REMARKS ON FREE PRODUCTS WITH RESPECT TO NON-TRACIAL STATES

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1. Introduction

Since the appearance of D. Voiculescu's free probability theory detailed study on free products of von Neumann algebras (based on his theory) has been carried by several authors, and free group factors are their typical examples. Indeed, his theory gives us very powerful tools to analyze free group factors.

On the other hand, at the beginning of the 80's (i.e., before the appearance of D. Voiculescu's theory) S. Popa studied in detail free group factors and related type II_1 factors and got several deep results ([P1], [P2]). He used there some techniques for type II_1 factors related to his notion "orthogonal pairs". Then, at the mid 90's L. Ge generalized them to more general (but type II_1) setting and solved some problems on maximal injective subalgebras ([G]).

In this note, we will try to generalize the techniques to more general and not necessary type II_1 setting. These generalizations are rather natural, but enable us to obtain some results on free products with respect to non-tracial states. In fact, as applications a factoriality and type classification result for free products and a result on (normal) conditional expectations from free products onto their subalgebras will be obtained. Based on the latter result we will discuss the recent question posed by M. Izumi, R. Longo and S. Popa ([ILP]): Let $M \supseteq L \supseteq N$ be von Neumann algebras with a normal conditional expectation from M onto N. If $N' \cap M = \mathsf{C}1$, does there exist a normal conditional expectation from M onto L?

2. Preliminaries

Let N_i, $i = 1, 2$, be a σ-finite von Neumann algebra with a faithful normal state φ_i respectively. The free product

$$(M, \varphi) = (N_1, \varphi_1) * (N_2, \varphi_2)$$

is a pair of a von Neumann algebra M and a faithful normal state φ characterized by the conditions ([V1], [VDN]):
   (1) M is generated by N_1 and N_2;
   (2) $\varphi|_{N_i} = \varphi_i$, $i = 1, 2$;
   (3) N_1, N_2 are free with respect to φ, i.e.,

$$\varphi(x_1^\circ \cdots x_n^\circ) = 0$$

whenever $x_j^\circ \in N_{i_j}^\circ := \mathrm{Ker}\, \varphi_{i_j}$, $i_1 \neq \cdots \neq i_n$.

It is known that the modular automorphism σ_t^φ satisfies

$$(1) \qquad\qquad\qquad \sigma_t^\varphi|_{N_i} = \sigma_t^{\varphi_i}, \quad i = 1, 2.$$

(See [Br], [D1], [V1], [VDN].) Therefore, there exists a φ-conditional expectations $E_i : M \to N_i$, $i = 1, 2$.

3. Some Properties of Free Products

In this section some properties of free products (not necessary in the type II_1 setting) will be shown.

Let $N_1, N_2, M, \varphi_1, \varphi_2, \varphi, E_1$ and E_2 be as in §2. Moreover, in what follows, $N_i \neq \mathbb{C}1$, $i = 1, 2$ will be assumed.

PROPOSITION 1. *If A is a non-atomic von Neumann subalgebra of the centralizer $(N_1)_{\varphi_1}$ and x is an element in M such that $xAx^* \subseteq N_1$, then x is in N_1.*

PROOF. We make use of the idea in the proof of 2.5. Lemma of [P1] with some modifications.

Let y° be an alternating word in N_1°, N_2° beginning and ending in N_2°, a, b be elements in N_1 with possibly $a = 1$ or $b = 1$, and $\{e_i\}_{i=1}^n$ be a partition of the unity of projections in A. We estimate

$$|\varphi(x^*(ay^\circ b))|^2 = |\varphi((\sum_i e_i)x^*(ay^\circ b))|^2$$

$$= |\varphi(\sum_i e_i x^* ay^\circ be_i)|^2 \quad (e_i \in A \subseteq (N_1)_{\varphi_1} \subseteq M_\varphi \text{ thanks to } (1))$$

$$\leq ||\sum_i e_i x^* ay^\circ be_i||_\varphi^2 \quad (\text{Cauchy-Schwartz})$$

$$= \sum_i ||e_i x^* ay^\circ be_i||_\varphi^2 \quad (\text{use } e_i e_j = 0, \ i \neq j)$$

$$= \sum_i \varphi((e_i b^*)y^{\circ*}(a^* xe_i x^* a)y^\circ(be_i))$$

$$= \sum_i \varphi_1(a^* xe_i x^* a)\varphi(e_i b^* y^{\circ*} y^\circ be_i)$$

(by $xe_i x^* \in N_1$ and the usual computation based on freeness)

$$\leq ||y^\circ||^2 ||b||^2 \max_i \varphi_1(e_i) \ \varphi_1(a^* x(\sum_i e_i)x^* a)$$

$$= ||y^\circ||^2 ||b||^2 \max_i \varphi_1(e_i) \ \varphi_1(a^* xx^* a)$$

$$\leq (||a|| ||y^\circ|| ||b|| ||x||)^2 \max_i \varphi_1(e_i).$$

Since A is non-atomic, for each $\varepsilon > 0$ we can choose $\{e_i\}_{i=1}^n$ in such a way that

$$\varphi_1(e_i) < \frac{\varepsilon^2}{(||a|| ||y^\circ|| ||b|| ||x||)^2},$$

and then the above estimate shows $|\varphi(x^*(ay^\circ b))| < \varepsilon$. Therefore, x is orthogonal to the word $ay^\circ b$.

Notice that the $*$-algebra M_0 (algebraically) generated by N_1, N_2 is

$$M_0 = N_1(= \mathsf{C}1 + N_1^\circ) + \text{span}(\Lambda^\circ(N_1^\circ, N_2^\circ) \setminus N_1^\circ),$$

and then

$$L^2(M, \varphi) = L^2(N_1) \oplus \overline{\text{span}(\Lambda^\circ(N_1^\circ, N_2^\circ) \setminus N_1^\circ)}^{||\cdot||_\varphi}.$$

Here $\Lambda^\circ(N_1^\circ, N_2^\circ)$ is the set of alternating words in N_1°, N_2°. The previous computation means that x is orthogonal to every element in $\Lambda^\circ(N_1^\circ, N_2^\circ) \setminus N_1^\circ$, and hence $(1 - e_{N_1})x\xi_\varphi = 0$, where ξ_φ is an implementing vector for φ and e_{N_1} is the Jones projection associated with E_1. Therefore, $x\xi_\varphi = e_{N_1}(x\xi_\varphi) = E_1(x)\xi_\varphi$, and hence $x = E_1(x) \in N_1$.

COROLLARY 2. *If A is a non-atomic von Neumann subalgebra of the centralizer $(N_1)_{\varphi_1}$, then $A' \cap M \subseteq N_1$.*

PROOF. Let u be an arbitrary unitary in $A' \cap M$, and then $uAu^* = A \subseteq N_1$. Hence, by the previous proposition it is in N_1.

More generally, by the same reason as above we have the following corollary:

COROLLARY 3. *If A is a non-atomic von Neumann subalgebra of the centralizer $(N_1)_{\varphi_1}$, then $\mathscr{N}_M(A) = \mathscr{N}_{N_1}(A)$, where $\mathscr{N}_M(A)$ and $\mathscr{N}_{N_1}(A)$ mean the normalizers of A in M and N_1 respectively.*

In his article ([R]) F. Rădulescu showed that the free product

$$(L(\mathsf{Z}), \tau_{\mathsf{Z}}) * (M_2(\mathsf{C}), \varphi_\lambda)$$

is a factor of type III_λ. Here φ_λ is the state associated with the density matrix $\mathrm{diag}\left(\dfrac{1}{1+\lambda}, \dfrac{\lambda}{1+\lambda}\right)$. The above corollary implies that (the canonical image of) $L(\mathsf{Z})$ is a singular MASA in the free product.

Set

$$\tilde{M} := M \rtimes_{\sigma^\varphi} \mathsf{R} \supseteq \tilde{N}_i := N_i \rtimes_{\sigma^{\varphi_i}} \mathsf{R} \quad (i = 1, 2).$$

There exists a faithful normal conditional expectation $\hat{E}_i : \tilde{M} \to \tilde{N}_i$ such that

$$\hat{E}_i\Big(\int_{-\infty}^{\infty} x(t)\lambda(t)dt\Big) = \int_{-\infty}^{\infty} E_i(x(t))\lambda(t)dt$$

for a smooth element $\int_{-\infty}^{\infty} x(t)\lambda(t)dt \in \tilde{M}$, where E_i is as in §2. (See [KL] for example.) Note that it is the restriction of $E_i \otimes \mathrm{Id}$ to $\tilde{M}$.

COROLLARY 4. *If A is a non-atomic von Neumann subalgebra of the centralizer $(N_1)_{\varphi_1}$, then $A' \cap \tilde{M} \subseteq \tilde{N}_1$.*

PROOF. Since $\tilde{M}$ is contained in $M \otimes B(L^2(\mathsf{R}))$, we have

(2) $A' \cap \tilde{M} \subseteq (A' \cap M) \otimes B(L^2(\mathsf{R})) \subseteq N_1 \otimes B(L^2(\mathsf{R}))$

thanks to Corollary 2. Let x be in $A' \cap \tilde{M}$ ($\subseteq \tilde{M}$). Now approximate x by elements of the form $\int_{-\infty}^{\infty} x(t)\lambda(t)dt$ in the σ-weak topology. Then we have

$$x = E_1 \otimes \mathrm{Id}(x) = \lim \hat{E}_1\left(\int_{-\infty}^{\infty} x(t)\lambda(t)dt\right) = \lim \int_{-\infty}^{\infty} E_1(x(t))\lambda(t)dt \in \tilde{N}_1.$$

Here the first equality comes from (2) and the second comes from the fact that $\hat{E}_1$ is the restriction of $E_1 \otimes \mathrm{Id}$ to $\tilde{M}$.

REMARK 5. By the similar method as in the proof of Proposition 1 we can also prove the following generalization of Theorem 4.1 of [P3]: Let Q be a σ-finite von Neumann algebra with a faithful normal state ψ and $N \supseteq L$ be σ-finite von Neumann algebras with a faithful normal conditional expectation F. Let

$$(M, E) = (Q \otimes L, \psi \otimes \text{Id}) *_L (N, F)$$

be the amalgamated free product (see [P3], [U2], [VDN] for definition). If A is a non-atomic von Neumann subalgebra of the centralizer Q_ψ, then

$$(A \otimes \mathsf{C}1)' \cap M = (A' \cap Q) \otimes L.$$

The following proposition, which is a generalization of the result of L. Ge ([G: Lemma 5.1]), is worth pointing out:

PROPOSITION 6. *If A is a non-atomic von Neumann subalgebra of the centralizer $(N_1)_{\varphi_1}$ and x is an element in M such that $xAx^* \subseteq N_2$, then $x = 0$.*

PROOF. Let $\Lambda^\circ(N_1^\circ, N_2^\circ)$ be as in the proof of Proposition 1, and the $*$-algebra

$$M_0 = N_2(= \mathsf{C}1 + N_2^\circ) + \text{span}(\Lambda^\circ(N_1^\circ, N_2^\circ) \setminus N_2^\circ)$$

is σ-weakly dense in M.

Let $\{e_i\}_{i=1}^n$ be a partition of the unity of projections in A. For an alternating word y° in N_1°, N_2° beginning and ending in N_1° and $a, b \in N_2$ with possibly $a = 1$ or $b = 1$, we estimate, by the similar way as in the proof of Proposition 1,

$$|\varphi(x^*(ay^\circ b))|^2 = |\varphi(\sum_i e_i x^* a y^\circ b e_i)|^2$$

$$\leq \sum_i \|e_i x^* a y^\circ b e_i\|_\varphi^2$$

$$= \sum_i \varphi(e_i b^* y^{\circ *}(a^* x e_i x^* a) y^\circ b e_i)$$

$$= \sum_i \varphi_2(a^* x e_i x^* a) \varphi(e_i b^* y^{\circ *} y^\circ b e_i)$$

(by $x e_i x^* \in N_2$ and the usual computation based on freeness)

$$\leq \|y^\circ\|^2 \|b\|^2 \max_i \varphi_1(e_i) \, \varphi_2(a^* x(\sum_i e_i) x^* a)$$

$$\leq (\|a\| \|y^\circ\| \|b\| \|x\|)^2 \max_i \varphi_1(e_i).$$

Thus, by the same way as in the proof of Proposition 1 we see that x is or-

thogonal to the word $ay°b$, and hence to every word in $\Lambda°(N_1°, N_2°) \setminus N_2°$. Next, for $z \in N_2 (\ni 1)$ we estimate

$$
\begin{aligned}
|\varphi(x^*z)|^2 &\leq \sum_i \varphi(e_i z^* x e_i x^* z e_i) \\
&= \sum_i \varphi(e_i(z^* x e_i x^* z)) \quad (e_i \in A \subseteq (N_1)_{\varphi_1} \subseteq M_\varphi \text{ thanks to (1)}) \\
&= \sum_i \varphi_1(e_i) \varphi(z^* x e_i x^* z) \quad \text{(by the same reason as above)} \\
&\leq \max_i \varphi_1(e_i) \, \varphi(z^* x(\sum_i e_i) x^* z) \\
&\leq (\|z\|\|x\|)^2 \max_i \varphi_1(e_i).
\end{aligned}
$$

Hence, x is orthogonal to z.

Therefore, x is orthogonal to M_0, and hence $x = 0$.

4. Factoriality and type classification for free products

Let N_i, $i = 1, 2$, be a σ-finite von Neumann algebra with a faithful normal state φ_i, and

$$(M, \varphi) = (N_1, \varphi_1) * (N_2, \varphi_2)$$

be the free product. Assume that $N_i \neq \mathsf{C}1$, $i = 1, 2$.

Set

$$\tilde{M} := M \rtimes_{\sigma^\varphi} \mathsf{R} \supseteq \tilde{N}_i := N_i \rtimes_{\sigma^{\varphi_i}} \mathsf{R} \supseteq D := \mathsf{C} \rtimes \mathsf{R}.$$

There exists a faithful normal conditional expectation $E : \tilde{M} \to D$, which is the restriction of the Fubini map $\varphi \otimes \mathrm{Id}$ to $\tilde{M}$. In [U2], it was shown that

$$(3) \qquad (\tilde{M}, E) = (\tilde{N}_1, E|_{\tilde{N}_1}) *_D (\tilde{N}_2, E|_{\tilde{N}_2}).$$

THEOREM 7. *If either $(N_1)_{\varphi_1}$ or $(N_2)_{\varphi_2}$ contains a non-atomic von Neumann subalgebra, then M is a factor and*

$$\mathscr{Z}(\tilde{M}) = (\mathsf{C} \rtimes \mathsf{R}) \cap \mathscr{Z}(\tilde{N}_1) \cap \mathscr{Z}(\tilde{N}_2),$$

where $\tilde{M}, \tilde{N}_1, \tilde{N}_2$ are as in §2.

PROOF. By symmetry we may and do assume that $(N_1)_{\varphi_1}$ contains a non-atomic von Neumann subalgebra. Since $N_2 \neq \mathsf{C}1$, we can choose orthogonal nonzero projections $e_1, e_2 \in N_2$ with $e_1 + e_2 = 1$. Set

$$y := \frac{1}{\varphi_1(e_1)} e_1 - \frac{1}{\varphi_1(e_2)} e_2,$$

and then it is in Ker $E \cap \tilde{N}_2$.

Let x be an element in the center $\mathscr{Z}(\tilde{M})$, and then it is in $\tilde{N}_1$ by Corollary 4. We have

$$(x - E(x))y + E(x)y = xy = yx = yE(x) + y(x - E(x)).$$

By freeness with amalgamation D (see (3)), the elements $(x - E(x))y$, $y(x - E(x))$ and $\{E(x)y, yE(x)\}$ are orthogonal with respect to E so that

$$\|y(x - E(x))\|_{\psi \circ E} = 0$$

for a fixed faithful normal state ψ on $\mathsf{C} \rtimes \mathsf{R}$. Thus we have

$$C \times \psi \circ E((x - E(x))^* e_i (x - E(x)))$$
$$\leq \psi \circ E((x - E(x))^* y^* y (x - E(x))) = 0, \quad i = 1, 2,$$

where $C = \min\left\{\left(\dfrac{1}{\varphi_1(e_i)}\right)^2 : i = 1, 2\right\}$, and hence

$$(x - E(x))^* e_i (x - E(x)) = 0 \quad i = 1, 2.$$

Since $e_1 + e_2 = 1$, this implies $x - E(x) = 0$, i.e., $x = E(x) \in D = \mathsf{C} \rtimes \mathsf{R}$.

The factoriality of M follows from

$$\mathscr{Z}(M) = \mathscr{Z}(\tilde{M})^{\theta^M} \subseteq (\mathsf{C} \rtimes \mathsf{R})^{\theta^M} = \mathsf{C}1,$$

where θ^M is the dual action of M.

In the above argument, information on the flow of weights was used to show the factoriality. However, it can be directly proved.

COROLLARY 8. *The above factor M is not of type* III_0, *and hence its type can be known by the Connes T-set. In this case, the Connes T-set is computed as follows:*

$$T(M) = \{t \in \mathsf{R} : \sigma_t^\varphi = \mathrm{Id}\}.$$

PROOF. By the above theorem, the (smooth) flow of weights ([CT]) of M is a factor flow of the transitive one arising from $\mathsf{C} \rtimes \mathsf{R}$. Hence, the flow of weights of M is also transitive. Thus, M is not of type III_0. This well-known fact can be seen as follows: Since the flow of weights is transitive, than it must be the translation on R/H with a closed subgroup H of R. (See [M, §2,3], [Z, p. 373] for example.) Since any closed subgroup of R is one of $\{0\}$, $(-\log \lambda)\mathsf{Z}$ $(0 < \lambda < 1)$ and R, the flow of weights must be a periodic flow or the trivial one, and hence M is not of type III_0. (This fact, i.e., M is not of type III_0, also follows from the fact that the flow of weights of a type III_0

factor (i.e., an aperiodic flow) can be written as a flow built under a function with an ergodic base transformation on a non-atomic measure space. See [T4, Chap. III §3] for example.)

By symmetry we may and do assume that $(N_1)_{\varphi_1}$ contains a non-atomic von Neumann subalgebra. First we further suppose that $(N_2)_{\varphi_2} \neq \mathsf{C}1$. Then, by the similar (actually simpler in this case) argument based on Corollary 2 as in the proof of the previous theorem we have $(M_\varphi)' \cap M = \mathsf{C}$, and get the assertion on the T-set. Next, we suppose that $(N_2)_{\varphi_2} = \mathsf{C}1$. If $((N_1)_{\varphi_1})' \cap N_1 = \mathsf{C}1$, then the same relative commutant property as above can be obtained directly by Corollary 2, and hence we get the assertion on the T-set. Therefore, the case that the relative commutant $((N_1)_{\varphi_1})' \cap N_1$ is not trivial and $(N_2)_{\varphi_2} = \mathsf{C}1$ remains. However, we can prove the following:

CLAIM. If u is a unitary in N_1 such that $\sigma_t^\varphi = \mathrm{Ad}\, u$, then u is in $\mathsf{C}1$.

Indeed, Set $u^\circ := u - \varphi_1(u)1$, and let x° be a non-zero element in N_2°. Then we compute

$$ux^\circ u^* = \varphi_1(u)\varphi_1(u^*)x^\circ + \varphi_1(u)x^\circ u^{\circ*} + \varphi_1(u^*)u^\circ x^\circ + u^\circ x^\circ u^{\circ*}.$$

Since the left-hand side is in N_1°, the second, third and fourth terms in the right-hand side are zero by the structure of the free product representation. Thus we easily have $u^\circ = 0$, i.e., $u = \varphi_1(u)1$.

This claim immediately implies the assertion on the T-set.

It is unknown whether $(M_\varphi)' \cap M = \mathsf{C}1$ or not in the case that the relative commutant $((N_1)_{\varphi_1})' \cap N_1$ is not trivial and $(N_2)_{\varphi_2} = \mathsf{C}1$.

REMARK 9. The above result on the T-set was already obtained by K. Dykema under a weaker condition than ours ([D1]). However, it is unknown whether the T-sets determine the types of the free products he considered there or not, and our proof gave another information on relative commutants of centralizers.

REMARK 10. The problem on factoriality for free products was ''almost'' completed by K. Dykema ([D1], [D2]) (at least when given von Neumann algebras are factors). However, type classification for free products is obtained in only the following cases:

1. (L. Barnett, [Br]) The centralizers of given faithful normal states contain ''enough'' unitaries;

2. (K. Dykema, [D2]) Each of given von Neumann algebras with faithful normal states is one of the following:

(i) a finite dimensional algebra with any faithful state,

(ii) $B(\mathscr{H})$ with any faithful normal state,

(iii) a type III factor with an almost periodic state having the centralizer isomorphic to the AFD type II_1 factor or to the free group factor $L(F_\infty)$,

(iv) a non-atomic AFD von Neumann algebra with a faithful normal tracial state,

(v) an interpolated free group factor $L(F_r)$ with a faithful normal tracial state,

(vi) a possibly infinite direct sum of algebras from (i)–(v).

Therefore, our Corollary 8 is a new type classification result for free products. In all the known cases the resulting free product von Neumann algebras are not of type III_0 or have no type III_0 direct summand. Hence it is an interesting problem to decide whether there exists a type III_0 factor as a free product or not. Theorem 5.1 of [U2] might be useful for this problem. At least it suggests us that the (smooth) flows of weights of free products are factor flows of the transitive one arising from $C \rtimes R$ as in Theorem 7 except for several cases (for example, the finite dimensional case etc.).

Our Corollary 8 can be applied to many cases; for example, one of given von Neumann algebras with faithful normal states comes from a group of non-singular transformations on a Lebesgue space with a probability measure. It might be worth pointing out the following lemmas (and their corollary):

LEMMA 11. *Let N be a σ-finite von Neumann algebra with a faithful normal state φ. If N is semi-finite and has no type I direct summand, then the centralizer N_φ contains a non-atomic von Neumann subalgebra.*

PROOF. Let $H = \dfrac{d\varphi}{d\mathrm{tr}}$ be the Radon-Nikodym derivative of φ with respect to a faithful normal semi-finite trace tr on N and A_0 be the abelian von Neumann subalgebra of N generated by the spectral projections of H. It is known that A_0 is of the form

$$A_0 = A_c \oplus A_d,$$

where A_c is non-atomic and A_d is atomic, i.e.,

$$A_d = \sum_{p \text{ projection}}^{\oplus} Cp.$$

If p is a minimal projection in A_d ($\subseteq A_0$), then $H^{it}p = \omega_t p$ for some $\omega_t \in T$. Hence,

$$\sigma_t^\varphi(pxp) = H^{it}pxpH^{-it} = (\omega_t p)x(\bar\omega_t p) = pxp$$

for $x \in N$ so that pNp is contained in N_φ. Of course, A_c is also contained in N_φ. Set

$$A = A_c \oplus \sum_p{}^{\oplus} pNp \ (\supseteq A_0).$$

Since N has no type I direct summand, pNp is non-atomic. Therefore, A is also non-atomic.

The following lemma can be easily observed based on the proof of the structure theorem of A. Connes ([C]) (and the similar argument as above):

LEMMA 12. *Let N be a σ-finite type* III *factor with a faithful normal state φ. If N is not of type* III$_1$*, then the centralizer N_φ contains a non-atomic von Neumann subalgebra.*

In the proof of the lemma the existence of a lacunary weight (or state) is crucial, and hence the above fact cannot be expected in the type III$_1$ case. In fact, there exists a faithful normal state ϕ on the AFD type III$_1$ factor R_∞ whose centralizer is trivial ([HT]). Moreover, there exists a faithful normal state φ on R_∞ whose centralizer is just $M_n(\mathsf{C})$, $n \in \mathsf{N}$. Indeed, let

$$N = R_\infty \otimes M_n(\mathsf{C}), \quad \varphi = \phi \otimes \tau,$$

where τ is a normalized trace on $M_n(\mathsf{C})$. Then, $N \cong R_\infty$ and the centralizer N_φ of φ is the fixed-point algebra $N^{\sigma^\varphi} = (R_\infty)^{\sigma^\phi} \otimes M_n(\mathsf{C}) = \mathsf{C}1 \otimes M_n(\mathsf{C}) \cong M_n(\mathsf{C})$.

By combining Theorem 7 (also Corollary 8) and the above two lemmas we have the following corollary:

COROLLARY 13. *If either N_1 or N_2 is one of the following:*
(1) a σ-finite semi-finite von Neumann algebra having no type I direct summand;
(2) a σ-finite von Neumann algebra having the non-atomic center;
(3) a σ-finite type III$_\lambda$ *factor ($\lambda \neq 1$);*
(4) a possibly (infinite) direct sum of algebras from (1)–(3), then M is a factor, not of type III$_0$ *and its (smooth) flow of weights can be computed as in Theorem 7.*

By the above corollary and K. Dykema's results (see Remark 10), the remaining cases of the type classification for free products of two factors are (i) both are of type III$_1$; (ii) one is of type I and the other is of type III$_1$.

5. A comment on Izumi-Longo-Popa's paper

In their recent article [ILP], M. Izumi, R. Longo and S. Popa obtained the complete Galois correspondence for compact group minimal actions and more generally for compact Kac algebra minimal actions. The most crucial part of their arguments is to show the existence of a normal conditional expectation onto any intermediate subfactor of the irreducible inclusion arising from such an action, and hence they asked the question stated in §1. They probably conjectured that the question is not always true. However, only several partial affirmative answers (the finite Jones index case etc.) are known. In this section, we at first obtain a result on conditional expectations from free product von Neumann algebras onto their subalgebras (Theorem 14), which is of independent interest. Next, as an application of the result a counter-example to the above-mentioned question is given. The example shows that their result on the existence of conditional expectations is very non-trivial.

THEOREM 14. *Let N_i be a σ-finite von Neumann algebra with a faithful normal state φ_i, $i = 1, 2$, and $(M, \varphi) = (N_1, \varphi_1) * (N_2, \varphi_2)$ be the free product. Suppose that $(N_1)_{\varphi_1}$ contains a non-atomic von Neumann subalgebra and $N_2 \neq \mathsf{C}1$. Let K be a von Neumann subalgebra of N_2, and set $L := \{N_1, K\}''$. If there exists a normal conditional expectation $F : M \to L$, then*

$$\sigma_t^{\varphi_2}(K) = K, \quad t \in \mathsf{R},$$

and $F = \mathrm{Id} * E_K^{N_2}$ in the sense of [Bc], [Ch], where $E_K^{N_2}$ is the φ_2-conditional expectation from N_2 onto K.

PROOF. Let $E_1 : M \to N_1$ be the normal conditional expectation conditioned by φ. Since $(N_1)' \cap M \subseteq N_1$ thanks to Corollary 2, this is a unique normal conditional expectation thanks to Connes' result ([C]), and hence $E_1 \circ F = E_1$. Since

$$\varphi \circ F = (\varphi \circ E_1) \circ F = \varphi \circ (E_1 \circ F) = \varphi \circ E_1 = \varphi,$$

we conclude $\sigma_t^{\varphi|_L} = \sigma_t^{\varphi}|_L \quad (t \in \mathsf{R})$.

On the other hand, it is clear (by checking the freeness) that

$$(4) \qquad\qquad (L, \varphi|_L) = (N_1, \varphi_1) * (K, \varphi_2|_K).$$

Therefore, we have

$$\begin{aligned} \sigma_t^{\varphi_2}(x) &= \sigma_t^{\varphi}(x) \quad (x \in K \subseteq N_2 \text{ and } (1)) \\ &= \sigma_t^{\varphi|_L}(x) \quad (x \in K \subseteq L) \\ &= \sigma_t^{\varphi_2|_K}(x) \quad (\text{by } (4) \text{ and } (1)) \end{aligned}$$

for $x \in K$, and hence

$$\sigma_t^{\varphi_2}(K) = \sigma_t^{\varphi_2|_K}(K) = K \quad (t \in \mathsf{R}).$$

The formula $F = \mathrm{Id} * E_K^{N_1}$ is clear from the construction of a φ-conditional expectation in [T2].

Let R_λ be the Powers factor of type III$_\lambda$ ($0 < \lambda < 1$) with the usual product state ϕ_λ. We define the state φ_λ on $M_2(\mathsf{C})$ by

$$\varphi_\lambda(\cdot) = \mathrm{tr}\left(\begin{bmatrix} \dfrac{1}{1+\lambda} & \\ & \dfrac{\lambda}{1+\lambda} \end{bmatrix} \cdot \right).$$

Let

$$(M, \varphi) = (R_\lambda, \phi_\lambda) * (M_2(\mathsf{C}), \varphi_\lambda)$$

be the free product. It is known that $(R_\lambda)_{\phi_\lambda}$ is non-atomic (or more precisely, the AFD type II$_1$ factor). Thus $M \supseteq N := R_\lambda$ is an irreducible inclusion thanks to Corollary 2. Also, there exists a (unique faithful) normal conditional expectation $E : M \to N$ ($= \mathrm{Id}_{R_\lambda} * \varphi_\lambda$).

REMARK 15. By using Theorem 7 it can be proved that the factors $M \supseteq N$ have the same flow of weights, i.e.,

$$\mathscr{L}(\tilde{M}) = \mathscr{L}(\tilde{N}) \subseteq \mathsf{C} \rtimes \mathsf{R},$$

where $\tilde{M}, \tilde{N}$ are as in §2. Hence M and N are of type III$_\lambda$.

Let $K := \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}'' \left(= \mathsf{C}1 + \mathsf{C}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$, and set $L := \{N, K\}''$. If there existed a normal conditional expectation from M onto L, then Theorem 14 would force

$$\begin{bmatrix} & \lambda^{-it} \\ \lambda^{it} & \end{bmatrix} = \sigma_t^{\varphi_\lambda}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \in K.$$

for each $t \in \mathsf{R}$, a contradiction. Therefore, we get the following corollary:

COROLLARY 16. *There exists no normal conditional expectation from M onto L.*

Here, the following fact might be worth pointing out:

FACT 17. *Let $M \supseteq N$ be an irreducible inclusion of σ-finite factors with a (unique faithful) normal conditional expectation $E : M \to N$. If L is an intermediate subfactor which is the range of a faithful normal semi-finite operator-*

valued weight from M, then there exists a (unique faithful) normal conditional expectation from M onto L.

PROOF. Let F be a faithful normal semi-finite operator-valued weight from M onto L. Fix a faithful normal state φ on N. The restriction of E to L is denoted by E_L, and let $u_t := [D\varphi \circ E : D\varphi \circ E_L \circ F]_t$ be the Connes cocycle Radon-Nikodym derivative. Since $\sigma_t^{\varphi \circ E}|_N = \sigma_t^\varphi$ and $\sigma_t^{\varphi \circ E_L \circ F}|_N = \sigma_t^\varphi$ (see [H]), we have

$$\sigma_t^\varphi(x) = \sigma_t^{\varphi \circ E}(x) = u_t \sigma_t^{\varphi \circ E_L \circ F}(x) u_t^* = u_t \sigma_t^\varphi(x) u_t^*$$

for $x \in N$. Thus we have $u_t \in N' \cap M = \mathbf{C}1$, and hence

$$\sigma_t^{\varphi \circ E}(L) = \sigma_t^{\varphi \circ E_L \circ F}(L) = \sigma_t^{\varphi \circ E_L}(L) = L.$$

Hence, we are done thanks to Takesaki's result ([T2]).

This fact implies that our L is not the range of any faithful normal semi-finite operator-valued weight from M. Indeed, if so was L, then the above fact would imply the existence of a normal conditional expectation from M onto L. This contradicts Corollary 16.

REMARK 18. Let $M \supseteq N$ be an irreducible inclusion of σ-finite factors with a (unique faithful) normal conditional expectation and M_1 be the Jones basic extension. In [ILP], the structure of the relative commutant $N' \cap M_1$ was investigated. In our case, we can show, by the same argument as in p.23 in [ILP], that the relative commutant $N' \cap M_1$ has a type III$_\lambda$ direct summand. Indeed, let $_NX_N$ be the N-N bimodule generated by the element $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ of $M_2(\mathbf{C})$ regarded as a vector in $L^2(M, \varphi)$. Then $_NX_N \cong {}_N L^2(N) \otimes L^2(N)_N$, and hence $\text{End}(_NX_N) \cong N^{\text{op}} \otimes N$.

REMARK 19. In the above example, N is AFD, while M is not. (See [D2].) However we can construct $M \supseteq N$ in such a way that M and N are isomorphic to the each other by considering the free product

$$(N, \phi) = \underset{n=1}{\overset{\infty}{*}} (M_2(\mathbf{C}), \varphi_\lambda),$$

instead of the Powers factor R_λ with the product state ϕ_λ.

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