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ON LANDAU'S PHENOMENON IN R^n

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1. Completeness problem for sparse exponential systems on large sets

Let $\Lambda = \{\lambda_n\}_{n\in\mathsf{Z}}$ be a sequence of distinct real numbers. Set

$$E(\Lambda) := \{e^{i\lambda_n t}\}_{n\in\mathsf{Z}}.$$

Such exponential systems have been intensively investigated since Paley and Wiener discovered the possibility of non-harmonic Fourier expansion in $L^2(-\pi,\pi)$.

In his remarkable paper [2] H. Landau revealed a striking phenomenon concerning the completeness property of exponential systems in L^2 on a union of disjoint intervals (we formulate this theorem in a slightly different form than in [1]).

THEOREM 1 (Landau). *Given $\delta > 0$ there exists a real sequence $\Lambda = \{\lambda_n\}_{n\in\mathsf{Z}}$ with $|\lambda_n - 2\pi n| < \delta$ such that the system $E(\Lambda)$ is complete in L^2 on every finite union of the intervals $(k+\epsilon, k+1-\epsilon)$, for every $0 < \epsilon < \frac{1}{2}$.*

This theorem shows that the completeness problem for exponential systems on a union of intervals is quite different from the one on a single interval. A sequence Λ in Theorem 1 is a 'small' perturbation of the set $2\pi\mathsf{Z} = \{2\pi n\}_{n\in\mathsf{Z}}$. This sequence yields the system $E(\Lambda)$ which is complete in L^2 on open sets with arbitrarily large measures. The set $2\pi\mathsf{Z}$ itself yields the trigonometrical system

$$E(2\pi\mathsf{Z}) = \{\exp(2\pi i n t)\}_{n\in\mathsf{Z}}.$$

Clearly, this system is not complete in L^2 on any open set whose measure is greater than 1. Indeed, if the measure of an open set I is greater than one, then there exist points $x, y \in I$ such that $x - y \in \mathsf{Z}$. Since every function in $E(2\pi\mathsf{Z})$ has period one, no continuous function $F \in L^2(I)$ with $F(x) \neq F(y)$ can be approximated by linear combinations of the functions of $E(2\pi\mathsf{Z})$.

Observe also that since a sequence Λ in Theorem 1 is as 'dense' as the set

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$2\pi\mathbb{Z}$, the system $E(\Lambda)$ *is not* complete in L^2 on any single interval of length greater than one (see [1, chapter 9] for the details). On the other hand, it is complete in L^2 on any finite union of intervals $(k+\epsilon, k+1-\epsilon)$ whose length is only slightly less than one.

A simplified version of Landau's result was obtained in [3]. It says: if only $|\lambda_n - 2\pi n|$ tends 'fast enough' to zero then Landau's phenomenon occurs.

The purpose of this note is to obtain a multidimensional variant of Landau's result. Set

$$2\pi\mathbb{Z}^N = \{(2\pi n_1, \dots, 2\pi n_N)\}_{n_1,\dots,n_N \in \mathbb{Z}}$$

and consider the corresponding trigonometrical system:

$$E(2\pi\mathbb{Z}^N) = \{\exp[2\pi i(n_1 t_1 + \dots + n_N t_N)]\}_{n_1,\dots,n_N \in \mathbb{Z}}.$$

It is well-known that $E(2\pi\mathbb{Z}^N)$ is complete in L^2 on the open unit cube $(0,1)^N = \{(t_1, \dots, t_N) : 0 < t_k < 1, k = 1, \dots, N\}$. It is also easy to check that this system is not complete in L^2 on any open set of $N-$dimentional measure greater than one.

In connection with Theorem 1 one may expect that perturbations of $2\pi\mathbb{Z}^N$ may yield systems which are complete in L^2 on large open sets in $\mathbb{R}^N$. It turns out that the multidimensional Landau's phenomenon is in a way even more surprising then the one-dimensional. Namely, Theorem 2 gives examples of 'perturbed' systems which are complete in L^2 on any bounded open set that does not contain a neighborhood of $\mathbb{Z}^N$. Such open sets can be connected and have arbitrarily large N-dimentional measure.

THEOREM 2. *Suppose* $\{\delta_{n_1,\dots,n_N}\}_{n_1,\dots,n_N \in \mathbb{Z}}$ *is any sequence satisfying*

$$0 < |\delta_{n_1,\dots,n_N}| < Cr^{|n_1|+\dots+|n_N|}, \qquad n_1, \dots, n_N \in \mathbb{Z},$$

with some $0 < r < 1$ *and* $C > 0$. *Suppose also that* $s_1, s_2, \dots, s_N$ *are real numbers linearly independent over the set of integers. Then the system*

$$\{\exp i[(2\pi n_1 + s_1\delta_{n_1,\dots,n_N})t_1 + \dots + (2\pi n_N + s_N\delta_{n_1,\dots,n_N})t_N]\}_{n_1,\dots,n_N \in \mathbb{Z}}$$

is complete in L^2 *on any open bounded set in* $\mathbb{R}^N$ *whose closure has empty intersection with the set* $\mathbb{Z}^N$.

The proof of Theorem 1 in [2] is based on a good understanding of the Beurling-Malliavin density (for definition see [1, chapter 9]). In particular, the author makes use of the fact that the integers can be partitioned into an infinite number of disjoint sequences each of which has Beurling-Malliavin density one. In our proof we use the approach suggested in [3]. The proof is fairly simple, and does not use any 'deep' facts.

2. Proof of Theorem 2

We prove Theorem 2 for the case $N = 2$. The case $N > 2$ is similar to this case.

LEMMA 1. *Suppose a function* $G \in L^2((0,1)^2)$ *and there exists a number* $0 < \epsilon < 1$ *such that*

$$G(t_1, t_2) = 0, \quad t_1, t_2 \in (1-\epsilon, 1). \tag{1}$$

Suppose also that the Fourier transform $\hat{G}$ *of* G *satisfies:*

$$|\hat{G}(2\pi m, 2\pi n)| \leq C_1 r_1^{|m|+|n|}, \quad m, n \in \mathsf{Z}, \tag{2}$$

with some $0 < r_1 < 1$ *and* $C_1 > 0$. *Then* $G = 0$ *a.e.*

PROOF. Set

$$c_{m,n} := \int_0^1 \int_0^1 e^{i2\pi(mt_1+nt_2)} G(t_1, t_2) dt_1 dt_2 = \hat{G}(2\pi m, 2\pi n)$$

and

$$G_0(t_1, t_2) := \sum_{m,n \in \mathsf{Z}} c_{m,n} e^{-2i\pi(mt_1+nt_2)}.$$

By the definition of G_0, its restriction to the unit square has the same Fourier coefficients as G. It follows that $G(t_1, t_2) = G_0(t_1, t_2)$ a.e. in $(0,1)^2$. By (2), the coefficients $c_{m,n}$ are so small that the function $G_0(t_1, t_2)$ is determined and complex-analytic in the domain $|\Im t_1| + |\Im t_2| < -\log r_1$. Moreover, assumption (1) shows that G_0 vanishes in the open domain $(1-\epsilon, 1)^2 \subset \mathsf{R}^2$. It is well-known that a domain in R^2 has positive capacity while the zero set of a nontrivial analytic function has zero capacity. We conclude[1] that $G_0 \equiv 0$, so that $G = 0$ a.e.

LEMMA 2. *Suppose that a function* $G \in L^2((0,1)^2)$ *and satisfies* (1). *Suppose also that sequences* $\{\delta_{m,n}^{(1)}\}_{m,n\in\mathsf{Z}}$ *and* $\{\delta_{m,n}^{(2)}\}_{m,n\in\mathsf{Z}}$ *are such that*

$$|\delta_{m,n}^{(j)}| < C_2 r_2^{|m|+|n|}, \quad m, n \in \mathsf{Z}, \quad j = 1, 2, \tag{3}$$

and

$$|\hat{G}(2\pi m + \delta_{m,n}^{(1)}, 2\pi n + \delta_{m,n}^{(2)})| \leq C_2 r_2^{|m|+|n|}, \quad m, n \in \mathsf{Z}, \tag{4}$$

with some $0 < r_2 < 1$ *and* $C_2 > 0$. *Then* $G = 0$ *a.e.*

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[1] One can also use the Taylor series for G_0 about $(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2})$. Since G_0 vanishes in a real neighborhood of $(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2})$, all the partial derivatives at $(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2})$ and so all the coefficients in the series are equal to zero. This establishes that $G_0 \equiv 0$.

PROOF. Observe that

$$\left|e^{i2\pi(mt_1+nt_2)} - e^{i[(2\pi m+\delta^{(1)}_{m,n})t_1+(2\pi n+\delta m,n^{(2)})t_2]}\right| =$$

$$2\left|\sin\frac{\delta^{(1)}_{m,n}t_1 + \delta^{(2)}_{m,n}t_2}{2}\right| \leq \left|\delta^{(1)}_{m,n}t_1 + \delta^{(2)}_{m,n}t_2\right|.$$

Hence,

$$|\hat{G}(2\pi m, 2\pi n)| \leq |\hat{G}(2\pi m + \delta^{(1)}_{m,n}, 2\pi n + \delta^{(2)}_{m,n})|$$

$$+\left|\int_0^1\int_0^1\left(e^{i2\pi(mt_1+nt_2)} - e^{i[(2\pi m+\delta^{(1)}_{m,n})t_1+(2\pi n+\delta^{(2)}_{m,n})t_2}\right)G(t_1,t_2)dt_1dt_2\right|$$

$$\leq C_2 r_2^{|m|+|n|} + \int_0^1\int_0^1 |\delta^{(1)}_{m,n}t_1 + \delta^{(2)}_{m,n}t_2||G(t_1,t_2)|dt_1dt_2$$

$$\leq C_2 r_2^{|m|+|n|} + \|G\|_{L^2}(|\delta^{(1)}_{m,n}| + |\delta^{(2)}_{m,n}|).$$

This and (3) show that G satisfies the assumptions of Lemma 1, and so $G = 0$ a.e.

PROOF OF THEOREM 2 (case $N = 2$). We must show that every system

$$\{\exp i[(2\pi m + s_1\delta_{m,n})t_1 + (2\pi n + s_2\delta_{m,n})t_2]\}_{m,n\in\mathsf{Z}} \tag{5}$$

is complete in $L^2(I)$ when $I \subset \mathsf{R}^2$ is open, bounded and $\bar{I}\bigcap\mathsf{Z}^2 = \emptyset$, and s_1, s_2 are linearly independent over the integers and $\delta_{m,n}$ satisfy

$$|\delta_{m,n}| \leq Cr^{|m|+|n|}, \quad m,n\in\mathsf{Z}, \quad 0<r<1, \quad C>0. \tag{6}$$

By the Hahn-Banach theorem, it is enough to check that every function $F \in L^2(I)$ which is orthogonal to the functions in (5) must vanish a.e.

Let $F(t_1, t_2)$ be such a function, and set

$$F_{k,l}(t_1,t_2) = \begin{cases} F(t_1+k, t_2+l) & \text{if } 0 < t_1, t_2 < 1 \\ 0 & \text{otherwise.}\end{cases}$$

Since the support of the function F belongs to I and I is bounded, there exists an integer $M > 0$ such that $F_{k,l} = 0$ for all $|k|, |l| > M$. Since $\bar{I}\bigcap\mathsf{Z}^2 = \emptyset$, F vanishes in some neighborhood of Z^2. Hence, there exists a number $0 < \epsilon < 1$ such that

$$F_{k,l}(t_1,t_2) = 0, \quad t_1,t_2 \in (1-\epsilon, 1), \quad |k|,|l| \leq M. \tag{7}$$

It is also clear that $F_{k,l} \in L^2((0,1)^2)$ and that

$$F(t_1,t_2)=\sum_{-M\le k,l\le M} F_{k,l}(t_1-k,t_2-l).$$

Now, by a change of variables,

$$\hat{F}(z_1,z_2)=\sum_{-M\le k,l\le M} e^{i(kz_1+lz_2)}\hat{F}_{k,l}(z_1,z_2),$$

where $\hat{F}_{k,l}$ is the Fourier transform of $F_{k,l}$. The fact that F is orthogonal to the system (5) can be written as

$$\hat{F}(2\pi m+s_1\delta_{m,n},2\pi n+s_2\delta_{m,n})=0,\quad m,n\in\mathsf{Z}.$$

This and the last equality give:

$$\sum_{-M\le k,l\le M} e^{i(ks_1\delta_{m,n}+ls_2\delta_{m,n})}\hat{F_{k,l}}(2\pi m+s_1\delta_{m,n},2\pi n+s_2\delta_{m,n})=0,\ m,n\in\mathsf{Z}. \tag{8}$$

Let us consider the functions

$$H_p(t_1,t_2)=\sum_{-M\le k,l\le M}(ks_1+ls_2)^p F_{k,l}(t_1,t_2),\quad p=0,1,\ldots$$

Clearly, $H_p\in L^2((0,1)^2)$ and by (7),

$$H_p(t_1,t_2)=0,\quad t_1,t_2\in(1-\epsilon,1),\quad p=0,1,\ldots \tag{9}$$

Now, by (8),

$$\left|\hat{H}_0(2\pi m+s_1\delta_{m,n},2\pi n+s_2\delta_{m,n})\right|=\left|\sum_{-M\le k,l\le M}\hat{F}_{k,l}(2\pi m+s_1\delta_{m,n},2\pi n+s_2\delta_{m,n})\right|$$

$$\le\sum_{-M\le k,l\le M}\left|e^{i(ks_1\delta_{m,n}+ls_2\delta_{m,n})}-1\right|\left|\hat{F}_{k,l}(2\pi m+s_1\delta_{m,n},2\pi n+s_2\delta_{m,n})\right|$$

$$\le M|\delta_{m,n}|(|s_1|+|s_2|)\max_{k,l,x_1,x_2}|\hat{F}_{k,l}(x_1,x_2)|.$$

Since

$$|\hat{F}_{k,l}(x_1,x_2)|\le\int_k^{k+1}\int_l^{l+1}|F(t_1,t_2)|dt_1dt_2\le\|F\|_{L^2}<\infty,$$

we conclude by (6), (9) and Lemma 2 that

$$\hat{H}_0=\sum_{-M\le k,l\le M}\hat{F}_{k,l}\equiv 0.$$

This and (8) give

$$0 = \sum_{-M \le k,l \le M} (e^{i(ks_1\delta_{m,n}+ls_2\delta_{m,n})} - 1)\hat{F}_{k,l}(2\pi m + s_1\delta_{m,n}, 2\pi n + s_2\delta_{m,n})$$

$$= \sum_{-M \le k,l \le M} \sum_{p=1}^{\infty} \frac{(i\delta_{m,n}(ks_1 + ls_2))^p}{p!} \hat{F}_{k,l}(2\pi n + s_1\delta_{m,n}, 2\pi m + s_2\delta_{m,n}).$$

Hence,

$$\left|\hat{H}_1(2\pi m + s_1\delta_{m,n}, 2\pi n + s_2\delta_{m,n})\right|$$

$$= \left| \sum_{-M \le k,l \le M} (ks_1 + ls_2)\hat{F}_{k,l}(2\pi m + s_1\delta_{m,n}, 2\pi n + s_2\delta_{m,n}) \right|$$

$$\le |\delta_{m,n}| \sum_{-M \le k,l \le M} \sum_{p=2}^{\infty} \frac{|\delta_{m,n}|^{p-2}|ks_1 + ls_2|^p}{p!} |\hat{F}_{k,l}(2\pi m + s_1\delta_{m,n}, 2\pi n + s_2\delta_{m,n})|$$

$$\le |\delta_{m,n}| \sum_{p=2}^{\infty} \frac{|\delta_{m,n}|^{p-2} M^p (|s_1| + |s_2|)^p}{p!} \max_{k,l,x_1,x_2} |\hat{F}_{k,l}(x_1, x_2)|.$$

We conclude by (6), (9) and Lemma 2 that

$$\hat{H}_1 = \sum_{-M \le k,l \le M} (ks_1 + ls_2)\hat{F}_{k,l} \equiv 0.$$

In a similar fashion one establishes that

$$\hat{H}_p = \sum_{-M \le k,l \le M} (ks_1 + ls_2)^p \hat{F}_{k,l} \equiv 0 \tag{10}$$

for every $p = 0, 1, 2, \ldots$

Let us numerate the pairs (k, l) where $|k|, |l| \le M$:

$$j = (k, l), \quad j = 0, \ldots, M^2 - 1,$$

and set $y_j = (ks_1 + ls_2), j = 0, \ldots, M^2 - 1$. By assumption, the numbers s_1 and s_2 are linearly independent over the integers. Hence, the numbers y_j are different for different j and are not equal to zero. It follows that the determinant of the $M^2 \times M^2$ matrix

$$\alpha_{j,p} := y_j^p, \quad 0 \le j, p \le M^2 - 1,$$

is not zero. Since this matrix corresponds to the first $0 \le p \le M^2 - 1$ equations in (10), we deduce that this system has only trivial solutions, that is all $\hat{F}_{k,l} \equiv 0$, and so $\hat{F} \equiv 0$ and $F = 0$ a.e.

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