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CONTINUITY AND DIFFERENTIABILITY OF THE MOORE–PENROSE INVERSE IN C^*-ALGEBRAS

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1. Continuity of the Moore–Penrose inverse

The paper gives an elementary proof of the theorem on the continuity of the Moore–Penrose inverse in a C^*-algebra that does not require the concept of the conorm, but uses instead a C^* modification of Izumino's inequality $\|b^\dagger\| \le 4\|a^\dagger\|$ valid when a, b have the Moore–Penrose inverse and satisfy the inequalities $\|b-a\| < \frac{1}{2}\|a^\dagger\|^{-1}$ and $\|bb^\dagger - aa^\dagger\| < 1$. The paper then studies the conditions for the differentiability of the Moore–Penrose inverse in a C^*-algebra and gives an explicit formula for the derivative.

The *Moore–Penrose inverse* of an element a of a unital C^*-algebra A with the unit e is the unique element $a^\dagger$ of A satisfying the equations

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (a^\dagger a)^* = a^\dagger a, \quad (aa^\dagger)^* = aa^\dagger \tag{1.1}$$

(see [10, 5, 11, 13]). The set of all $a \in A$ that possess the Moore–Penrose inverse will be denoted by $A^\dagger$. It is shown in [5, Theorem 6] that $a \in A^\dagger$ if and only if $a \in aAa$. The elements $a^\dagger a$ and $aa^\dagger$ are Hermitian idempotents. We also write A^{-1} for the set of all invertible elements in A.

It is well known that the following two results hold for the ordinary inverse in Banach algebras.

Theorem A. If a is an invertible element of the Banach algebra A and if $a_n \to a$, then a_n are invertible for all sufficiently large n, and $a_n^{-1} \to a^{-1}$.

Theorem B. If a_n are invertible elements of the Banach algebra A such that $a_n \to a$ and that the norms $\|a_n^{-1}\|$ are bounded, then a is invertible and $a_n^{-1} \to a^{-1}$.

We discuss the validity of Theorems A and B when the ordinary inverse is replaced by the Moore–Penrose inverse in C^*-algebras.

Example 1.1. Theorem A is, in general, false for the Moore–Penrose inverse in a C^*-algebra A as the set $A^\dagger$ need not be open in A.

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First we observe that if $a \in A$ is Hermitian, then $a \in A^\dagger$ if and only if 0 is not an accumulation spectral point of a; this follows from [6, Theorem 7]. Let A be the C^*-algebra of all complex valued functions continuous on the set $[0,1] \cup [2,3]$, equipped with the supremum norm. Define a and a_n by $a(t) = 0$ if $t \in [0,1]$, $a(t) = t$ if $t \in [2,3]$, $a_n(t) = t/n$ if $t \in [0,1]$ and $a_n(t) = t$ if $t \in [2,3]$. Note that a and a_n are Hermitian elements of A and that 0 is an isolated spectral point for a. Then $a \in A^\dagger$, and $a^\dagger$ is defined by $a^\dagger(t) = 0$ if $t \in [0,1]$, $a^\dagger(t) = 1/t$ if $t \in [2,3]$. We have $\|a_n - a\| = 1/n \to 0$, however, $a_n \notin A^\dagger$ for all n since $\sigma(a_n) = [0, 1/n] \cup [2,3]$, and 0 is an accumulation spectral point of a_n. (See [8, Example 2.1].)

Before we can show that Theorem B holds for the Moore–Penrose inverse in any C^*-algebra A, we need to derive some auxiliary results.

Theorem 1.2. *Let $a, b \in A^\dagger$. Then*

$$b^\dagger - a^\dagger = -b^\dagger(b-a)a^\dagger + (e - b^\dagger b)(b^* - a^*)(a^\dagger)^* a^\dagger + b^\dagger (b^\dagger)^*(b^* - a^*)(e - aa^\dagger). \tag{1.2}$$

The foregoing identity was first obtained by Wedin [18] for matrices and by Harte and Mbekhta [6, Theorem 5] for C^*-algebras. When we observe that $\|e - b^\dagger b\| \le 1$, $\|e - aa^\dagger\| \le 1$ and $\|a^* - b^*\| = \|a - b\|$, we obtain the following result.

Theorem 1.3. *Let $a, b \in A^\dagger$. Then*

$$\|a^\dagger - b^\dagger\| \le 3\max\{\|a^\dagger\|^2, \|b^\dagger\|^2\}\|a - b\|. \tag{1.3}$$

We can now give an elementary proof of the validity of Theorem B for the Moore–Penrose inverse.

Theorem 1.4. *Let $a_n \in A^\dagger$ be such that $a_n \to a$ and that the norms $\|a_n^\dagger\|$ are bounded. Then $a \in A^\dagger$, and $a_n^\dagger \to a^\dagger$.*

Proof. Let $\|a_n^\dagger\| \le M$ for all n. By Theorem 1.3,

$$\|{a_m}^\dagger - a_n^\dagger\| \le 3M^2\|a_m - a_n\|,$$

the sequence $(a_n^\dagger)$ is Cauchy, and hence convergent to some element $c \in A$. From the continuity of the product in A we get $aca = \lim_n a_n a_n^\dagger a_n = \lim_n a_n = a$; then $a \in aAa$, and $a \in A^\dagger$. Another application of Theorem 1.3 yields $\|a_n^\dagger - a^\dagger\| \le 3M^2\|a_n - a\| \to 0$, and the result follows.

The conclusion that $a \in A^\dagger$ if $a_n \to a$ and the norms $\|a_n^\dagger\|$ are bounded was obtained by Harte and Mbekhta [6, Theorems 7 and 8] as a consequence of the upper semicontinuity of the conorm on $A \backslash \{0\}$.

The equivalence of the four conditions given in Theorem 1.6 below for C^*-algebras has attracted considerable attention in recent literature, and many different proofs have been given [5, 6, 9, 11, 12, 13, 14]. The arguments used in these proofs involve the concept of the conorm in a Banach algebra or, equivalently, the reduced minimum modulus of the induced regular representation of an element $a \in A$, and the concept of the gap between two subspaces of A. Our aim is to find more elementary arguments rooted in Banach algebra techniques, rather than relying on the concepts of the conorm and of the gap between subspaces which are motivated by operator theory. To this end we adapt an inequality originally obtained by Izumino [7, Lemma 2.2] for Hilbert space operators.

Theorem 1.5. *If $a, b \in A^\dagger$ are such that $\|b - a\| < \frac{1}{2}\|a^\dagger\|^{-1}$ and $\|bb^\dagger - aa^\dagger\| <$ 1, then*

$$\|b^\dagger\| \leq 4\|a^\dagger\|. \tag{1.4}$$

Proof. Let $p = aa^\dagger$ and $q = bb^\dagger$; p and q are Hermitian idempotents. We have

$$\begin{aligned}\|(e-p)q\|^2 &= \|q^*(e-p^*)(e-p)q\| = \|q(e-p)q\| \\ &= \|q(q-p)q\| \leq \|q\|\|p-q\|\|q\| = \|p-q\| < 1;\end{aligned}$$

hence $e - (e-p)q \in A^{-1}$. Further, $e + a^\dagger(b-a) \in A^{-1}$ as $\|a^\dagger(b-a)\| \leq \|a^\dagger\|\|b-a\| < 1$.

Since $(e-(e-p)q)b = aa^\dagger b = a(e + a^\dagger(b-a))$, we can express b as the product

$$b = (e-(e-p)q)^{-1}a(e + a^\dagger(b-a)) = uav,$$

where $u, v \in A^{-1}$. Then $b = bv^{-1}a^\dagger u^{-1}b$, $b^\dagger = b^\dagger bb^\dagger = (b^\dagger b)(v^{-1}a^\dagger u^{-1})(bb^\dagger)$, and

$$\begin{aligned}\|b^\dagger\| &\leq \|b^\dagger b\|\|v^{-1}\|\|a^\dagger\|\|u^{-1}\|\|bb^\dagger\| \leq \|u^{-1}\|\|v^{-1}\|\|a^\dagger\| \\ &\leq \|e-(e-p)q\|\|(e+a^\dagger(b-a))^{-1}\|\|a^\dagger\| \\ &\leq 2(1 - \|a^\dagger\|\|b-a\|)^{-1}\|a^\dagger\| \leq 4\|a^\dagger\|.\end{aligned}$$

We now give an elementary proof of the main result on the continuity of the Moore–Penrose inverse, which subsumes the results of Izumino [7, Proposition 2.3] for the case of bounded linear operators in Hilbert spaces. Simultaneously we recover Mbekhta [9, Théorème 2.2], Rakočević [13, Theorem 2.2] and Harte and Mbekhta [6, Theorem 6]. For the continuity of the

Moore–Penrose inverse in Banach algebras or C^*-algebras see also [11, 12, 14].

THEOREM 1.6. *Let a_n, a be nonzero elements of $A^\dagger$ such that $a_n \to a$ in A. Then the following conditions are equivalent.*

$$a_n^\dagger \to a^\dagger, \tag{1.5}$$

$$a_n a_n^\dagger \to a a^\dagger, \tag{1.6}$$

$$a_n^\dagger a_n \to a^\dagger a, \tag{1.7}$$

$$\sup_n \|a_n^\dagger\| < \infty. \tag{1.8}$$

PROOF. The implications (1.5) $\Rightarrow$ (1.6) $\Rightarrow$ (1.8) $\Rightarrow$ (1.5) follow from the continuity of the algebra multiplication in A, Theorem 1.5, and Theorem 1.3, respectively. The preceding arguments applied to $a_n{}^*$ and a^* yield (1.5) $\Rightarrow$ (1.7) $\Rightarrow$ (1.8) $\Rightarrow$ (1.5) since $(c^\dagger)^* = (c^*)^\dagger$ for $c \in A^\dagger$.

NOTE 1.7. The argument used in the proof of Theorem 1.5 is essentially due to Izumino [7, Lemma 1.2 and Lemma 2.2] (for Hilbert space operators). We note that the hypotheses of the general case of [7, Lemma 1.2] should be supplemented by the assumption that the operator AB has closed range; this does not follow from the other assumptions. For the special case when B is invertible and A has closed range this is not needed as the equation $(AB)(B^{-1}A^\dagger)(AB) = AB$ implies that AB has closed range.

There are many publications dealing with the continuity of the Moore–Penrose inverse for complex matrices, both square and rectangular, such as [1, 16, 17, 18]. We recover the following fundamental result of Penrose [10, p. 408].

COROLLARY 1.8. *Let a_n, a be nonzero $p \times p$ matrices such that $a_n \to a$. Then $a_n^\dagger \to a^\dagger$ if and only if there is n_0 such that* $\operatorname{rank}(a_n) = \operatorname{rank}(a)$ *for all $n \geq n_0$.*

PROOF. Let $a_n^\dagger \to a^\dagger$. By the preceding theorem, $a_n a_n^\dagger \to a a^\dagger$. Then

$$\operatorname{rank}(a_n) = \operatorname{rank}(a_n a_n^\dagger) = \operatorname{tr}(a_n a_n^\dagger) \to \operatorname{tr}(a a^\dagger) = \operatorname{rank}(a a^\dagger) = \operatorname{rank}(a)$$

by the continuity of the trace.

Conversely, suppose that $\operatorname{rank}(a_n) = \operatorname{rank}(a)$ for all $n \geq n_0$. By a result of Wedin [18], inequality (1.3) reduces to

$$\|a_n^\dagger - a^\dagger\| \leq 3\|a_n^\dagger\|\|a^\dagger\|\|a_n - a\|, \quad n \geq n_0.$$

Write $\varepsilon_n = 3\|a^\dagger\|\|a_n - a\|$. Then $\|a_n^\dagger\| \leq \|a^\dagger\| + \|a_n^\dagger\|\varepsilon_n$, and $\|a_n^\dagger\| \leq$

$(1-\varepsilon_n)^{-1}\|a^\dagger\| \le 2\|a^\dagger\|$ whenever $n \ge n_0$ and $0 \le \varepsilon_n \le \frac{1}{2}$. The result then follows from the preceding theorem.

We restate the theorem on the continuity of the Moore–Penrose inverse for C^*-algebra-valued functions, mainly because of its application in the next section to the differentiation of the Moore–Penrose inverse. In the following, J denotes an interval, t_0 an element of J, and $a(t)$ a C^*-algebra valued function defined for all $t \in J$. By $a^\dagger(t)$ we denote the Moore–Penrose inverse $a(t)^\dagger$ of $a(t)$.

THEOREM 1.9. *Let $a(t)$ be a function with values in a C^*-algebra A defined on an interval J such that $0 \neq a(t) \in A^\dagger$ for all $t \in J$, and that $a(t)$ is continuous at t_0. The following conditions are equivalent.*

$$a^\dagger(t) \text{ is continuous at } t_0, \tag{1.9}$$

$$a(t)a^\dagger(t) \text{ is continuous at } t_0, \tag{1.10}$$

$$a^\dagger(t)a(t) \text{ is continuous at } t_0, \tag{1.11}$$

$$\text{there is } \delta > 0 \text{ such that } \sup_{|t-t_0|<\delta} \|a^\dagger(t)\| < \infty. \tag{1.12}$$

2. Differentiability of the Moore–Penrose inverse

The differentiation of the Moore–Penrose inverse for matrices was first studied by Golub and Pereyra in [4] and by Decell in [2]; Wedin obtained the equation (2.2) which leads to the explicit formula for the derivative. Drazin [3] investigated the problem in the setting of associative rings, and gave a unified derivation of the differentiation formulae for the Moore–Penrose inverse and the Drazin inverse.

In this section J again denotes an interval, t_0 an element of J, and $a : J \to A$ a C^*-algebra valued function. By $a'(t)$ we denote the derivative of $a(t)$ at t, and by $a^\dagger(t)$ the Moore–Penrose inverse $a(t)^\dagger$.

THEOREM 2.1. *Let $a(t)$ be a C^*-algebra valued function defined on an interval J such that $0 \neq a(t) \in A^\dagger$ for all $t \in J$ and that $a(t)$ is differentiable at t_0. Then the function $a^\dagger(t)$ is differentiable at t_0 if and only if one of the conditions* (1.9)–(1.12) *is satisfied. The derivative $(a^\dagger)' = (a^\dagger)'(t_0)$ is given by*

$$(a^\dagger)' = -a^\dagger a' a^\dagger + (e - a^\dagger a)(a')^*(a^\dagger)^* a^\dagger + a^\dagger (a^\dagger)^*(a')^*(e - aa^\dagger), \tag{2.1}$$

where a, a^*, $a^\dagger$, a' stand for $a(t_0)$, $a^*(t_0)$, $a^\dagger(t_0)$, $a'(t_0)$, respectively.

PROOF. First we observe that if $a(t)$ is differentiable, then so is $a^*(t)$, and

$$(a^*)'(t) = (a')^*(t).$$

From Theorem 1.2 we get

$$\begin{aligned}\frac{a^\dagger(t) - a^\dagger(t_0)}{t - t_0} &= -a^\dagger(t)\frac{a(t) - a(t_0)}{t - t_0}a^\dagger(t_0) \\ &\quad + (e - a^\dagger(t)a(t))\frac{a^*(t) - a^*(t_0)}{t - t_0}(a^\dagger)^*(t_0)a^\dagger(t_0) \\ &\quad + a^\dagger(t)(a^\dagger)^*(t)\frac{a^*(t) - a^*(t_0)}{t - t_0}(e - a(t_0)a^\dagger(t_0)).\end{aligned} \tag{2.2}$$

If one of the conditions (1.9)–(1.12) is satisfied, then $a^\dagger(t)$ is continuous at t_0, and we can take the limit as $t \to t_0$ in (2.2). This proves (2.1).

Conversely, if $a^\dagger(t)$ is differentiable at t_0, then it is also continuous at t_0, and all of the equivalent conditions (1.9)–(1.12) are satisfied.

Note 2.2. The arguments in the foregoing proof depend on the fact that t is a real variable as we make use of the formula

$$\frac{da^*(t)}{dt} = \left(\frac{da(t)}{dt}\right)^*,$$

which is false when t is complex; this suggests that in the preceding theorem the differentiability with respect to a real variable cannot be replaced by analyticity. This is confirmed by observing that if $a(t)$ and $a^\dagger(t)$ were analytic, $a^\dagger(t)a(t)$ and $a(t)a^\dagger(t)$ would be constant, as $\|a^\dagger(t)a(t)\| = \|a(t)a^\dagger(t)\| = 1$.

For finite matrices, the preceding theorem together with Corollary 1.8 yields the following result due to Golub and Pereyra [4, Theorem 4.3].

Corollary 2.3. *Let $a(t)$ be a function defined on the interval J whose values are nonzero $p \times p$ matrices, differentiable at t_0. Then $a^\dagger(t)$ is differentiable at t_0 if and only if* rank $(a(t))$ *is constant in some interval $|t - t_0| < \delta$. The derivative $(a^\dagger)'(t_0)$ is given by* (2.1).

We mention that applications of the differentiation of the Moore–Penrose inverse include optimization with nonlinear equality constraints, generalized Newton's method and stability of perturbed least squares problems (see [4]).

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