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TOEPLITZ OPERATORS ON GENERALIZED BERGMAN-HARDY SPACES

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Abstract

We study the Toeplitz operators $T_f : H_2 \to H_2$, for $f \in L_\infty$, on a class of spaces H_2 which includes, among many other examples, the Hardy and Bergman spaces as well as the Fock space. We investigate the space X of those elements $f \in L_\infty$ with $\lim_j ||T_f - T_{f_j}|| = 0$ where (f_j) is a sequence of vector-valued trigonometric polynomials whose coefficients are radial functions. For these T_f we obtain explicit descriptions of their essential spectra. Moreover, we show that $f \in X$, whenever T_f is compact, and characterize these functions in a simple and straightforward way. Finally, we determine those $f \in L_\infty$ where T_f is a Hilbert-Schmidt operator.

1. Introduction

Let $\mathsf{T}^n = \{(z_1, \ldots, z_n) \in \mathsf{C}^n : |z_k| = 1, k = 1, \ldots, n\}$ and consider the normalized Haar measure $d\varphi$ on T^n. For $z = (z_1, \ldots, z_n) \in \mathsf{C}^n$ and $m = (m_1, \ldots, m_n) \in \mathsf{Z}^n$, $k = (k_1, \ldots, k_n) \in \mathsf{Z}^n$ we use the following notation. Put $z^m = \prod_{j=1}^n z_j^{m_j}$. We write $r \cdot z = (r_1 z_1, \ldots, r_n z_n)$ if $r = (r_1, \ldots, r_n)$. Furthermore we put $z = r \cdot \exp(i\varphi)$ if $z_j = r_j e^{i\varphi_j}$ and $\varphi = (\varphi_1, \ldots, \varphi_n)$. Finally, we define $|m| = |m_1| + \ldots + |m_n|$.

Let μ be a bounded positive measure on R_+^n with supp $\mu \cap$ interior of $\mathsf{R}_+^n \neq \emptyset$ and consider, for $f, g : \mathsf{C}^n \to \mathsf{C}$,

$$\langle f, g\rangle = \int\int f(r \cdot \exp(i\varphi))\overline{g(r \cdot \exp(i\varphi))} d\varphi d\mu(r), \quad ||f||_2 = \sqrt{\langle f, f\rangle}.$$

We only deal with those μ which are such that all polynomials on C^n are elements of $L_2(d\varphi \otimes d\mu)$. (This is always satisfied if μ has compact support.)

Let $H_2(\mu)$ be the closure of the subspace of all polynomials in $L_2(d\varphi \otimes d\mu)$. $H_2(\mu)$ may be interpreted as a space of holomorphic functions where

$$M_2(f, r) := \left(\int |f(r \cdot \exp(i\varphi))|^2 d\varphi\right)^{1/2}$$

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is $L_2(\mu)-$bounded with respect to r.

EXAMPLES. Let λ be the Lebesgue measure on R_+^n.

(1) $d\mu(r) = (\prod_{j=1}^n r_j)e^{-\Sigma_{j=1}^n r_j^2/2}d\lambda(r)$. Here $H_2(\mu)$ is the Fock space ([8]).

(2) $d\mu(r) = (\prod_{j=1}^n r_j)1_{[0,1]^n}(r)d\lambda(r)$. Here $H_2(\mu)$ can be identified with the Bergman space on the polydisc D^n ([5,6,11]), i.e.

$$H_2(\mu) \cong \left\{ f : D^n \to \mathsf{C} : f \text{ holomorphic}, \int_{D^n} |f|^2 d\tilde{\lambda} < \infty \right\},$$

where $\tilde{\lambda}$ is the Lebesgue measure on C^n.

(3) $\mu = \delta_{(1,\dots,1)}$ (Dirac measure at $(1,\dots,1)$). Here $H_2(\mu)$ yields the classical Hardy space on the polydisc D^n ([6,11]), i.e.

$$H_2(\mu) \cong \left\{ f : D^n \to \mathsf{C} : f \text{ holomorphic}, \sup_{r \in [0,1[^n} M_2(f,r) < \infty \right\}.$$

(4) $\mu = \sum_{j=1}^\infty 2^{-k} f_k \nu_k$ where ν_k is a product of measures of the preceding kind and the $f_k \in L_1(d\nu_k)$ are non-negative.

It is one of our goals to give a unifying approach to these and to similar examples.

1.1. DEFINITION. Let $f \in L_\infty := L_\infty(d\varphi \otimes d\mu)$ and consider the orthogonal projection $P : L_2(d\varphi \otimes d\mu) \to H_2(\mu)$. The Toeplitz operator $T_f : H_2(\mu) \to H_2(\mu)$ is defined by $T_f h = P(f \cdot h),\ h \in H_2(\mu)$.

Clearly, $||T_f|| \le ||f||_\infty$. However, equality does not hold in general.

A function $f : \mathsf{C}^n \to \mathsf{C}$ is called radial if $f(r \cdot \exp(i\varphi)) = f(r)$ for all $r \cdot \exp(i\varphi) \in \mathsf{C}^n$. f is called angular if $f(r \cdot \exp(i\varphi)) = f(\exp(i\varphi))$ whenever $r \cdot \exp(i\varphi) \in \mathsf{C}^n \setminus \{0\}$. Put, for $k \in \mathsf{Z}^n$,

$$\xi_k(r \cdot \exp(i\varphi)) = \prod_{j=1}^n e^{ik_j\varphi_j}.$$

So ξ_k is angular.

Note that any $f \in L_\infty(d\varphi \otimes d\mu)$ has a Fourier series expansion $\sum_{k\in \mathsf{Z}^n} F_k \cdot \xi_k$, where the Fourier coefficients F_k are radial functions. Here

$$F_k(r) = \int f(r \cdot \exp(i\varphi))\xi_{-k}(r \cdot \exp(i\varphi))d\varphi.$$

This series converges, for fixed r, $\mu - a.e.$ in the $L_2(d\varphi)$-sense. Using the dominated convergence theorem we see that the series converges to f in $L_2 := L_2(d\varphi \otimes d\mu)$. We sometimes write $f \overset{(L_2)}{=} \sum_k F_k \xi_k$.

Define

$$e_m(r \cdot \exp(i\varphi)) = \frac{r}{\sqrt{\int r^{2m} d\mu}} \xi_m(r \cdot \exp(i\varphi)), \quad r \cdot \exp(i\varphi) \in \mathsf{C}, \ m \in \mathsf{Z}_+^n.$$

Then $\{e_m : m \in \mathsf{Z}_+^n\}$ is a complete ON-system for $H_2(\mu)$. For $h = \sum_{l \in \mathsf{Z}_+^n} \beta_l e_l \in H_2(\mu)$ put $P_j h = \sum_{|l|<j} \beta_l e_l$, $j \in \mathsf{Z}_+$, in particular, $P_0 = 0$.

1.2. Proposition. *Let $f \in L_\infty$ and $h \in H_2(\mu)$.*
If $f \stackrel{(L_2)}{=} \sum_{k \in \mathsf{Z}^n} F_k \xi_k$, F_k radial, and $h = \sum_{l \in \mathsf{Z}_+^n} \beta_l e_l$ then we have

$$T_f h = \sum_{m \in \mathsf{Z}_+^n} \left(\sum_{l \in \mathsf{Z}_+^n} \frac{\int F_{m-l} r^{m+l} d\mu}{\sqrt{\int r^{2m} d\mu \int r^{2l} d\mu}} \beta_l \right) e_m.$$

In particular, for radial F,

$$T_F h = \sum_{m \in \mathsf{Z}_+^n} \left(\frac{\int F r^{2m} d\mu}{\int r^{2m} d\mu} \right) \beta_m e_m.$$

Proof. By definition of T_f we obtain

$$T_f h = \sum_{m \in \mathsf{Z}_+^n} \langle f \cdot h, e_m \rangle e_m.$$

Using the Fourier expansion of f and the fact that

$$\langle F_k \xi_k e_l, e_m \rangle = \begin{cases} \frac{\int F_k r^{l+m} d\mu}{\sqrt{\int r^{2l} d\mu \int r^{2m} d\mu}} & \text{if } k + l = m \\ 0 & \text{else} \end{cases}$$

we derive the first assertion. The second equation follows from the first one by putting $l = m$.

2. The spaces X and X_c

Let $\mathscr{L}(H_2(\mu))$ be the space of all bounded linear operators on $H_2(\mu)$ and $\mathscr{K} \subset \mathscr{L}(H_2(\mu))$ the space of all compact operators. Moreover let $q : \mathscr{L}(H_2(\mu)) \to \mathscr{L}(H_2(\mu))/\mathscr{K}$ be the quotient map and define $\tau : L_\infty \to \mathscr{L}(H_2(\mu))$ by $\tau(f) = T_f$, $f \in L_\infty$. τ is a linear map.

Recall that $\mathscr{L}(H_2(\mu))$ is the dual Banach space for the trace class operators on $H_2(\mu)$. With respect to this duality, $\mathscr{L}(H_2(\mu))$ is the bidual of $\mathscr{K}$ ([7]).

Functions of the form $\sum_{|k| \le j} F_k \xi_k$ for some integer j and radial L_∞-functions F_k will be called $L_\infty(d\mu)$-valued trigonometric polynomials.

Now we introduce our main objects of study.

2.1. Definition. Put

$$X = \{f \in L_\infty : \text{ there is a sequence of } L_\infty(d\mu)\text{-valued trigonometric polynomials } f_j \text{ with } \lim_j ||qT_{f_j} - qT_f|| = 0\},$$

$$X_c = \{f \in L_\infty : \text{ there is a sequence of } L_\infty(d\mu)\text{-valued trigonometric polynomials } f_j \text{ with } \lim_j ||f_j - f||_\infty = 0\}.$$

We have $X_c \subset X$. Note, X_c contains all $L_\infty(d\mu)$-valued trigonometric polynomials. So there are many discontinuous functions which are elements of X_c (and hence of X), for example all radial L_∞-functions. The most important property of X is the following: If T_f is compact then f is always an element of X (by definition of X).

If $n = 1$ we give an explicit description of the maximal ideal space of the C^*-algebra generated by $\{qT_f : f \in X\}$, which turns out to be commutative under some restrictions on μ (Theorem 5.3.). In particular we describe $||qT_f||$ and determine the essential spectrum of T_f for $f \in X$ (Corollary 5.4.). Finally, for arbitrary n, we characterize those $f \in X$ where T_f is compact and those $f \in L_\infty$ where T_f is a Hilbert-Schmidt operator (section 6).

2.1. Lemma. (a) *Let* $f, f_j \in L_\infty$ *such that* $\lim_j ||f - f_j||_2 = 0$ *and* $\sup_j ||f_j||_\infty < \infty$. *Then, for any* $h \in H_2(\mu)$, *we have* $\lim_j T_{f_j}h = T_f h$. *Furthermore,* $T_f = w^* - \lim_j T_{f_j}$ *with respect to the* w^**-topology on* $\mathscr{L}(H_2(\mu))$.

(b) *Assume that, for* $f_j, f \in L_\infty$, $\lim_j ||qT_f - qT_{f_j}|| = 0$ *and* $\lim_j T_{f_j}h = T_f h$, $h \in H_2(\mu)$. *Then there is a sequence of convex combinations* g_k *of* f_j *such that* $\lim_k ||T_f - T_{g_k}|| = 0$.

Proof. (a) Fix $h \in H_2(\mu)$ and take, for $\epsilon > 0$, $\tilde{h} \in L_\infty$ with $||h - \tilde{h}||_2 \leq \epsilon$. We have

$$||T_f h - T_{f_j}h||_2 \leq \epsilon \sup_j ||f - f_j||_\infty + ||\tilde{h}||_\infty ||f - f_j||_2.$$

Hence

$$\limsup_{j\to\infty} ||T_f h - T_{f_j}h||_2 \leq \epsilon \sup_j ||f - f_j||_\infty.$$

We obtain $\lim_j ||T_{f_j}h - T_f h||_2 = 0$ since ϵ was arbitrary. For the second part of (a) let T be a trace class operator on $H_2(\mu)$ with complete ON-systems (f_k), (g_l) and singular numbers λ_k such that

$$Th = \sum_k \lambda_k \langle h, f_k\rangle g_k, \; h \in H_2(\mu), \text{ and } \sum_k |\lambda_k| < \infty.$$

Then according to the duality on $\mathscr{L}(H_2(\mu))$ ([7]),

$$\langle T, T_{f_j}\rangle := \mathrm{trace}(TT_{f_j}) = \sum_m \langle TT_{f_j}g_m, g_m\rangle = \sum_k \lambda_k\langle T_{f_j}g_k, f_k\rangle.$$

Since $\lim_j\langle T_{f_j}g_k, f_k\rangle = \langle T_f g_k, f_k\rangle$ for all k we see that $\lim_j\langle T, T_{f_j}\rangle = \langle T, T_f\rangle$.

(b) We find $K_j \in \mathscr{K}$ with $\lim_j ||T_f - T_{f_j} + K_j|| = 0$. Since $T_{f_j} \to T_f$ in the strong operator topology, applying the basis projections P_k, we obtain $\lim_j ||(T_f - T_{f_j})P_k|| = 0$ for all k. Moreover $\lim_j ||(T_f - T_{f_j})P_k + K_jP_k|| = 0$, so $\lim_j ||K_jh||_2 = 0$ for all $h \in H_2(\mu)$. We infer, as in (a), that $K_j \to 0$ weakly since $\mathscr{K}^*$ is the space of all trace class operators. By Mazur's theorem ([2]) there is a suitable sequence $H_k = \sum_{j=a_k}^{b_k}\lambda_{j,k}K_j$ of convex combinations of K_j with $\lim_k ||H_k|| = 0$ and $a_k \to \infty$. Denote the corresponding convex combinations of the f_j by g_k. We conclude $\lim_k ||T_f - T_{g_k}|| = 0$.

For $f \overset{(L_2)}{=} \sum_{k\in \mathsf{Z}^n} F_k\xi_k$, F_k radial, define the Cesaro means $\sigma_j f$ by

$$\sigma_j f = \sum_{|k|\le j}\frac{j-|k|}{j}F_k\xi_k.$$

We always have $||\sigma_j f||_p \le ||f||_p$, if $p = 2$ or $p = \infty$ and $\lim_j ||f - \sigma_j f||_2 = 0$ ([3], apply σ_j to the function $f_z(w) = f(wz)$ for fixed $z \in \mathsf{C}^n$ and $w \in \mathsf{C}$).

Put, for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathsf{T}^n$ and $z = (z_1, \dots, z_n) \in \mathsf{C}^n$,

$$f_\lambda(z) = f(\lambda_1 z_1, \dots, \lambda_n z_n).$$

Then we obtain $||f||_p = ||f_\lambda||_p$ if $p = 2$ or $p = \infty$.

Let $T \in \mathscr{L}$. Frequently, we make use of the fact that

$$||qT|| = \inf_j ||T(\mathrm{id} - P_j)|| = \inf_k ||(\mathrm{id} - P_k)T||.$$

2.2. Lemma. *We have*

(a) $$T_{f_\lambda}h = (T_f h_{\bar\lambda})_\lambda \text{ if } \lambda \in \mathsf{T}^n \text{ and } h \in H_2(\mu),$$

(b) $$||T_{\sigma_j f}|| \le ||T_f|| \text{ and } ||qT_{\sigma_j f}|| \le ||qT_f|| \text{ for every } j \in \mathsf{Z}_+.$$

Proof. (a) Here $f_\lambda \overset{(L_2)}{=} \sum_k F_k\lambda^k\xi_k$ if $f \overset{(L_2)}{=} \sum_k F_k\xi_k$. Hence, (a) follows from Proposition 1.2.

(b) Let $\Gamma_j(w)$ be the Fejer kernel with

$$\Gamma_j(w) = \sum_{|k|\le j}\frac{j-|k|}{j}w^k,\ w \in \mathsf{T}.$$

Extending the preceding notation we define, for $h \in H_2(\mu)$ and $w \in \mathsf{T}$,

$$h_w(z) = h(w\cdot z), \quad \text{if } z \in \mathsf{C}^n,$$

(i.e. $h_w = h_{(w,\dots,w)}$ in the former notation). Then, using Fubini's theorem and Cauchy-Schwarz inequality, we have, for $h \in H_2(\mu)$,

$$||T_{\sigma_j f} h||_2^2 = \int\int \left| \int_{\mathsf{T}} (T_f h_{e^{-i\psi}})_{e^{i\psi}} \Gamma_j(e^{-i\psi}) d\psi \right|^2 d\varphi d\mu$$
$$\leq \sup_\psi ||T_f h_{e^{-i\psi}}||_2^2.$$

This implies $||T_{\sigma_j f}|| \leq ||T_f||$. Moreover, if $h \in (\mathrm{id} - P_j)H_2(\mu)$ then $h_\lambda \in (\mathrm{id} - P_j)H_2(\mu)$ for any $\lambda \in \mathsf{T}^n$. Hence the preceding yields $||T_{\sigma_j f}(\mathrm{id} - P_j)|| \leq ||T_f(\mathrm{id} - P_j)||$ for any j from which we infer $||qT_{\sigma_j f}|| \leq ||qT_f||$.

2.3. Proposition. *We obtain*

$$X = \{f \in L_\infty : \lim_j ||T_f - T_{f_j}|| = 0 \text{ for some } L_\infty(d\mu)\text{-valued trigonometric polynomials } f_j\}$$
$$= \{f \in L_\infty : \lim_j ||T_f - T_{\sigma_j f}|| = 0\}.$$

Proof. Put

$$Y = \{f \in L_\infty : \lim_j ||T_f - T_{f_j}|| = 0 \text{ for some } L_\infty(d\mu)\text{-valued trigonometric polynomials } f_j\}.$$

Then clearly, $Y \subset X$. Conversely, let $f \in X$ and let f_j be $L_\infty(d\mu)$-valued trigonometric polynomials with $\lim_j ||qT_f - qT_{f_j}|| = 0$. We obtain easily $\lim_k ||f_j - \sigma_k f_j||_\infty = 0$ for each j. Fix $\epsilon > 0$ and j with $||qT_{f-f_j}|| \leq \epsilon/3$ and find k_j with $||f_j - \sigma_k f_j||_\infty \leq \epsilon/3$ for all $k \geq k_j$. We conclude, using Lemma 2.2.(b),

$$||qT_f - qT_{\sigma_k f}|| \leq ||qT_{f-f_j}|| + ||qT_{f_j - \sigma_k f_j}|| + ||qT_{\sigma_k(f-f_j)}|| \leq \epsilon.$$

Thus $\lim_k ||qT_f - qT_{\sigma_k f}|| = 0$. In view of Lemma 2.1. we find suitable convex combinations g_j of the $\sigma_k f$ such that $\lim_k ||T_f - T_{g_k}|| = 0$. This yields the first part of the proposition. Finally, a 3ϵ−proof as before now shows that even $\lim_k ||T_f - T_{\sigma_k f}|| = 0$.

3. Conditions on the measure μ

Before we come to the main results in sections 4 and 5 we dicuss moment conditions on μ which are needed in the proofs lateron. Here we restrict ourselves to the case of $n = 1$. So let μ be a measure on R_+.

3.1. Definition. Consider

$$\text{(I)}\qquad \lim_{m\to\infty}\int\left|\frac{s^{m-k}}{\int r^{m-k}d\mu}-\frac{s^m}{\int r^m d\mu}\right|d\mu(s)=0 \text{ for all } k\in \mathsf{Z}_+$$

and

$$\text{(II)}\qquad \lim_{m\to\infty}\frac{\int r^m d\mu\int r^{m-l-k}d\mu}{\int r^{m-k}d\mu\int r^{m-l}d\mu}=1 \text{ for all } k,l\in \mathsf{Z}_+$$

Examples. If μ is a Dirac measure then (I) and (II) are satisfied. An elementary calculation shows that μ of the Fock space (section 1) satisfies (I) and (II), too. Similarly $d\mu(r)=e^{-r}dr$ fulfils the conditions of Definition 3.1. The next Proposition implies that the measure of the Bergman space is also included. Indeed, we have

3.2. Proposition. *Let* μ *have bounded support and assume that* $a=\sup(\operatorname{supp}\mu)$. *Then* μ *satisfies* (I) *and* (II).

Proof. We show

$$(\star)\qquad \lim_{m\to\infty}\frac{\int r^{m-k}d\mu}{\int r^m d\mu}=a^{-k} \text{ for all } k\in \mathsf{Z}_+.$$

(II) is a direct consequence of $(\star)$. By assumption, for $0<\delta<1$, we have $0<\int_{(1-\delta)a}^a d\mu$. Moreover,

$$\mu([0,a])\le\mu([0,(1-\delta)a])+\mu([(1-\delta)a,a]).$$

Hence

$$\begin{aligned}a^{-k}&\le\frac{\int_0^a r^{m-k}d\mu}{\int_0^a r^m d\mu}\\&\le\frac{(1-\delta)^m a^m}{(1-\delta/2)^m a^m}(1-\delta)^{-k}a^{-k}\frac{\int_0^{(1-\delta)a}d\mu}{\int_{(1-\delta/2)a}^a d\mu}+(1-\delta)^{-k}a^{-k}\frac{\int_{(1-\delta)a}^a r^m d\mu}{\int_{(1-\delta)a}^a r^m d\mu}.\end{aligned}$$

The right-hand side converges to $(1-\delta)^{-k}a^{-k}$ as $m\to\infty$. Since δ was arbitrary we obtain $(\star)$ and hence (II). To prove (I) observe that

$$\int\left|\frac{s^{m-k}}{\int r^{m-k}d\mu}-\frac{s^m}{\int r^m d\mu}\right|d\mu=\frac{\int s^{m-k}\left|1-s^k\frac{\int r^{m-k}d\mu}{\int r^m d\mu}\right|d\mu}{\int r^{m-k}d\mu}.$$

With $C=\sup_m(\int_0^a r^{m-k}d\mu/\int_0^a r^m d\mu)$ and $0<\delta<1$ as above we obtain

$$0 \leq \frac{\int_0^a s^{m-k}\left|1 - s^k \frac{\int_0^a r^{m-k} d\mu}{\int_0^a r^m d\mu}\right| d\mu}{\int_0^a r^{m-k} d\mu}$$

$$\leq \frac{(1-\delta)^{m-k} a^{m-k}}{(1-\delta/2)^{m-k} a^{m-k}} (1 + a^k C) \frac{\int_0^{(1-\delta)a} d\mu}{\int_{(1-\delta/2)a}^a d\mu}$$

$$+ \max\left(\left|a^k \frac{\int_0^a r^{m-k} d\mu}{\int_0^a r^m d\mu} - 1\right|, \left|1 - a^k (1-\delta)^k \frac{\int_0^a r^{m-k} d\mu}{\int_0^a r^m d\mu}\right|\right) \frac{\int_{(1-\delta)a}^a s^{m-k} d\mu}{\int_{(1-\delta)a}^a r^{m-k} d\mu}.$$

With ($\star$) the right-hand side tends to $1 - (1-\delta)^k$ as $m \to \infty$. Since δ was arbitrary we obtain (I).

4. The algebra generated by $q\tau(X)$

Here we study $q\tau(X) \subset \mathscr{L}(H_2(\mu))/\mathscr{K}$. Again, let $n = 1$. At first we show

4.1. Proposition. *Let μ satisfy* (I) *and* (II)*. Then for any radial F and $k, l \in \mathsf{Z}$ we have*

(a) $$q(T_{F\xi_k}) = q(T_F) \cdot q(T_{\xi_k}) = q(T_{\xi_k}) \cdot q(T_F) \quad \textit{and}$$

(b) $$q(T_{\xi_{k+l}}) = q(T_{\xi_k}) \cdot q(T_{\xi_l}).$$

Proof. Let $h = \sum_{l \in \mathsf{Z}_+} \beta_l e_l \in H_2(\mu)$. Then, in view of Proposition 1.2.,

$$T_{F\xi_k} h = \sum_{m \geq \max(k,0)} \frac{\int F r^{2m-k} d\mu}{\sqrt{\int r^{2m} d\mu \int r^{2m-2k} d\mu}} \beta_{m-k} e_m \text{ and}$$

$$T_{\xi_k} h = \sum_{m \geq \max(k,0)} \frac{\int r^{2m-k} d\mu}{\sqrt{\int r^{2m} d\mu \int r^{2m-2k} d\mu}} \beta_{m-k} e_m$$

Hence

$$T_F T_{\xi_k} h = \sum_{m \geq \max(k,0)} \left(\frac{\int F r^{2m} d\mu}{\int r^{2m} d\mu}\right)\left(\frac{\int r^{2m-k} d\mu}{\sqrt{\int r^{2m} d\mu \int r^{2m-2k} d\mu}}\right) \beta_{m-k} e_m.$$

We obtain

$$(T_{F\xi_k} - T_F T_{\xi_k}) h = \sum_{m \geq \max(k,0)} \frac{\int F(s) s^{2m-k}\left(1 - s^k \frac{\int r^{2m-k} d\mu}{\int r^{2m} d\mu}\right) d\mu(s)}{\sqrt{\int r^{2m} d\mu \int r^{2m-2k} d\mu}} \beta_{m-k} e_m.$$

So, for $j \in \mathsf{Z}_+$ and the basis projections P_j (section 1),

$$||(\mathrm{id}-P_j)(T_{F\xi_k}-T_FT_{\xi_k})||\le\sup_{m\ge j}\left|\frac{\int Fs^{2m-k}\left(1-s^k\frac{\int r^{2m-k}d\mu}{\int r^{2m}d\mu}\right)d\mu}{\sqrt{\int r^{2m}d\mu\int r^{2m-2k}d\mu}}\right|$$

$$\le||F||_\infty\sup_{m\ge j}\frac{\int s^{2m-k}\left|1-s^k\frac{\int r^{2m-k}d\mu}{\int r^{2m}d\mu}\right|d\mu}{\int r^{2m-k}d\mu}.$$

(Here we used the Cauchy-Schwarz inequality.) In view of condition (I) the right-hand side tends to 0 as $j\to\infty$. This implies $T_{F\xi_k}-T_FT_{\xi_k}\in\mathscr{K}$. Similarly we obtain

$$(T_{F\xi_k}-T_{\xi_k}T_F)h=\sum_{m\ge\max(k,0)}\frac{\int Fs^{2m-2k}\left(s^k-\frac{\int r^{2m-k}d\mu}{\int r^{2m-2k}d\mu}\right)d\mu}{\sqrt{\int r^{2m}d\mu\int r^{2m-2k}d\mu}}\beta_{m-k}e_m$$

and

$$||(\mathrm{id}-P_j)(T_{F\xi_k}-T_{\xi_kT_F})||\le||F||_\infty\sup_{m\ge j}\frac{\int s^{2m-2k}\left|s^k-\frac{\int r^{2m-k}d\mu}{\int r^{2m-2k}d\mu}\right|d\mu}{\int r^{2m-k}d\mu}.$$

Again by (I), $T_{F\xi_k}-T_{\xi_k}T_F\in\mathscr{K}$. We conclude (a). To prove (b) we derive from Proposition 1.2.

$$T_{\xi_l}T_{\xi_k}h=$$
$$\sum_{m\ge\max(k+l,l,0)}\left(\frac{\int r^{2m-l}d\mu}{\sqrt{\int r^{2m}d\mu\int r^{2m-2l}d\mu}}\right)\left(\frac{\int r^{2m-2l-k}d\mu}{\sqrt{\int r^{2m-2l}d\mu\int r^{2m-2k-2l}d\mu}}\right)\beta_{m-k-l}e_m$$

and hence, for $j\in\mathsf{Z}_+$ with $j>\max(k+l,l,0)$,

$$(\mathrm{id}-P_j)(T_{\xi_{l+k}}-T_{\xi_l}T_{\xi_k})h=$$

$$\sum_{m\ge j}\left(\frac{\int r^{2m-l-k}d\mu}{\sqrt{\int r^{2m}d\mu\int r^{2m-2l-2k}d\mu}}\right)\left(1-\frac{\int r^{2m-l}d\mu\int r^{2m-2l-k}d\mu}{\int r^{2m-l-k}d\mu\int r^{2m-2l}d\mu}\right)\beta_{m-k-l}e_m.$$

This implies

$$||(\mathrm{id} - P_j)(T_{\xi_{l+k}} - T_{\xi_l} T_{\xi_k})||$$
$$\leq \sup_{m \geq j} \left(\frac{\int r^{2m-l-k} d\mu}{\sqrt{\int r^{2m} d\mu \int r^{2m-2l-2k} d\mu}} \right) \left| 1 - \frac{\int r^{2m-l} d\mu \int r^{2m-2l-k} d\mu}{\int r^{2m-l-k} d\mu \int r^{2m-2l} d\mu} \right|$$
$$\leq \sup_{m \geq j} \left| 1 - \frac{\int r^{2m-l} d\mu \int r^{2m-2l-k} d\mu}{\int r^{2m-l-k} d\mu \int r^{2m-2l} d\mu} \right|.$$

(For the latter estimate we used the Cauchy-Schwarz inequality.) The right-hand side tends to 0 as $j \to \infty$ according to condition (II). We obtain $T_{\xi_{l+k}} - T_{\xi_l} T_{\xi_k} \in \mathscr{K}$ which yields (b).

REMARK. Proposition 4.1.(a) remains valid for arbitrary n with an analoguous proof. However 4.1.(b) is no longer true for $n > 1$. Here $T_{\xi_k} T_{\xi_{-k}} - \mathrm{id}$ is not compact in general.

4.2. COROLLARY. *If* μ *satisfies* (I) *and* (II) *then* $q\tau(X)$ *generates a commutative* C^**-algebra, hence a* $C(K)$*-space.*

PROOF. This is an easy consequence of Proposition 4.1. and the fact that $\{qT_f : f \text{ a } L_\infty(d\mu)\text{-valued trigonometric polynomial}\}$ is dense in $q\tau(X)$.

5. The functions $\Phi_{\mathscr{U}}(f)$

Here we want to characterize the maximal ideal space of the algebra generated by $q\tau X$. *Throughout this section let* $n = 1$ *and let* μ *satisfy* (I) *and* (II) .

Let $f \in L_\infty = L_\infty(d\varphi \otimes \mu)$. Recall, $\int f(r \cdot \exp(i\varphi)) r^{2m} d\mu(r) / \int r^{2m} d\mu(r)$ is an element of $L_\infty(d\varphi) = L_1^*(d\varphi)$. Let $\mathscr{U}$ be a free ultrafilter on Z_+. The limit along $\mathscr{U}$ will be denoted by $\lim_{m,\mathscr{U}}$. Put, for $z = \exp(i\varphi) \in \mathsf{T}$,

$$\Phi_{\mathscr{U}}(f)(z) = w^* - \lim_{m,\mathscr{U}} \left(\frac{\int f(r \cdot \exp(i\varphi)) r^{2m} d\mu}{\int r^{2m} d\mu} \right).$$

Then $\Phi_{\mathscr{U}}$ is linear in f. Moreover, $\Phi_{\mathscr{U}}(f) \in L_\infty(d\varphi)$ and $||\Phi_{\mathscr{U}}(f)||_\infty \leq ||f||_\infty$.

5.1. LEMMA. (a) *If* $f \overset{(L_2)}{=} \sum_{k \in \mathsf{Z}} F_k \xi_k$ *for radial* F_k *then we have*

$$\Phi_{\mathscr{U}}(f) \overset{(L_2)}{=} \sum_{k \in \mathsf{Z}} \left(\lim_{m,\mathscr{U}} \frac{\int F_k(r) r^{2m} d\mu}{\int r^{2m} d\mu} \right) \xi_k.$$

(b) *For any* $\mathscr{U}$ *there is a suitable sequence* $N \subset \mathsf{Z}_+$ *with* $\Phi_{\mathscr{U}}(f) = w^* - \lim_{m \in N} \left(\int f r^{2m} d\mu / \int r^{2m} d\mu \right)$.

(c) $\Phi_{\mathscr{U}}(f) = f$ *if* f *is angular.*

(d) $\Phi_{\mathscr{U}}(F) = \lim_{m,\mathscr{U}} (\int F(r) r^{2m} d\mu / \int r^{2m} d\mu)$ *if* F *is radial. Hence* $\Phi_{\mathscr{U}}(F)$ *is a constant function.*

(e) *Let* $a = \sup(\operatorname{supp}\mu)$ *(a can be ∞). Assume that* $\lim_{r\to a} f(r\cdot\exp(i\varphi))$ *exists* a.e. *on* T. *Then*

$$\Phi_{\mathscr{U}}(f)(\exp(i\varphi)) = \lim_{r\to a} f(r\cdot\exp(i\varphi)).$$

Proof. Put $\Phi_m(f) = \frac{\int fr^{2m}d\mu}{\int r^{2m}d\mu}$. Then (Φ_m) is uniformly bounded in $L_\infty(d\varphi)$ and

$$\Phi_m(f) \overset{(L_2)}{=} \sum_k \left(\frac{\int F_k r^{2m}d\mu}{\int r^{2m}d\mu}\right)\xi_k.$$

Since the unit ball of $L_\infty(d\varphi)$ is w^*-sequentially compact we find a sequence $N \in \mathscr{U}$ such that $\Phi_{\mathscr{U}}(f) = w^* - \lim_{m\in N}\Phi_m(f)$. The Fourier coefficients of $\Phi_{\mathscr{U}}(f)$ are $\lim_{m\in N}\frac{\int F_k r^{2m}d\mu}{\int r^{2m}d\mu} = \lim_{m,\mathscr{U}}\frac{\int F_k r^{2m}d\mu}{\int r^{2m}d\mu}$. This proves (a) and (b). The remaining assertions are straightforward.

5.2. Lemma. *For any* $f \in L_\infty$ *with* $f \overset{(L_2)}{=} \sum_k F_k\xi_k$, F_k *radial, we have*

$$\lim_{m,\mathscr{U}}\frac{\int F_k r^{2m}d\mu}{\int r^{2m}d\mu} = \lim_{m,\mathscr{U}}\langle T_f e_{m-k}, e_m\rangle, \quad k \in \mathsf{Z}.$$

In particular $\left|\lim_{m,\mathscr{U}}\frac{\int F_k r^{2m}d\mu}{\int r^{2m}d\mu}\right| \leq ||qT_f||$ *for all* k.

Proof. We have, with Proposition 1.2.,

$$\langle T_f e_{m-k}, e_m\rangle = \frac{\int F_k r^{2m-k}d\mu}{\sqrt{\int r^{2m}d\mu\int r^{2m-2k}d\mu}} =$$

$$\left(\frac{\int r^{2m-k}d\mu\int r^{2m-k}d\mu}{\int r^{2m}d\mu\int r^{2m-2k}d\mu}\right)^{1/2}\left(\frac{\int F_k r^{2m}d\mu}{\int r^{2m}d\mu} + \int F_k\left(\frac{r^{2m-k}}{\int s^{2m-k}d\mu} - \frac{r^{2m}}{\int s^{2m}d\mu}\right)d\mu\right).$$

The first result follows by applying (I) and (II) (with $l = k$). Finally, we obtain for any $j \in \mathsf{Z}_+$,

$$|\lim_{m,\mathscr{U}}\langle T_f e_{m-k}, e_m\rangle| = |\lim_{m,\mathscr{U}}\langle T_f(\mathrm{id} - P_j)e_{m-k}, e_m\rangle| \leq ||T_f(\mathrm{id} - P_j)||.$$

Since $||qT_f|| = \inf_j ||T_f(id - P_j)||$ we infer the second result.

If $\Phi_{\mathscr{U}}(f) \in L_\infty(d\varphi)$ can be represented by a continuous function, we shall always identify $\Phi_{\mathscr{U}}(f)$ with its continuous representative. For a commutative Banach algebra A let Spec(A) be the the maximal ideal space. Finally, let $\mathscr{A}$ be the closed subalgebra of $\mathscr{L}(H_2(\mu))/\mathscr{K}$ generated by $q\tau X$.

5.3. THEOREM. *For any $f \in X$ the function $\Phi_{\mathscr{U}}(f)$ is continuous. Moreover,*

$$\mathrm{Spec}(\mathscr{A}) \circ q \circ \tau|_X = \{\Phi_{\mathscr{U}}(\cdot)(z)|_X : z \in \mathsf{T},\ \mathscr{U} \text{ a free ultrafilter on } \mathsf{Z}_+\}.$$

PROOF. (a): At first, a few introductory remarks.
Let $Y =$ closed span of $\{\xi_k : k \in \mathsf{Z}\} \subset L_\infty$. Then, in view of the Weierstrass theorem, Y can be identified with $C(\mathsf{T})$, the continuous functions on T. By Proposition 4.1. $\overline{q\tau Y}$ is a commutative C^*-algebra.

Put $\mathscr{B} =$ closed subalgebra of $\mathscr{L}(H_2(\mu))$ generated by $\{T_F : F \text{ radial}\}$. According to Proposition 1.2., $\mathscr{B}$ is a commutative C^*-algebra which consists of multipliers, i.e. if $T \in \mathscr{B}$ then there is a bounded sequence (a_k) with $T(\sum_k \beta_k e_k) = \sum_k a_k \beta_k e_k$. Put $\Phi_k(T) = a_k$. Then $\Phi_k \in \mathrm{Spec}(\mathscr{B})$. Moreover $||T|| = \sup_k |\Phi_k(T)|$. Hence $\mathrm{Spec}(\mathscr{B}) = w^*$-closure of $\{\Phi_k : k \in \mathsf{Z}_+\}$.

The definition of $\mathscr{A}$ and Proposition 4.1. imply $\mathscr{A} = \overline{q\mathscr{B} \otimes q\tau Y}$. We have $\mathrm{Spec}(\mathscr{A})|_{\overline{q\mathscr{B}}} = \mathrm{Spec}(\overline{q\mathscr{B}})$ and $\mathrm{Spec}(\mathscr{A})|_{\overline{q\tau Y}} = \mathrm{Spec}(\overline{q\tau Y})$. Put

$$\Omega = \{\Phi_{\mathscr{U}}(\cdot)(z)|_X : z \in \mathsf{T},\ \mathscr{U} \text{ a free ultrafilter on } \mathsf{Z}_+\}.$$

(b): Now let $\Psi \in \mathrm{Spec}(\mathscr{A})$. For radial $F \in L_\infty$ and $k \in \mathsf{Z}$ we obtain, by Proposition 4.1.,

$$\Psi(qT_{F\xi_k}) = \Psi(qT_F) \cdot \Psi(qT_{\xi_k})$$

and $\Psi \circ q \circ \tau|_Y \in \mathrm{Spec}(Y)$. Hence there is $z \in \mathsf{T}$ such that $\Psi \circ q \circ \tau|_Y$ is the Dirac functional δ_z. Moreover $\Psi \circ q|_{\mathscr{B}} \in \mathrm{Spec}(\mathscr{B})$ and $\Psi \circ q|_{\mathscr{K}} = 0$. This implies, for any $T \in \mathscr{B}$ and $j \in \mathsf{Z}_+$, $\Psi(qP_jT) = 0$. Hence there is a free ultrafilter $\mathscr{U}$ on Z_+ with $\Psi \circ q|_{\mathscr{B}} = w^* - \lim_{k,\mathscr{U}} \Phi_k$ and therefore $(\Psi q\tau)(F) = \Phi_{\mathscr{U}}(F)$ if F is radial (in view of Proposition 1.2.). Thus, if $f = \sum_{|k| \le j} F_k \xi_k$ is a $L_\infty(d\mu)-$valued trigonometric polynomial we have $\Psi(qT_f) = \Phi_{\mathscr{U}}(f)(z)$.

(c): Conversely, let $\mathscr{U}$ be a free ultrafilter on Z_+. Then there is $\Psi \in \mathrm{Spec}(\mathscr{A})$ with $\Psi \circ q|_{\mathscr{B}} = w^* - \lim_{m,\mathscr{U}} \Phi_m$. Hence, for radial F, $\Psi(qT_F) = \Phi_{\mathscr{U}}(F)$. Since $\Psi \in \mathrm{Spec}(\mathscr{A})$ there exists some $z \in \mathsf{T}$ with $\Psi(qT_f) = f(z) = \Phi_{\mathscr{U}}(f)(z)$ if $f \in Y$. We have

$$\Phi_{\mathscr{U}}(f_\lambda)(z) = \Phi_{\mathscr{U}}(f)(\lambda z)$$

if $z \in \mathsf{T}$ and $\lambda \in \mathsf{T}$. So, using Lemma 2.2.(a) and Proposition 4.1., we obtain, for any $w \in \mathsf{T}$, an element $\tilde{\Psi} \in \mathrm{Spec}(\mathscr{A})$ with $\tilde{\Psi}(qT_f) = \Phi_{\mathscr{U}}(f)(w)$ if f is a $L_\infty(d\mu)-$valued trigonometric polynomial.

(d): (b) and (c) imply that $\mathrm{Spec}(\mathscr{A}) \circ q \circ \tau$ and Ω coincide on the $L_\infty(d\mu)$-valued trigonometric polynomials. Now let $f \in X$ and let (f_j) be a sequence of $L_\infty(d\mu)$-valued trigonometric polynomials with $\lim_j ||qT_f - qT_{f_j}|| = 0$. Since $\mathscr{A}$ is a commutative C^*-algebra we conclude

$$\lim_{j} \sup_{\Psi \in \mathrm{Spec}(\mathscr{A})} |\Psi(qT_f) - \Psi(qT_{f_j})| = 0.$$

(b) and (c) yield $||qT_{f_j} - qT_{f_k}|| = \sup_{\mathscr{U}} ||\Phi_{\mathscr{U}}(f_j) - \Phi_{\mathscr{U}}(f_k)||_\infty$. This implies that, for any $\mathscr{U}$, $(\Phi_{\mathscr{U}}(f_j))_j$ is a $||\cdot||_\infty$-Cauchy sequence of trigonometric polynomials on T. Let $\Phi = \lim_j \Phi_{\mathscr{U}}(f_j)$. According to the second assertion of Lemma 5.2., the Fourier coefficients of Φ coincide with those of $\Phi_{\mathscr{U}}(f)$. Hence $\Phi_{\mathscr{U}}(f) = \Phi$. In particular $\Phi_{\mathscr{U}}(f)$ is continuous. Finally, with (b) and (c), $\mathrm{Spec}(\mathscr{A}) \circ q \circ \tau|_X$ and Ω coincide.

For $T \in \mathscr{L}(H_2(\mu))$ let $\sigma_{\mathrm{ess}}(T)$ be the spectrum of $q(T)$ in $\mathscr{L}(H_2(\mu))/\mathscr{K}$.

5.4. Corollary. *Let $f \in X$. Then*

$$\sigma_{\mathrm{ess}}(T_f) = \{\Phi_{\mathscr{U}}(f)(z) : z \in \mathsf{T},\ \mathscr{U} \text{ a free ultrafilter on } \mathsf{Z}_+\}.$$

Moreover, $||qT_f|| = \sup_{\mathscr{U}} ||\Phi_{\mathscr{U}}(f)||_\infty$.

In particular, T_f is a Fredholm operator if and only if $\Phi_{\mathscr{U}}(f)(z) \neq 0$ for all $z \in \mathsf{T}$ and all free ultrafilters $\mathscr{U}$.

5.5. Corollary. *Let $f \in L_\infty$ be an angular function. Then*

$$||f||_\infty = ||qT_f|| = ||T_f||.$$

Moreover, if f is continuous on T and angular then $\sigma_{\mathrm{ess}}(T_f) = f(\mathsf{T})$.

Proof. If f is angular and continuous on T then $f \in X$ and $\Phi_{\mathscr{U}}(f) = f$. Hence $\sigma_{\mathrm{ess}}(T_f) = f(\mathsf{T})$ and $||f||_\infty = ||qT_f|| = ||T_f||$. Now let $f \in L_\infty$ be arbitrarily angular. Then $\sigma_j f \to f$ a.e. on T ([10]). Moreover, all $\sigma_j f$ are angular and continuous on T. We obtain, in view of Lemma 2.2.,

$$||f||_\infty \leq \limsup_j ||\sigma_j f||_\infty = \limsup_j ||qT_{\sigma_j f}|| \leq ||qT_f|| \leq ||T_f|| \leq ||f||_\infty,$$

hence equality.

6. Compact Toeplitz operators

Now, again, let n be an arbitrary positive integer. Throughout this section let $f \in L_\infty$ and $f \overset{(L_2)}{=} \sum_{k \in \mathsf{Z}^n} F_k \xi_k$.

At first we characterize those Toeplitz operators which are Hilbert-Schmidt operators.

6.1. Proposition. *T_f is a Hilbert-Schmidt operator if and only if*

$$\sum_{l \in \mathsf{Z}_+^n} \sum_{m \in \mathsf{Z}_+^n} \frac{|\int F_{m-l} r^{m+l} d\mu|^2}{\int r^{2m} d\mu \int r^{2l} d\mu} < \infty.$$

PROOF. T_f is a Hilbert-Schmidt operator if and only if $\sum_{l\in\mathbb{Z}_+^n} ||T_f e_l||_2^2 < \infty$. Proposition 1.2. yields

$$||T_f e_l||_2^2 = \sum_{m\in\mathbb{Z}_+^n} \frac{|\int F_{m-l} r^{m+l} d\mu|^2}{\int r^{2m} d\mu \int r^{2l} d\mu}$$

which proves Proposition 6.1.

Now we determine those f among the elements of X where T_f is compact. Recall that $f \in X$ whenever $f \in L_\infty$ and T_f is compact.

6.2. PROPOSITION. (a) *T_f is compact if and only if $f \in X$ and*

$$\lim_{m\to\infty} \frac{\int F_k r^{2m-k} d\mu}{\sqrt{\int r^{2m} d\mu \int r^{2m-2k} d\mu}} = 0 \ \textit{for all} \ k \in \mathbb{Z}^n.$$

(b) *Let $n = 1$ and let μ satisfy* (I) *and* (II). *Then T_f is compact if and only if $f \in X$ and*

$$\lim_{m\to\infty} \frac{\int F_k r^{2m} d\mu}{\int r^{2m} d\mu} = 0 \ \textit{for all} \ k \in \mathbb{Z}.$$

PROOF. (a) If T_f is compact then $f \in X$. Proposition 1.2. yields

$$\frac{\int F_k r^{2m-k} d\mu}{\sqrt{\int r^{2m} d\mu \int r^{2m-2k} d\mu}} = \langle T_f e_{m-k}, e_m \rangle.$$

Since (e_{m-k}) converges weakly to 0 as $m \to \infty$ and T_f is compact we see that $\lim_m \langle T_f e_{m-k}, e_m \rangle = 0$.

Conversely, if $\lim_m \langle T_f e_{m-k}, e_m \rangle = 0$ then Proposition 1.2. shows that $T_{F_k \xi_k}$ is compact for all k. Hence, by definition, $T_{\sigma_j f}$ is compact for all j. Since $f \in X$ Proposition 2.3. shows that T_f is compact.

(b) follows from Theorem 5.3. and Lemma 5.1.(a). Here T_f is compact if and only if $f \in X$ and $\Phi_{\mathcal{U}}(f) = 0$ for all $\mathcal{U}$.

For other conditions which characterize compact Toeplitz operators on the Bergman and on the Fock space see [8,9].

EXAMPLE. Let $\mu_1 = 1_{[0,1[} d\lambda + \delta_1$ and $\mu_2 = \delta_1$ (λ the Lebesgue measure on $\mathbb{R}_+$). It follows from the maximum principle that $H_2(\mu_1)$ and $H_2(\mu_2)$ are isomorphic and can be identified as sets of holomorphic functions. There are many non-trivial compact Toeplitz operators on $H_2(\mu_1)$, for example T_F with $F(r) = 1_{[0,1/2]}(r)$. On the other hand, in view of Corollary 5.5., the only compact Toeplitz operator on $H_2(\mu_2)$ is the zero operator.

6.3. COROLLARY. *Let* $n = 1$ *and let* μ *satisfy* (I) *and* (II). *If* T_f *is compact then all* T_{F_k} *are compact.*

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