E and called the *Hodge bundle* relative to π.

1.5. A theorem by Harer ([18]) states that $\mathrm{Pic}(M_g) \otimes \mathbb{Q}$ is 1-dimensional for $g \geq 3$. It is generated by the class λ. For $g \geq 4$ the class λ may be defined exactly as in 1.4 by observing that the singular locus of M_g occurs in codimension $g-2$ in M_g ([19], p. 102) and that $C_g^0 \longrightarrow M_g^0$, the *tautological family* over the smooth locus M_g^0, is a smooth curve over M_g^0 exactly in the sense of Sect. 1.4. For $g = 3$ one may circumvent the difficulty by defining a λ-class $\tilde{\lambda}$ on a smooth ramified covering of M_3, $p : \tilde{M}_3 \longrightarrow M_3$, and setting:

$$\lambda = \frac{1}{\deg(p)} p_*(\tilde{\lambda}). \tag{5}$$

Such smooth ramified coverings exist (they are moduli spaces parametrizing curves with some *n-level structures*: see [30]); moreover definition (5) does not depend on the chosen covering, by ([11], p. 143).

From this, it follows that $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ is $(h+2)$-dimensional, $h = [g/2]$, generated by λ and the boundary components of $\overline{M}_g$ (see [AC]). The boundary of $\overline{M}_g$, $\overline{M}_g - M_g$ is the union $\cup_{i=0}^h \Delta_i$, where:

$$\Delta_0 = \{\text{closure of the locus of the uninodal irreducible curves}\}$$

and

$$\Delta_i = \{\text{closure of the locus of the connected curves having two irreducible components, one of genus } i \text{ and the other of genus } g-i\}.$$

1.6. Let $\pi : \mathfrak{X} \longrightarrow S$ be a stable curve of genus g over S and assume that the general fiber of $\pi : \mathfrak{X} \longrightarrow S$ is not singular (the only case used in the sequel). Then Mumford shows ([26], p. 101) that the locus of the singular points on fibers of π has codimension 2 in $\mathfrak{X}$. We may then define divisors δ_i's on $\mathrm{Pic}(S)$ related with the divisors Δ_i in $\overline{M}_g$ as follows. Locally around each singular point of a singular fiber (see e.g. [9] or [21]), the family is given as $xy = \pi^*(\gamma(s))$, $s \in S$, where γ is a regular function on S. We define δ_i as the divisor associated to the zero scheme $\mathscr{Z}(\prod_{\mathrm{type}\, i} \gamma) = 0$.

1.7. Let (Sch/C) be the category whose objects are schemes of finite type over C and the morphisms are C-morphisms of schemes over C. The *moduli functor* of the stable curves $\overline{\mathscr{M}}_g$ is a contravariant functor:

$$\overline{\mathscr{M}}_g : (Sch/\mathsf{C}) \rightsquigarrow (Sets),$$

associating to each C-scheme S of finite type, the set:

$$\overline{\mathscr{M}}_g(S) = \{\textit{isomorphism classes of stable curves over } S\}.$$

The functor $\overline{\mathscr{M}}_g$ is *coarsely represented* by the moduli space of stable curves $\overline{M}_g$, which is a normal projective variety. One can attach to the functor $\mathscr{M}_g$ an abelian group, $\mathrm{Pic}(\mathscr{M}_g)$, and a deep theorem by Mumford ([26], Lemma 5.8), states that $\mathrm{Pic}(\overline{M}_g) \otimes \mathsf{Q}$ is isomorphic to $\mathrm{Pic}(\overline{\mathscr{M}}_g) \otimes \mathsf{Q}$. The idea of $\mathrm{Pic}(\overline{\mathscr{M}}_g)$ is to consider line bundles on families of curves all at once. More precisely, a line bundle on the moduli functor of stable curves is a line bundle $L(\pi)$ on the base S of every proper flat family $\pi : \mathfrak{X} \longrightarrow S$ of stable curves parametrized by a scheme S, enjoying the following property. If:

$$\begin{tikzcd}
\mathfrak{X}_1 \arrow[r] \arrow[d, "\pi_1"'] & \mathfrak{X}_1 \arrow[d, "\pi_2"'] \\
S_1 \arrow[r, "f_1"'] & S_2
\end{tikzcd}$$

is a morphism of families with *cartesian* square (i.e. $\mathfrak{X}_1 = S_1 \times_{S_2} \mathfrak{X}_2$), then there is an isomorphism between $L(\pi_1)$ and $f^*L(\pi_2)$. The isomorphisms should be compatible in a obvious sense. Namely, if both the squares of the diagram:

$$\begin{tikzcd}
\mathfrak{X}_1 \arrow[r] \arrow[d, "\pi_1"'] & \mathfrak{X}_2 \arrow[r] \arrow[d, "\pi_2"'] & \mathfrak{X}_3 \arrow[d, "\pi_3"'] \\
S_1 \arrow[r, "f_1"'] & S_2 \arrow[r, "f_2"'] & S_3
\end{tikzcd}$$

are cartesian, then

$$L(\pi_1) \cong f_1^* L(\pi_2) \cong f_1^* f_2^* L(\pi_3) \cong (f_2 \circ f_1)^* L(\pi_3).$$

Two lines bundles L_1 and L_2 on the moduli functor are isomorphic iff for any family $\pi : X \longrightarrow S$, $L_1(\pi)$ and $L_2(\pi)$ are isomorphic. As the reader can easily check, the tensor product is also well defined and it is compatible with the relation of isomorphism, so that we can attach an abelian group $\mathrm{Pic}(\overline{\mathscr{M}}_g)$ to the moduli functor $\overline{\mathscr{M}}_g$, the *Picard group of the moduli functor*. Analogously one can define $\mathrm{Pic}(\mathscr{M}_g)$: one simply considers families of smooth curves of genus g instead of families of stable curves. Hence, the classes $\lambda, \delta_0, \ldots, \delta_{[g/2]}$ previously defined on each $\pi : \mathfrak{X} \longrightarrow S$ stable curve over S, induce divisors in $\mathrm{Pic}(\overline{\mathscr{M}}_g)$ (denoted by the same symbol) and they are in fact a basis of the vector space $\mathrm{Pic}(\overline{\mathscr{M}}_g) \otimes \mathbf{Q}$. Their images, denoted by the same symbols, through the isomorphism between $\mathrm{Pic}(\overline{\mathscr{M}}_g) \otimes \mathbf{Q}$ and $\mathrm{Pic}(\overline{M}_g) \otimes \mathbf{Q}$, are a basis of $\mathrm{Pic}(\overline{M}_g) \otimes \mathbf{Q}$ as well.

1.8. The purpose of this paper is to compute the class of $[\overline{wt(2)}]$ in the *Picard group of the moduli functor* $\mathrm{Pic}(\overline{\mathscr{M}}_g) \otimes \mathbf{Q}$, of the stable curves of genus g defined above. To this purpose, if $\pi : \mathfrak{X} \longrightarrow S$ is a stable curve over S whose general fiber is not singular, we shall use the following relation: (Cfr. [26], p. 102):

$$\kappa_1 = 12\lambda - \delta, \tag{6}$$

where $\delta = \sum \delta_i$ is the class of the divisor of S corresponding to singular fibers of π and $\lambda = \lambda_\pi$.

2. Jets of Relative Line Bundles

2.1. The purpose of this section is to provide a reference for the construction of jets extensions of relative line bundles defined on the total space of a family $\pi : \mathfrak{X} \longrightarrow S$ of stable curves with possibly singular fibers. When the family has singular fibers, the sheaf of relative differentials is no longer locally free, so that the sheaf of the principal parts of a relative line bundle is, in general, only a coherent sheaf. By the way, one may provide a useful "substitute" of the *bundle of principal parts* by letting the (invertible) dualizing sheaf of the family, ω_π, play the role of the sheaf of the differentials. Such a construction makes possible the definition of a suited notion of *relative wronskian* in the sense, e.g., of [24], [25], even for families of stable curves.

2.2. Our parameter spaces will be taken in the category of smooth schemes of finite type over the complex field $\mathbf{C}$. Let S be one such. By a stable (resp. smooth) curve of genus $g \geq 2$ over S we shall intend a flat proper surjective morphism $\pi : \mathfrak{X} \longrightarrow S$ such that the scheme theoretical fibers $\mathfrak{X}_s = \mathfrak{X} \times_S \mathrm{Spec}(\mathbf{k}(s)) := \pi^{-1}(s)$ are stable (resp. smooth) projective connected curves of genus g over $\mathbf{k}(s) \cong \mathbf{C}$. In an analytic language, π can be seen as a holomorphic proper map between complex analytic spaces, such that the fibers are (possibly singular) Riemann surfaces of genus g. However, in this section we shall work mainly algebraically, since it is more convenient from a formal point of view.

2.3. As it is well known, a stable curve of genus g over S, comes equipped with a sheaf of $O_{\mathfrak{X}}$-modules, $\Omega^1_{\mathfrak{X}/S}$ ([4], p. 108), the so-called *sheaf of the relative differentials* of $\mathfrak{X}$ over S, together with a *universal derivation*:

$$d_{\mathfrak{X}/S} : O_{\mathfrak{X}} \longrightarrow \Omega^1_{\mathfrak{X}/S}. \tag{7}$$

Unless $\mathfrak{X}$ is a family of smooth curves, $\Omega^1_{\mathfrak{X}/S}$ is not in general invertible. However there is a natural map:

$$c : \Omega^1_{\mathfrak{X}/S} \longrightarrow \omega_\pi,$$

where ω_π is the relative dualizing sheaf of the family which is invertible. We shall denote, in the sequel, by d_π the composition of c with $d_{\mathfrak{X}/S}$. In other words:

$$d_\pi = c \circ d_{\mathfrak{X}/S} : O_{\mathfrak{X}} \longrightarrow \omega_\pi$$

The map d_π will be said, in the following, by a slight abuse of terminology, the *exterior derivative along the fibers* of π (compare with [23], where in the case of families of smooth curves, an analytic description of d_π is provided). As it is well known, ω_π and d_π enjoy some nice functorial properties. More precisely, if:

$$\begin{array}{ccc} \mathfrak{X}_T & \xrightarrow{p_2} & \mathfrak{X} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow[\phi]{} & S \end{array}$$

is a cartesian diagram, i.e. $\mathfrak{X}_T$ is the induced family over T, defined by $T \times_S \mathfrak{X}$, then ([4], p. 110):

$$p_2^*\omega_\pi = \omega_{p_1}, \tag{9}$$

and

$$p_2^* d_\pi = d_{p_1}. \tag{10}$$

In particular, if $s \in S$ and $T = \operatorname{Spec}(\mathbf{k}(s))$, then $p_2^*\omega_\pi = \omega_{p_1} = \omega_{\mathfrak{X}_s}$, the dualizing bundle of the fiber $\mathfrak{X}_s$ and $d_{p_2} \equiv d : O_{\mathfrak{X}_s} \longrightarrow \omega_{\mathfrak{X}_s}$. This in particular means that ω_π is a *dualizing bundle along the fibers*, i.e. its fiber at a point $P \in \mathfrak{X}$ is $\omega_{\mathfrak{X}_{\pi(P)}}$.

2.4. The purpose of the present subsection is to define a suitable notion of

n-th jets extension of a relative line bundle $\mathscr{L}$ on $\mathfrak{X}/S$ along the fibers of π, where $\pi : \mathfrak{X} \longrightarrow S$ is a stable curve over S of genus $g \geq 2$. By a relative line bundle $\mathscr{L}$ on $\mathfrak{X}/S$ we shall intend a line bundle over $\mathfrak{X}$ modulo pull-backs of line bundles over S. Since this is the only case we are interested in, we shall assume, in the following, that the bundle $\mathscr{L}$ possesses at least one non zero global section.

Let $\mathscr{U} = \{U_\alpha : \alpha \in \mathscr{A}\}$ be an open affine covering of $\mathfrak{X}$ such that $\omega_\pi(U_\alpha)$ and $\mathscr{L}(U_\alpha)$ are generated by σ_α and ψ_α, respectively, over $O_{\mathfrak{X}}(U_\alpha)$.

If $0 \neq \lambda \in H^0(\mathfrak{X}, \mathscr{L})$, we can then write $\lambda_{|_{U_\alpha}} = \ell_\alpha \psi_\alpha$, for some $\ell_\alpha \in O_{\mathfrak{X}}(U_\alpha)$. The purpose is now to define the higher order derivatives of λ with respect to the generator σ_α (see also [24] and [25] for a more elegant and abstract approach for families of smooth curves). We set $\ell_\alpha^{(0)} = \ell_\alpha$ and, recursively,

$$d_\pi(\ell_\alpha^{(n-1)}) = \ell_\alpha^{(n)} \sigma_\alpha. \tag{11}$$

It is now a standard patching game to show that the collection:

$$\{U_\alpha; \ell_\alpha, \ell'_\alpha, \dots, \ell_\alpha^{(n)}\},$$

defines a section, written $D^n\lambda$, of a vector bundle which (following Lax, [23]) shall be denoted as $J_\pi^n \mathscr{L}$, and which is, by definition, the n-th jets bundle of $\mathscr{L}$ along the fibers of π. The above outlined construction is the relative version of the jet bundles for Gorenstein curves constructed in [12]. If $\pi : \mathfrak{X} \longrightarrow S$ is a family of smooth curves and $\mathscr{L} = \omega_\pi$ is the relative canonical sheaf, then $J^n \omega_\pi$ is exactly the n-th jet of the relative canonical bundle along the fibers defined by Lax in [23]. There, the construction was performed by using Patt's local coordinates ([28]) in the Teichmüller space.

Now, by virtue of the property enjoyed by d_π (formula (10)), we also have, referring to the diagram (8),

$$p_2^*(D^k \tau) = D^k(p_2^* \tau), \tag{12}$$

and, hence

$$p_2^*(J_\pi^k \mathscr{L}) = J_{p_1}^k (p_2^* \mathscr{L})$$

where the D on the right hand side is precisely the D relatively to the induced family $p_1 : \mathfrak{X}_T \longrightarrow T$. In particular, if $T = \mathrm{Spec}(\mathbf{k}(s))$, D is defined exactly as above and $J_{p_1}^k \mathscr{L}_s$ is the k-th jet of the bundle of $\mathscr{L}_{|_{\mathfrak{X}^s}} = p_2^* \mathscr{L} = \mathscr{L}_s$.

We need one more (easy) technical result that establishes a (well-known in the case of smooth families) exact sequence of vector bundles to be used for computing Chern classes in section 5. It is formally the same as formula (2.1.1) in [24], p. 138, rephrased in the language of jets bundles.

2.5. PROPOSITION. *For each line bundle $\mathscr{L}$ over $\mathfrak{X}/S$ and each $n \geq 1$, the following exact sequence holds:*

$$0 \longrightarrow \mathscr{L} \otimes {\omega_\pi}^{\otimes n} \longrightarrow J^n_\pi \mathscr{L} \longrightarrow J^{n-1}_\pi \mathscr{L} \longrightarrow 0. \tag{13}$$

PROOF (Sketch of). Let $(P; \ell_{0,\alpha}, \ell_{1,\alpha}, \ldots, \ell_{n,\alpha})$ be the representation of a *point* of $J^n_\pi \mathscr{L}$ in a given trivialization (U_α, Φ_α). Define a projection $p_{n-1} : J^n_\pi \mathscr{L} \longrightarrow J^{n-1}_\pi \mathscr{L}$ as:

$$(P; \ell_{0,\alpha}, \ell_{1,\alpha}, \ldots, \ell_{n,\alpha}) \mapsto (P; \ell_{0,\alpha}, \ell_{1,\alpha}, \ldots, \ell_{n-1,\alpha}).$$

The kernel of such a map is the set of all the points:

$$(P, \underbrace{0, \ldots, 0}_{n \text{ times}}, \tilde{\ell}_{n,\alpha}),$$

which, passing to a trivialization (U_β, Φ_β) , must transform according the transition function of $J^n \mathscr{L}$. An easy exercise shows that under such a transformation, one has:

$$(P, \underbrace{0, \ldots, 0}_{n \text{ times}}, \ell_{n,\beta}) = (P, \underbrace{0, \ldots, 0}_{n \text{ times}}, t_{\beta\alpha} \ell_{n,\alpha}),$$

where $t_{\alpha\beta}$ are the transition functions of the line bundle $\mathscr{L} \otimes {\omega_\pi}^{\otimes n}$. Hence $\mathrm{Ker}(p_{n-1})$ may be identified with $\mathscr{L} \otimes {\omega_\pi}^{\otimes n}$ and the claim follows.

3. Weierstrass points degenerating to cusps

3.1. Let us consider a stable curve of genus g which is the union of a nodal rational curve X intersecting transversely an irreducible smooth curve Y of genus g-1 at a point P (see picture 3.1). We shall often denote such a curve as $X \cup_P Y$.

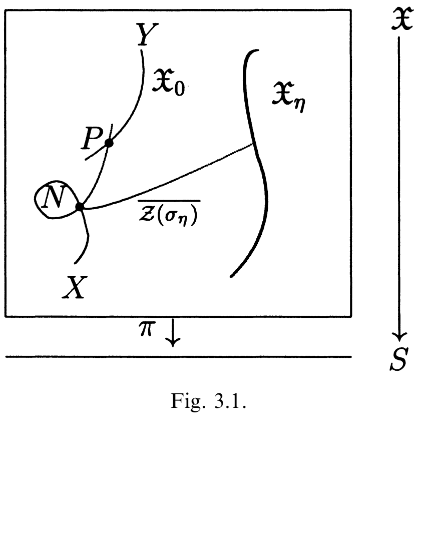


Fig. 3.1.

Suppose that P is not a Weierstrass point for the curve Y. The purpose of this section consists in showing that *the node N of X cannot be limit of a Weierstrass point of weight at least* 2 *on nearby smooth curves.* Notice that a Weierstrass point of weight at least 2 on a curve C is either of type $g-1$, i.e. its first non gap is $g-1$, or of type $g+1$, i.e. there exists a non-zero holomorphic differential σ vanishing at P with multiplicity at least $g+1$ (or, in other words $(\sigma) \geq (g+1)P$). The former case has already been studied by Diaz, by using the theory of admissible covers, introduced by Harris and Mumford in [21]. Assuming the result of Diaz (recalled below), we shall prove our claim by directly proving that the non separating node of a stable curve like in Fig. 3.1 cannot be limit of a WP of type $g+1$. To this goal we shall begin by proving a lemma on families of curves degenerating to a cuspidal curve which seems interesting in its own.

3.2. Lemma. *Suppose that $\pi : \mathfrak{X} \longrightarrow S$ is a flat proper family of curves of arithmetic genus $g \geq 3$ parametrized by some smooth scheme of finite type over* $\mathrm{Spec}(\mathbf{C})$. *Suppose that the general curve of the family is smooth and that the special fiber $\mathfrak{X}_0$ is a curve having a cusp at a point P_0. Let $n : \tilde{\mathfrak{X}}_0 \longrightarrow \mathfrak{X}_0$ be the normalization of $\mathfrak{X}_0$ and let $Q = n^{-1}(P_0)$. If there is a section of WP's having weight at least* 2 *degenerating to the cusp P_0, then Q is a Weierstrass point for $\tilde{\mathfrak{X}}_0$.*

Proof. Let ω_π be the relative dualizing sheaf of the family. Then, by hypothesis, there is a section $\sigma_\eta \in H^0(\mathfrak{X}_\eta, \omega_{\mathfrak{X}_\eta})$, such that $(\sigma_\eta) \geq (g+1)P_\eta$. This section extends to a section σ on all the family, such that $(\sigma_0) \geq (g+1)P_0$. The induced σ_0 is a section of the dualizing sheaf of the curve $\mathfrak{X}_0$. Let $n^*\sigma_0$ be the pull-back of σ_0 to $\tilde{\mathfrak{X}}$. It is a section of the sheaf $n^*\omega_{C_0}$, which is isomorphic, by adjunction theory, to $K_{\tilde{\mathfrak{X}}_0}(2Q)$, $K_{\tilde{\mathfrak{X}}_0}$ being the canonical sheaf of $\tilde{\mathfrak{X}}_0$. Hence σ_0 induces a section $\tilde{\sigma_0} \in H^0(\tilde{\mathfrak{X}}_0, K_{\tilde{\mathfrak{X}}_0})$ such that $(\tilde{\sigma_0}) \geq (g-1)Q$, which is the same as claiming that Q is a WP for $\tilde{\mathfrak{X}}_0$.

3.3. Lemma. *Let $X \cup_P Y =: C_0$ denote the union of a rational nodal curve X and an irreducible smooth curve Y of genus $g-1$ intersecting transversely at a point P. Assume that P is not a Weierstrass point for Y. Then the node N (see Fig.* 3.1) *is not a limit of a WP of type $g+1$.*

Proof. Suppose that there is a family $\pi : \mathfrak{X} \longrightarrow S$ parametrized by $S = \mathrm{Spec}(\mathbf{C}[[T]])$, $\mathfrak{X}$ a smooth surface, such that $\mathfrak{X}_\eta$ is geometrically smooth, and that there is a WP of type $g+1$, P_η, such that $P_0 \in \overline{\{P_\eta\}}$. We can assume, up to replacing the special fiber by an equivalent semistable model, that P_η is $\mathbf{C}((T))$-rational. Now we play with our family as follows. Let us consider the sheaf $\omega_\pi(-2Y)$. One has $\omega_\pi(-2Y)_{|\mathfrak{X}_\eta} = \omega_{\pi|\mathfrak{X}_\eta}$.

Therefore:

$$\pi_*[\omega_\pi(-2Y)] \otimes \mathsf{C}(0) \cong \pi_*[\omega_\pi(-2Y)] \otimes \mathsf{C} \cong H^0(X \cup Y, \omega_\pi(-2Y)_{|_{X\cup Y}}).$$

We now claim that:

$$H^0(X \cup Y, \omega_\pi(-2Y)_{|_{X\cup Y}}) \cong H^0(Y, \omega_Y(2P)).$$

In fact $h^0(X \cup Y, \omega_\pi(-2Y)_{|_{X\cup Y}}) \geq g$. Moreover the inclusion $Y \hookrightarrow X \cup Y$ induces a natural restriction map:

$$\rho : H^0(X \cup Y, \omega_\pi(-2Y)_{|_{X\cup Y}}) \longrightarrow H^0(Y, \omega_Y(2P)), \tag{14}$$

defined by $\sigma \mapsto \sigma_{|_Y}$. The map ρ is injective. In fact, suppose that:

$$\sigma \in H^0(X \cup Y, \omega_\pi(-2Y)_{|_{X\cup Y}}),$$

vanishes identically on Y, i.e.:

$$\sigma_{|_Y} = 0.$$

Hence $\sigma(P) = 0$, and since $\deg(\sigma_{|_X}) = 0$, it follows that $\sigma_{|_X} = 0$, i.e. that $\sigma = 0$. This proves injectivity. By dimension reasons, (14) is actually an isomorphism.

Now the sheaf $\pi_*\omega_\pi(-2Y)$ embeds the family $\pi : \mathfrak{X} \longrightarrow S$ in $\mathsf{P}(\pi_*\omega_\pi(-2Y))$, i.e. we have the following diagram:

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "\phi_{\pi_*(\omega_\pi(-2Y))}"] \arrow[dr] & & \mathsf{P}(\pi_*(\omega_\pi(-2Y))) \arrow[dl] \\ & S & \end{tikzcd}$$

The generic fiber is a geometrically smooth curve of genus g while the special fiber is a cuspidal curve having a cusp in P_0 with the rational nodal component of C_0 contracted in P_0 by the map $\phi_{\pi_*(\omega_\pi(-2Y))}$. In fact such a map has degree 0 when restricted to X. The generic fiber has a WP P_η of type $g+1$ degenerating onto the cusp P_0 (because it degenerated onto X in the initial family and X has been contracted in the cusp). But then P_0 would be a Weierstrass point by Lemma 3.2., contradicting the hypothesis. Hence the node of X is not a limit of a Weierstrass point of type $g+1$.

Now we recall a theorem by Diaz (Lemma 7.2, p. 40 in [7]): *Suppose that $C_0 = X \cup_P Y$ and that X is a rational nodal curve intersecting transversely an irreducible smooth curve Y of genus $g-1$ at a point P. Assume that P is not a Weierstrass point for Y. Then the node N (see Fig. 3.1) cannot be a limit of a Weierstrass point of type $g-1$.*

Since a Weierstrass point of weight at least 2 is either a point of type $g-1$ or a point of type $g+1$, patching together Lemma 7.2, p. 40 in [7], and Lemma 3.3 we have proven the following:

THEOREM 3.4. *Let $g \geq 3$ and let $C_0 = X \cup_P Y$ such that:*

a) *X is a rational nodal irreducible curve. Let N be its non separating node.*

b) *Y is a connected smooth curve of genus $g-1$ intersecting X transversely at the point P which is not a Weierstrass point for Y.*

Then N is not a limit of a Weierstrass point of weight at least 2 on nearby smooth curves.

4. The closure of $wt(2)$ in $\overline{M}_g$

4.1. Let $\pi : \mathfrak{X} \longrightarrow S$ be a smooth proper curve of genus $g \geq 4$ over a smooth scheme S of finite type over C. In S one can detect three relevant divisors, defined in terms of curves possessing some special Weierstrass points. They are set-theoretically described as follows:

a) $D_{g-1}(S) = \{s \in S : \mathfrak{X}_s \text{ has a WP of type } g-1\}$

b) $wt(2)(S) = \{s \in S : \mathfrak{X}_s \text{ has a WP of weight at least } 2\}$

c) $E(1)(S) = \{s \in S : \mathfrak{X}_s \text{ has a WP of type } g+1\}$

It is clear that $wt(2)(S)$ is the set theoretical union of $E(1)(S)$ and $D_{g-1}(S)$.

Our aim is to show, now, that $E(1)$, D_{g-1} and $wt(2)$ can be given a natural structure of closed subschemes of S. Before defining them, let us introduce a piece of notation. Suppose that $\phi : E \longrightarrow F$ is a map of holomorphic vector bundles, of rank m and n respectively, over an algebraic scheme X of finite type. Let $p = \min\{m, n\}$. Set:

$$\mathscr{Z}(\phi) = \{x \in X : rk(\phi) < p\}.$$

For instance, if σ is a section of a rank m vector bundle over X, $\mathscr{Z}(\sigma)$ would mean the zero scheme associated to the section σ.

Let ω_π now be the relative dualizing sheaf of the family of smooth curves $\pi : \mathfrak{X} \longrightarrow S$ we started with (which, in this case, coincides with its relative canonical sheaf) and let $J^i_\pi \omega_\pi$ be the i-th jets extension of ω_π (Sect. 2). Let E be the *Hodge bundle* relative to π, i.e. the locally free sheaf (of rank g) $\pi_* \omega_\pi$, and consider the natural evaluation maps of vector bundles:

$$
\begin{array}{ccc}
a')\ \begin{tikzcd} \pi^*\mathsf{E} \arrow[rr, "D^{g-2}"] \arrow[dr] & & J_\pi^{g-2}\omega_\pi \arrow[dl] \\ & \mathfrak{X} \arrow[d, "\pi"] & \\ & S & \end{tikzcd}
&
b')\ \begin{tikzcd} \pi^*\mathsf{E} \arrow[rr, "D^{g-1}"] \arrow[dr] & & J_\pi^{g-1}\omega_\pi \arrow[dl] \\ & \mathfrak{X} \arrow[d, "\pi"] & \\ & S & \end{tikzcd}
&
c')\ \begin{tikzcd} \pi^*\mathsf{E} \arrow[rr, "D^{g}"] \arrow[dr] & & J_\pi^{g}\omega_\pi \arrow[dl] \\ & \mathfrak{X} \arrow[d, "\pi"] & \\ & S & \end{tikzcd}
\end{array}
$$

Since π is proper, the sets a) and c) above inherit a structure of closed subschemes of S by setting:

$$D_{g-1}(S) =: \pi(\mathscr{Z}(D^{g-2})), \tag{15}$$

$$E(1)(S) =: \pi(\mathscr{Z}(D^{g})), \tag{16}$$

while for the set b) one needs to work a little bit more on the map b').

The map D^{g-1} induces a map, $\mathsf{W}_\pi =: \wedge^g D^{g-1}$, between the top exterior product of the bundles $\pi^*\mathsf{E}$ and $J_\pi^{g-1}\omega_\pi$, namely:

$$
\begin{tikzcd} \wedge^g\pi^*\mathsf{E} \arrow[rr, "\wedge^g D^{g-1}"] \arrow[dr] & & \wedge^g J_\pi^{g-1}\omega_\pi \arrow[dl] \\ & \mathfrak{X} \arrow[d, "\pi"] & \\ & S & \end{tikzcd}
$$

In the following, such a map will be said the *wronskian relative to the family* π or, briefly, the *relative wronskian* of the family. Because $\wedge^g\pi^*\mathsf{E}$ is a line bundle, it turns out that W_π is a section of the line bundle $\omega_\pi^{\otimes\frac{g(g+1)}{2}} \otimes (\pi^* \bigwedge^g \mathsf{E})^\vee$. This was of course well known (see, e.g. the very first pages of [24], [25], or [30]). What it seems to be rather new, although very natural, is to consider the *derivatives* of the relative wronskian. In other words, relying on the constructions performed in Sect. 2, the section W_π of the bundle $\omega_\pi^{\otimes\frac{g(g+1)}{2}} \otimes (\pi^* \bigwedge^g \mathsf{E})^\vee$, induces a section $D^k\mathsf{W}_\pi$ of the rank $k+1$-bundle:

$$J_\pi^k\left(\omega_\pi^{\otimes\frac{g(g+1)}{2}} \otimes \left(\pi^* \bigwedge^g \mathsf{E}\right)^\vee\right).$$

Why such sections are important, at least for our present purposes, is explained in the following, still rather natural:

4.2. Proposition. *Let P be a point of $\mathfrak{X}$. Then P is a Weierstrass point on $\pi^{-1}(\pi(P))$ of weight at least k, if and only if $D^{k-1}\mathsf{W}_\pi$ vanishes at P.*

Proof. The key remark to be used, here, is that the relative wronskian W_π restricted to a fiber $\mathfrak{X}_s$ is (up to a multiplicative constant) a wronskian section of the bundle $\omega_s^{\otimes\frac{g(g+1)}{2}}$, ω_s being the canonical sheaf of the fiber $\mathfrak{X}_s$. Let us denote by W_s such a restriction. Suppose now that P is a WP for a fiber $\mathfrak{X}_s$ having weight at least k. If $D^{k-1}\mathsf{W}_s(P) = 0$, then:

$$D^{k-1}\mathsf{W}_\pi(P) = 0, \tag{17}$$

the latter equality holding because of the functoriality of the wronskian. Now, $D^{k-1}\mathsf{W}_s(P)$, in a local coordinate chart (U, z) of the curve $\mathfrak{X}_s$, around the point P and such that $z(P) = 0$, can be expressed as the k-tuple:

$$(w(0), w'(0), \ldots, w^{(k-1)}(0)), \tag{18}$$

where we set $\mathsf{W}_{s|_U} = w(z) \cdot dz^{\otimes\frac{g(g+1)}{2}}$ and the derivatives are taken with respect to the local parameter z. By hypothesis P is a WP of weight at least k, hence it is a zero of the wronskian of order k. Hence, the first $(k-1)$ derivatives of the wronskian, at $z = 0$, must vanish, too. This proves the first implication.

Conversely, suppose that $D^{k-1}\mathsf{W}_\pi(P) = 0$. Then, arguing as above, in a local coordinate chart around P in the curve $\mathfrak{X}_{\pi(P)}$, the local expression of the wronskian vanishes at P together with its $k-1$ derivatives, i.e. P is a WP on $\mathfrak{X}_{\pi(P)}$ of weight at least k.

Because of Proposition 4.2, we can put on $wt(2)(S)$ a natural closed subscheme structure by setting:

$$wt(2)(S) = \pi(\mathscr{Z}(D\mathsf{W}_\pi)). \tag{19}$$

We prove now a very important theorem for the computational purposes to be pursued in the two next sections.

4.3. Theorem. *Let $\pi : \mathfrak{X} \longrightarrow S$ be a smooth curve of genus $g \geq 4$ over a smooth curve over S (cf. 2.2.). Assume that the general fiber of π is a smooth curve having no special Weierstrass point. Then $wt(2)(S)$ is the scheme theoretic union of $E(1)(S)$ and $D_{g-1}(S)$.*

Proof. Let, following [8], $\mathscr{V}D_{g-1}(S)$ and $\mathscr{V}E(1)(S)$ be the closed subschemes of S set theoretically described as:

$$\mathscr{V}D_{g-1}(S) = \{(s, P) \in \mathfrak{X} : P \in \mathfrak{X}_s \text{ has a WP of type } g-1\},$$

and

$$\mathscr{V}E(1)(S) = \{(s, P) \in \mathfrak{X} : P \in \mathfrak{X}_s \text{ has a WP of type } g+1\}.$$

The last assumption of the theorem means that the family we are dealing with is general with respect to the property that the fibers have non special Weierstrass points.

In this case, following the notation of section 1, it turns out that:

$$\mathscr{V}D_{g-1}(S) = \mathscr{V}(1, 2, \ldots, g-2, g, g+1) \quad \text{and}$$
$$\mathscr{V}E(1)(S) = \mathscr{V}(1, 2, \ldots, g-1, g+2),$$

as sets. In fact $\{1, 2, \ldots, g-2, g, g+1\}$ and $\{1, 2, \ldots, g-1, g+2\}$ are the only WGSs that occur at points of $\mathfrak{X}$ in codimension 2. We shall equip the right hand sides of the above equalities of the same scheme structure of the left hand side, namely $\mathscr{Z}(D^{g-2})$ and $\mathscr{Z}(D^g)$ respectively. It may be worth to remark that $\mathscr{V}D_{g-1}(S) \cap \mathscr{V}E(1)(S) = \emptyset$ (a point of a curve cannot have two different WGSs!).

We contend that:

$$Z(D\mathsf{W}_\pi)(S) = \mathscr{V}D_{g-1}(S) \cup \mathscr{V}E(1)(S),$$

in the scheme theoretic setting.

To show that, let V be an open affine subset of S. Since π is proper, $\pi^{-1}(V)$ can be covered by (finitely many) open affine subsets of $\mathfrak{X}$. Let $U = \mathrm{Spec}(R)$ be one such. Up to shrinking V, we may assume that the restriction of the map π to U, $\pi_U : U \longrightarrow V$, is surjective as well. Let σ be a generator of $\Omega^1_{\mathfrak{X}/S}(U)$ over $R = O_{\mathfrak{X}}(U) = O_U$ and let $(\omega_1, \ldots, \omega_g)$ be a O_V-basis of $\Omega^1_{\mathfrak{X}/S}(U)$, so that $\omega_i = u_i\sigma$, where $u_i \in R$. According to the notation of 1.1., we shall write:

$$\mathbf{u} = (u_1, \ldots, u_g) \in O_{\mathfrak{X}}(V)^{\oplus g}.$$

The relative wronskian W_π restricted to V admits a local representation:

$$\mathsf{W}_{\pi|_U} = \mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-1)} \cdot \sigma^{\otimes \frac{g(g+1)}{2}},$$

where the derivatives of the u_i's are taken in the sense of 2.4.

In other words the principal ideal of R generated by $\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-1)}$ defines the closed subscheme of U of the Weierstrass points on fibers of π_U. Similarly, by virtue of Proposition 4.2, the ideal of R:

$$I_g = (\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-1)}, \mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-2)} \wedge \mathbf{u}^{(g)})$$

defines the closed (0-dimensional) subscheme of the points of U which are Weierstrass points on fibers of π_U of weight at least 2. Let I_{g-1} and I_{g+1} be, respectively, the defining ideals of $\mathscr{V}D_{g-1}(S)_{|_U} = \mathscr{V}D_{g-1}(V)$ and $\mathscr{V}E(1)(S)_{|_U} = \mathscr{V}E(1)(V)$. By the very definition of $\mathscr{V}D_{g-1}$ and $\mathscr{V}E(1)$, they are:

$$I_{g-1} = (\mathbf{u} \wedge \ldots \wedge \mathbf{u}^{(g-2)}), \tag{20}$$

and

$$I_{g+1} = (\mathbf{u} \wedge \ldots \wedge \mathbf{u}^{(g-1)}, \mathbf{u} \wedge \ldots \wedge \mathbf{u}^{(g-2)} \wedge \mathbf{u}^{(g)}, \ldots,$$
$$\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-1)} \wedge \mathbf{u}^{(g)}, \ldots, \mathbf{u}' \wedge \mathbf{u}'' \wedge \ldots \wedge \mathbf{u}^{(g)}),$$

where by the notation $(\mathbf{u} \wedge \ldots \wedge \mathbf{u}^{(g-2)})$ we mean the ideal of R generated by all the $(g-1) \times (g-1)$ minors of the $(g-1) \times g$ matrix:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{u}' \\ \vdots \\ \mathbf{u}^{(g-2)} \end{pmatrix}. \tag{21}$$

Similarly, the ideal I_{g+1} is generated, as the notation should suggest, by all the $g \times g$ minors (which are $g+1$) of the $(g+1) \times g$ matrix:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{u}' \\ \vdots \\ \mathbf{u}^{(g-2)} \\ \mathbf{u}^{(g-1)} \\ \mathbf{u}^{(g)} \end{pmatrix}. \tag{22}$$

We want to show that:

$$I_g = I_{g-1} \cap I_{g+1}.$$

a) $I_g \subseteq I_{g-1} \cap I_{g+1}$.

In fact, by its very definition:

$$I_g = (\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-1)}, \mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-2)} \wedge \mathbf{u}^{(g)}).$$

Now, on one hand each generator of I_g belongs to I_{g-1}, because we may write:

$$\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-1)} = (\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-2)}) \wedge \mathbf{u}^{(g-1)},$$

and

$$\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-2)} \wedge \mathbf{u}^{(g)} = (\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-2)}) \wedge \mathbf{u}^{(g)}.$$

On the other hand, the two generators of I_g form a subset of the generators of I_{g+1}. This ends the proof of the first inclusion.

b) $I_{g-1} \cap I_{g+1} \subseteq I_g$.

An element of $I_{g-1} \cap I_{g+1}$ must be expressed as a linear combination, with

R-coefficients, of the generators of I_{g+1}, and only of those generators which belong to I_{g-1}, too. These are the generators of I_{g+1} which contain the $(g-1) \times (g-1)$ minors of the matrix:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{u}' \\ \vdots \\ \mathbf{u}^{(g-2)} \end{pmatrix},$$

i.e. exactly $\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-1)}$ and $\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-2)} \wedge \mathbf{u}^{(g)}$, the generators of I_g. The second inclusion is hence proven and then:

$$I_g = I_{g-1} \cap I_{g+1},$$

as claimed.

We conclude that:

$$V(I_g) = V(I_{g-1}) \cup V(I_{g+1}),$$

i.e.

$$Z(D\mathsf{W}_\pi)(V) = \mathscr{V}D_{g-1}(V) \cup \mathscr{V}E(1)(V). \tag{23}$$

Since the ideals I_k's, $k \in \{g-1, g, g+1\}$ come from local representations of degeneracy loci of global maps of vector bundles, formula (20) actually shows that

$$Z(D\mathsf{W}_\pi)(S) = \mathscr{V}D_{g-1}(S) \cup \mathscr{V}E(1)(S), \tag{24}$$

in the scheme theoretic sense. Taking the projection via π of both sides of equality (24), one gets:

$$\pi(Z(D\mathsf{W}_\pi)(S)) = \pi(\mathscr{V}D_{g-1}(S) \cup \mathscr{V}E(1)(S)) = \pi(\mathscr{V}D_{g-1}(S)) \cup \pi(\mathscr{V}E(1)(S)),$$

i.e., by their very definition that

$$wt(2)(S) = D_{g-1}(S) \cup E(1)(S). \tag{25}$$

Let now $\pi : \mathfrak{X} \longrightarrow S$ be a family of stable curves such that the general curve is smooth and non hyperelliptic. Let Δ be the locus of S corresponding to the singular fibers. Over the base $S' = S \setminus \{\Delta\}$ we have a family of smooth curves whose total space is $\mathfrak{X} \setminus \pi^{-1}(\Delta)$. As explained in 4.1 and 4.2, $wt(2)(S')$, $E(1)(S')$, $D_{g-1}(S')$ do live in S' as closed subschemes of it. By (25) one hence has:

$$\overline{wt(2)(S')} = \overline{E(1)(S')} \cup \overline{D_{g-1}(S')},$$

where the closure is taken in S. For each stable curve over S, $\pi : \mathfrak{X} \longrightarrow S$, set, by definition, $\overline{wt(2)}(S) = \overline{wt(2)(S')}$. Define $\overline{E(1)}$ and $\overline{D_{g-1}}$ analogously.

The assignment of a Cartier divisor on the smooth base S for each stable curve over S, $\pi : \mathfrak{X} \longrightarrow S$, is the same as assigning a Cartier divisor in the moduli functor of the stable curves of genus $g \geq 4$, $\overline{\mathcal{M}}_g$. Let us denote by $E(1)$, D_{g-1} and $wt(2)$ the divisors defined by $E(1)(S)$, $D_{g-1}(S)$ and $wt(2)(S)$ for each family $\pi : \mathfrak{X} \longrightarrow S$. In particular, if M_g^0 is the smooth locus of M_g, and $C_g^0 \longrightarrow M_g^0$ is the tautological family, Theorem 4.3 proves that:

$$wt(2)(M_g^0) = E(1)(M_g^0) \cup D_{g-1}(M_g^0),$$

in the scheme theoretic sense. Taking the closures in $\overline{M}_g$ of both sides of the previous equality, we have hence proved that:

4.5. T*heorem. In the moduli space* $\overline{M}_g$ *of the stable curves of genus* $g \geq 4$ *one has*

$$\overline{wt(2)} = \overline{E(1)} \cup \overline{D_{g-1}}, \tag{26}$$

in the scheme theoretic sense, and, hence, the following equality holds in the group $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$

$$[\overline{wt(2)}] = [\overline{E(1)}] + [\overline{D_{g-1}}]. \tag{27}$$

Instead of computing the class $[\overline{wt(2)}]$ in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$, we shall do it in the Picard group of the moduli functor (Sect. 1.7) $\overline{\mathcal{M}}_g$ (Cf. Sect. 1.7). In such a group, let us write the class $[\overline{wt(2)}]$ as:

$$[\overline{wt(2)}] = a\lambda - b_0\delta_0 - b_1\delta_1 - \ldots - b_{[\frac{g}{2}]}\delta_{[\frac{g}{2}]}.$$

The coefficient a was computed by Ponza and it is known to be $3g^4 + 4g^3 + 9g^2 + 6g + 2$ ([29]; see also [15] and [6]). The coefficients $b_0, b_1, \ldots, b_{[\frac{g}{2}]}$ will be computed in the next section.

5. The class of $\overline{wt(2)}$ in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$

The rest of this section will be devoted to prove the following:

5.1. Theorem. *In the Picard group of the moduli functor* $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$ (*see Sect.* 1.7) , *the following equality holds:*

$$[\overline{wt(2)}] = (3g^4 + 4g^3 + 9g^2 + 6g + 2)\lambda - \tfrac{1}{6}g(g+1)(2g^2 + g + 3)\delta_0 + \\ - (g^3 + 3g^2 + 2g + 2)\sum_{i=1}^{[\frac{g}{2}]} i(g-i)\delta_i.$$

Proof. Let us write the expression $[\overline{wt(2)}]$ in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$:

$$[\overline{wt(2)}] = a\lambda - b_0\delta_0 - b_1\delta_1 - \ldots - b_{[g/2]}\delta_{[g/2]}, \tag{28}$$

where $a, b_0, b_1, \ldots, b_{[g/2]}$ are coefficients to be determined. Let us consider now the following *test families* $\pi_i : \mathfrak{X}_i \longrightarrow S_i$ $(1 \leq i \leq [g/2])$ defined as follows:

$\pi_i : \mathfrak{X}_i \longrightarrow S_i$ is a stable curve over S_i, where $S_i = \mathrm{Spec}(\mathbf{C}[[T]])$ such that:

i) the geometric generic fiber $\mathfrak{X}_{i,\overline{\eta}}$ is smooth and non hyperelliptic;

ii) the special fiber $\mathfrak{X}_{i,0}$ is a stable curve of genus g that is the union of an irreducible smooth curve X of genus i which intersects transversely at a point P an irreducible smooth curve Y of genus $g-i$. Moreover P is not a Weierstrass point neither for X nor for Y.

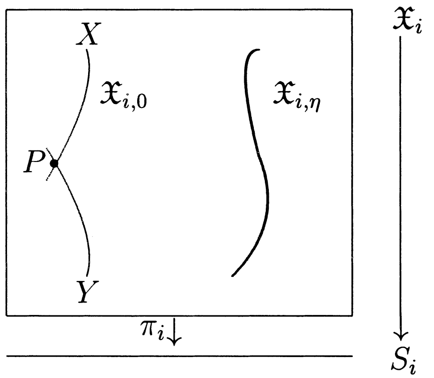
In other words, the only singular fiber of the test family $\mathfrak{X}_i$ is a fiber (the special one) of type Δ_i (see Sect. 1 for the basic definitions).

By the very definition of the Picard group of the moduli functor, "evaluating" expression (28) on the family π_i, one has:

$$[\overline{wt(S_i)}] = a\lambda_{\pi_i} - b_i\delta_i,$$

and there are no contributions coming from the δ_j, for $j \neq i$, because the only singular fiber of π_i is of type Δ_i.

Then we reduced our problem to determine (a and) b_i's for our test families π_i's.



Notice that, because of Section 2, in our tool-box we have a notion of relative wronskian even for families possessing singular stable fibers. In fact, the natural evaluation map:

$$D^{g-1} : \pi^*\pi_*\omega_\pi \longrightarrow J_\pi^{g-1}\omega_\pi,$$

is well defined, as well as the induced map $\mathsf{W}_\pi =: \wedge^g D^{g-1}$ between the top exterior powers of the bundles $\pi^*\pi_*\omega_\pi$ and $J_\pi^{g-1}\omega_\pi$ respectively.

In [6], p. 326, Cukierman computed exactly the order of vanishing of the relative wronskian $\mathscr{W}_{\pi_i}$ along X and Y as above. Actually, if $\mathscr{W}_{\pi_i}$ is the wronskian relative to the family π_i, then $\mathscr{W}_{\pi_i}$ vanishes $\binom{g-i+1}{2}$ along X and $\binom{i+1}{2}$ along Y.

Let us set, for notational convenience:

$$\alpha = \binom{g-i+1}{2} \quad \text{and} \quad \beta = \binom{i+1}{2}.$$

Then $\mathscr{W}_{\pi_i}$ induces a section, denoted in the same way by abuse of notation, of the bundle:

$$\omega_{\pi}^{\otimes\frac{g(g+1)}{2}} \otimes O_{\mathfrak{X}}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^{g} \mathsf{E}^{\vee}.$$

The aim, now, is to compute $\pi_*([\mathscr{Z}(D\mathscr{W}_{\pi})])$, where:

$$D\mathscr{W}_{\pi_i} \in H^0\left(\mathfrak{X}_i, J^1_{\pi}\left(\omega_{\pi_i}^{\otimes\frac{g(g+1)}{2}}\right) \otimes O_{\mathfrak{X}}(-\alpha X - \beta Y) \otimes \pi_i^* \bigwedge^{g} \mathsf{E}^{\vee}\right).$$

Clearly, because $\mathscr{W}_{\pi_i}$ does not vanish on the special fiber, the same holds for the section $D\mathscr{W}_{\pi_i}$ (in fact, $D\mathscr{W}_{\pi_i}$ locally looks like a pair (w, w') where w is a local equation for $\mathscr{W}_{\pi_i}$ and w' is its derivative (in the sense of Sect. 2) along the fibers). We use the following exact sequence to compute Chern classes (Cf. Prop. 2.5):

$$\begin{aligned}
0 \longrightarrow \omega_{\pi_i}^{\otimes\left(\frac{g(g+1)}{2}+1\right)}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^{g} \mathsf{E}^{\vee} \longrightarrow \\
\longrightarrow J^1_{\pi}\left(\omega_{\pi_i}^{\otimes\frac{g(g+1)}{2}}\right)(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^{g} \mathsf{E}^{\vee} \longrightarrow \\
\longrightarrow \omega_{\pi_i}^{\otimes\frac{g(g+1)}{2}}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^{g} \mathsf{E}^{\vee} \longrightarrow 0
\end{aligned}$$

so that:

$$c_2\left(J^1_\pi\left(\omega_{\pi_i}^{\frac{g(g+1)}{2}}\otimes O_{\mathfrak{X}}(-\alpha X-\beta Y)\otimes\pi^*\bigwedge^g \mathsf{E}^\vee\right)\right)=$$

$$=\left[\left(\frac{g(g+1)}{2}+1\right)c_1(\omega_{\pi_i})-\alpha X-\beta Y-\pi^*\lambda\right]\cdot\left[\frac{g(g+1)}{2}c_1(\omega_{\pi_i})+\right.$$

$$\left.-\alpha X-\beta Y-\pi^*\lambda\right]=$$

$$=\frac{g(g+1)}{2}\left(\frac{g(g+1)}{2}+1\right)c_1(\omega_{\pi_i})^2-(\alpha X+\beta Y)[g(g+1)+1]c_1(\omega_{\pi_i})+$$

$$+2(\alpha X+\beta Y)\pi^*\lambda-[g(g+1)+1]c_1(\omega_{\pi_i})\pi^*\lambda+(\alpha X+\beta Y)^2+(\pi^*\lambda)^2.$$

By pushing down the above equality via π_* one has:

$$J\pi_*\left[c_2\left(J^1_\pi\left(\omega_{\pi_i}^{\frac{g(g+1)}{2}}\otimes O_{\mathfrak{X}}(-\alpha X-\beta Y)\otimes\pi^*\bigwedge^g \mathsf{E}^\vee\right)\right)\right]=$$

$$=\frac{g(g+1)}{2}\left(\frac{g(g+1)}{2}+1\right)\kappa_1-(g(g+1)+1)[\alpha(2i-1)+\beta(2(g-i)+1)]\delta_i+$$

$$-2(g-1)[g(g+1)+1]\lambda-\alpha^2\delta_i-\beta^2\delta_i+2\alpha\beta\delta_i=$$

$$=\left[\frac{g^2(g+1)^2}{4}+\frac{g(g+1)}{2}\right]\kappa_1-\{[g(g+1)+1](\alpha(2i-1)+\beta(2(g-i)-1))+$$

$$=[3g^2(g+1)^2+6g(g+1)-2g(g-1)(2g+1)-2(g-1)]\lambda+$$

$$-\{[g(g+1)](\alpha(2i-1)+\beta(2(g-i)-1))+$$

$$\left.+\alpha^2+\beta^2-2\alpha\beta+\left(\frac{g^2(g+1)^2}{4}+\frac{g(g+1)}{2}\right)\right\}\delta_i.$$

In the previous formulas we skipped the subscript π_i from the class κ_1, in order not to make notation too heavy. In order to conclude the computations, one first replace α and β, respectively, by their values $\binom{g-i+1}{2}$ and $\binom{i+1}{2}$, and then uses the fundamental relation (8) (see e.g. [Mu1], p. 102):

$$\kappa_1=12\lambda-\delta, \tag{29}$$

where, as said in Sect. 1, $\delta=\sum\delta_i$ and the δ_i's are, respectively, the classes in $A^1(S)$ of the points corresponding to singular fibers of type Δ_i (cfr. Sect. 1.3). Since our family is supposed to contain only one singular fiber of type Δ_i (i.e. a reducible curve union of an irreducible curve of genus i and an irreducible curve of genus $g-i$), we can rewrite formula (29) as:

$$\kappa_1 = 12\lambda_{\pi_i} - \delta_{\pi_i} = 12\lambda - \sum_{j=1}^{[g/2]}(\delta_j)_{\pi_i} = 12\lambda_{\pi_i} - \delta_i,$$

Making the appropriate substitutions as indicated one gets, at last:

$$[\overline{wt(2)(S_i)}] = (3g^4+4g^3+9g^2+6g+2)\lambda\pi_i - (g^3+3g^2+2g+2)i(g-i)\delta_i. \tag{30}$$

By our previous remarks, we can conclude that in the Picard group of the moduli functor $\mathrm{Pic}(\overline{\mathscr{M}}_g) \otimes \mathbb{Q}$ the following equality holds:

$$\overline{wt(2)}] = (3g^4+4g^3+9g^2+6g+2)\lambda - b_0\delta_0 - (g^3+3g^2+2g+2)\sum_{i=1}^{[\frac{g}{2}]} i(g-i)\delta_i, \tag{31}$$

in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$.

5.2. We only need, now, to determine the coefficient b_0. To this purpose we shall argue as follows (imitating Diaz, [7], p. 40). Let us consider in $\overline{M}_g$ a family lying entirely in the divisor Δ_1, whose general point corresponds to a stable curve consisting of an elliptic curve E meeting a smooth curve X of genus $g-1$ transversely at a point P which is not a WP for X. The one parameter family one wants to construct is gotten by fixing the curve of genus $g-1$ and the point of intersection on it and varying the j-invariant of the elliptic curve (for a rigorous and detailed construction of such a family see [21], p. 83). It is a family of curves parametrized by the j-line (hence the parameter space is complete). To be consistent with our previous notation, we shall denote it by $\pi_0 : \mathfrak{X}_0 \longrightarrow S_0$ (while Diaz, in [D1], denotes it by $\mathscr{F}_2$), where $S_0 \cong \mathbb{P}^1$ is the j-line. We claim that:

$$[\overline{wt(2)}(S_0)] = 0,$$

in $\mathrm{Pic}(S_0)$. In fact if $E \cup_P X$ is a stable curve where E is elliptic, with X a general curve of genus $g-1$, we know, by a rough dimension count (shown below), that $E \cup_P X$ is not in the closure of $wt(2)$ in $\overline{M}_g$. In fact $wt(2)$ has dimension $3g-4$ in M_g, while the set of all stable uninodal curves of type $E \cup_P X$ is parametrized by $M_{1,1} \times M_{g-1,1}$ whose dimension is:

$$\dim(M_{1,1}) + \dim(M_{g-1,1}) = 1 + [(3(g-1)-3)+1] = 3g-4,$$

as well. So, if one found a family of curves $\pi : \mathfrak{X} \longrightarrow S$ with $\mathfrak{X}_\eta$ smooth and a WP $P_\eta \in \mathfrak{X}_\eta$ degenerating to $E \cup_P X$, the only possibility would be that P_η degenerated to the node N of $E \cup_P X$, when E is rational nodal. But Theorem 3.4 says us that such a curve cannot be in the closure of the locus of curves having a WP of weight at least 2. Hence:

$$[\overline{wt(2)}(S_0)] = 0. \tag{32}$$

Now, as computed by Harris and Mumford in [21], pp. 83-84 (see also [14]), one has that:

$$\deg(\lambda_{\pi_0}) = 1, \quad \deg(\delta_{0,\pi_0}) = 12, \quad \deg(\delta_{1,\pi_0}) = -1 \quad \text{and} \quad \deg(\delta_{j,\pi_0}) = 0,$$

for $j > 1$.

Hence, equality (32) (taking the degree of both sides of equality (28) "evaluated" on the family $\pi_0 : \mathfrak{X}_0 \longrightarrow S_0$), implies the relation:

$$a - 12b_0 + b_1 = 0. \tag{33}$$

Since we already know the values of a and b_1 in the expression of $[\overline{wt(2)}]$, we are also able to compute b_0. One gets:

$$b_0 = \tfrac{1}{12}(a + b_1) = \tfrac{1}{12}\big(3g^4 + 4g^3 + 9g^2 + 6g + 2 + (g^3 + 3g^2 + 2g + 2)(g - 1)\big) =$$
$$= \tfrac{1}{12}(4g^4 + 6g^3 + 8g^2 + 6g) = \tfrac{1}{6}g(g+1)(2g^3 + g + 3),$$

and Theorem 5.1 is now proven.

5.4. Remark. By virtue of the isomorphism, stated in [26], Lemma 5.8, between $\mathrm{Pic}(\overline{\mathscr{M}}_g) \otimes \mathbb{Q}$ and $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$, it follows that the same equality (28) holds in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$, where the class λ is defined according Sect. 1.5, and the δ_i's are the classes of the divisors Δ_i (Sect. 1.5) in the group $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$.

5.5. Remark. As it is clear from the proof above, the crucial step consists in computing the coefficients b_1, b_2 up to $b_{[g/2]}$. There is an alternative way to compute them, by using the *residual formula for top Chern classes* explained in [11], p. 245. The situation is in fact as follows. The *first derivative* of the wronskian, $D\mathsf{W}_\pi$, is a section of the rank 2 vector bundle $J^1_\pi(\omega_{\pi_i}^{\otimes\frac{g(g+1)}{2}}) \otimes (\pi^* \bigwedge^g \mathsf{E})^\vee$ on $\mathfrak{X}$, that vanishes along the divisor $\alpha X + \beta Y$. Here, $\mathfrak{X}$ is assumed to be a smooth surface, and so all the hypotheses for applying the *residual formula* are fulfilled. Let us call $D\mathsf{W}_\pi{}'$ the section of the bundle:

$$J^1_\pi\left(\omega_{\pi_i}^{\otimes\frac{g(g+1)}{2}}\right) \otimes O_{\mathfrak{X}}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^g \mathsf{E}^\vee,$$

induced by $D\mathsf{W}_\pi$. Then ([11], ex. 14.1.4, p. 245), one has:

$$\mathscr{Z}(D\mathsf{W}_\pi) = \mathscr{Z}(D\mathsf{W}_\pi{}') + c_1(\omega_{\pi_i}^{\otimes\frac{g(g+1)}{2}} \otimes \pi^* \bigwedge^g \mathsf{E}^\vee)(\alpha[X] + \beta[Y]) - (\alpha[X] + \beta[Y])^2,$$

i.e.:

$$\mathscr{X}(DW_{\pi}') = c_2(\omega_{\pi_i}^{\otimes\frac{g(g+1)}{2}} \otimes \pi^* \bigwedge^g E^\vee) +$$
$$- c_1(\omega_{\pi_i}^{\otimes\frac{g(g+1)}{2}} \otimes \pi^* \bigwedge^g E^\vee)(\alpha[X] + \beta[Y]) + (\alpha[X] + \beta[Y])^2.$$

Performing the easy computations involved, pushing down via π and using the relation $\kappa_1 = 12\lambda - \delta_i$, one gets exactly formula (30).

5.5. Recall now the expression of the class of the divisor $\overline{D_{g-1}}$ computed by Diaz, i.e.:

$$[\overline{D_{g-1}}] = \tfrac{1}{2}g^2(g-1)(3g-1)\lambda - \tfrac{1}{6}g(g-1)^2(g+1)\delta_0 - \tfrac{1}{2}g(g^2+g-4)\textstyle\sum_{i=1}^{[\frac{g}{2}]} i(g-i)\delta_i \tag{34}$$

in $\mathrm{Pic}(\overline{\mathscr{M}_g}) \otimes \mathbb{Q}$. Using (27) and Theorem 5.1 we re-obtain the result gotten by Cukierman in ([6], p. 344):

5.6. Corollary (Cukierman). *The class of* $\overline{E(1)}$ *in* $\mathrm{Pic}(\overline{\mathscr{M}_g}) \otimes \mathbb{Q}$ *is given by*:

$$[E(1)] = \tfrac{1}{2}(g+1)(g+2)(3g^2+3g+2)\lambda - \tfrac{1}{6}g(g+1)^2(g+2)\delta_0 + - \tfrac{1}{2}(g+1)(g+2)^2 \sum_{i=1}^{[\frac{g}{2}]} i(g-i)\delta_i. \tag{35}$$

Proof. The class of $E(1)$ in $\mathrm{Pic}(\overline{\mathscr{M}_g}) \otimes \mathbb{Q}$ is given by:

$$[E(1)] = [wt(2)] - [D_{g-1}].$$

The right hand side of the above equality is known and the easy computations give the result.

We stress once more that, differently from Cukierman, we got formula (35) with no use of the theory of the compactification of the Hurwitz scheme by means of the admissible coverings introduced by Harris and Mumford in [HM].

5.7. *The case* $g = 3$. For $g = 3$ and for each good family of smooth curves $\pi : \mathfrak{X} \longrightarrow S$ whose general fiber is not hyperelliptic, as proven by Diaz ([7], p. 59), $\mathscr{V}E(1)(S)$ is the scheme theoretical union of $\mathscr{V}\mathscr{H}(S)$ and of $\mathscr{V}(1,2,5)$, where:

$$\mathscr{V}\mathscr{H}(S) = \{(s,P) \in \mathfrak{X} : P \in \mathfrak{X}_s\, \mathfrak{X}_s \text{ is hyperelliptic and P is a WP of } \mathfrak{X}_s\},$$

and

$$\mathscr{V}(1,2,5) = \{(s,P) \in \mathfrak{X} : \mathrm{P} \in \mathfrak{X}_s \text{ is a WP with WGS } \{1,2,5\}\}.$$

By its very definition, $\mathscr{Z}(D\mathsf{W}_\pi)$ contains $\mathscr{V}\mathscr{H}(S)$ and $\mathscr{V}(1,2,5)$. Moreover, by direct computation, Ponza proved that ([29], [15]):

$$\pi_*[\mathscr{Z}(D\mathsf{W}_\pi)] = [W(1,2,5)] + 16[\mathscr{H}], \tag{36}$$

the equality holding in $\mathrm{Pic}(\mathscr{M}_3) \otimes \mathbb{Q}$.

As for $g \geq 4$, equality (36) may be extended to the equality:

$$\pi_*[\overline{\mathscr{Z}(D\mathsf{W}_\pi)}] = [\overline{W(1,2,5)}] + 16[\overline{\mathscr{H}}].$$

The way to prove it is by noticing that the scheme $Z(\mathsf{W}_\pi) \subset \mathfrak{X}$ is smooth along $\mathscr{V}(1,2,5)$ and $\mathscr{V}(1,3,5)$. This may be checked by a computation of the tangent space of $Z(\mathsf{W}_\pi)$ at points of $\mathscr{V}(1,2,5)$ and $\mathscr{V}(1,3,5)$ as shown, e.g., in [6], p. 339, and then imitating the reasoning of Diaz in [7], pp. 57-58. It follows that:

$$Z(D\mathsf{W}_\pi) = \mathscr{V}(1,2,5) \cup 2\mathscr{V}\mathscr{H},$$

in the scheme theoretic sense, where by the notation $2\mathscr{V}\mathscr{H}$ we mean that the hyperelliptic locus $\mathscr{V}\mathscr{H}$ in $\mathfrak{X}$ occurs in $Z(D\mathsf{W}_\pi)$ with multiplicity 2. Passing to the closures we have

$$\overline{Z(D\mathsf{W}_\pi)} = \overline{\mathscr{V}(1,2,5)} \cup 2\overline{\mathscr{V}\mathscr{H}}.$$

so that

$$\pi_*[\overline{\mathscr{Z}(D\mathsf{W}_\pi)}] = [\overline{W(1,2,5)}] + 16[\overline{\mathscr{H}}],$$

where we used the fact that $\pi(\overline{\mathscr{V}(1,2,5)}) = \overline{W(1,2,5)}$ and that $\pi(\overline{\mathscr{V}\mathscr{H}}) = \overline{\mathscr{H}}$ by the very definition of $\mathscr{V}(1,2,5)$ and $\mathscr{V}\mathscr{H}$. The factor $16 = 2 \cdot 8$ is due to the fact that $\mathscr{V}\mathscr{H}$ covers $\mathscr{H}$ 8 : 1.

Now, one has:

$$\pi_*[\overline{\mathscr{Z}(D\mathsf{W}_\pi)}] = 452\,\lambda - b_0\,\delta_0 - 124\,\delta_1$$

where $a = 452$ has been computed in [29] (see also [15]) and $b_1 = -124$ can be computed exactly by the same method shown in the last subsections. Actually the number 124 turns out to be the value of b_1 in formula (28) evaluated at $g = 3$. Using the relation (33) one finds $b_0 = -64$, so that:

5.8. Theorem. *For $g = 3$, the following equality holds in* $\mathrm{Pic}(\overline{\mathscr{M}}_3) \otimes \mathbb{Q}$:

$$\pi_*[\overline{\mathscr{Z}(D\mathsf{W}_\pi)}] = 452\lambda - 64\delta_0 - 124\delta_1. \tag{37}$$

By using the expression of the class of the hyperelliptic locus in $\mathrm{Pic}(\overline{\mathscr{M}}_3) \otimes \mathbb{Q}$:

$$[\overline{\mathscr{H}}] = 9\lambda - \delta_0 - 3\delta_1,$$

formula (37) yields the result by Cukierman for $g = 3$.

5.9. COROLLARY (Cukierman). *The class of the hyperflex divisor in* $\mathrm{Pic}(\overline{\mathscr{M}}_3) \otimes \mathbb{Q}$ *is given by:*

$$[W(1,2,5)] = 308\lambda - 32\delta_0 - 76\,\delta_1.$$

PROOF. Follows from the equality $[W(1,2,5)] = \pi_*[\overline{\mathscr{Z}(D\mathsf{W}_\pi)}] - 16[\overline{\mathscr{H}}]$.

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