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ON EQUIVALENCE OF SEMIGROUP IDENTITIES

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Abstract

For a given relation ρ on a free semigroup $\mathscr{F}$ we describe the smallest cancellative fully invariant congruence $\rho^{\#}$ containing ρ.

Two semigroup identities are *s-equivalent* if each of them is a consequence of the other on cancellative semigroups. If two semigroup identities are equivalent on groups, it is not known if they are s-equivalent. We give a positive answer to this question for all binary semigroup identities of the degree less or equal to 5. A poset of corresponding varieties of groups is given.

1. Introduction

Let $\mathscr{F}$ be a free semigroup (F_∞ be a free group) generated by $x_1, x_2, \ldots$. A *semigroup identity* of a group G (or a semigroup S) is a nontrivial identity of the form $u \equiv v$ where $u, v \in \mathscr{F}$, which becomes the equality under every substitution of generators by elements from G (elements from S).

There are several open problems concerning semigroup identities. By an old result of A. I. Mal'cev [8] a group, which is an extension of a nilpotent group by a group of finite exponent, satisfies a semigroup identity. Recently, after more then 40 years, it was shown that the converse is not true [9].

In 1966 A. I. Shirshov (see [6, problem 2.82]) posed the following problem: can the class of all groups with the n-Engel condition be defined by semigroup identities? This problem has a positive answer for residually finite n-Engel groups [2], but in general it is still open.

Another open problem is due to G. M. Bergman [1] (see also [10]): Let G be any group and S be any subsemigroup generating G. Must any semigroup identity satisfied in S be satisfied in G? For a large class of groups the solution of Bergman's problem is positive in particular for residually finite and soluble groups [2], however S. V. Ivanov and E. Rips believe that there exists a counterexample. It can be shown (unpublished) that the Bergman's problem is equivalent to the following:

QUESTION 1. Let a semigroup identity $u \equiv v$ imply a semigroup identity $a \equiv b$ for groups. Does the same implication hold in the class of cancellative semigroups?

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To illustrate the situation we give an example. The identity $(xy)^2 \equiv (yx)^2$ implies $xy^2 \equiv y^2x$ for groups, because the automorphism $\alpha : x \to x, y \to x^{-1}y$ changes $(xy)^2 \equiv (yx)^2$ into $xy^2 \equiv y^2x$.
For *semigroups* we can not use this automorphism.

So, to prove that $(xy)^2 \equiv (yx)^2$ implies $xy^2 \equiv y^2x$ for cancellative semigroups we need another way to go. The idea is to show first that $(xy)^2 \equiv (yx)^2$ implies:

(i) $(yx)^2y \equiv y(yx)^2$,

(ii) $(yx)^4y^2 \equiv ((yx)^2y)^2$,

(iii) $x((yx)^2y)^2 \equiv ((yx)^2y)^2x$,

(iv) $(xy)^4 \equiv (yx)^4$.

Then for some word p we start with $p \cdot xy^2$ and by using (i)–(iv) obtain $p \cdot y^2x$, which by cancellation, implies required $xy^2 \equiv y^2x$.

To be precise we introduce a relation ρ containing pairs:

(i) $((yx)^2y, y(yx)^2)$,

(ii) $((yx)^4y^2, ((yx)^2y)^2)$,

(iii) $(x((yx)^2y)^2, ((yx)^2y)^2x)$,

(iv) $((xy)^4, (yx)^4)$.

Definition 1.1. For a relation ρ we say that two words $a, b \in \mathscr{F}$ are connected by a ρ-step, if $a = c_1sc_2$, $b = c_1tc_2$, and $(s,t) \in \rho$. In this case we write $a \longleftrightarrow b$. A sequence of a finite number of ρ-steps is called a ρ-*sequence*. If a and b are connected by a ρ-sequence, we write $a \overset{\rho}{\Longleftrightarrow} b$ or $(a \equiv b)$.

In our case for some word p we shall find a ρ-sequence connecting pxy^2 and py^2x which after cancelling gives required $xy^2 \equiv y^2x$. Namely, for $p = (xy)^4$ we have

$$\begin{aligned} pxy^2 &= (xy)^4xy^2 = x(yx)^4y^2 = x(yx)^2((yx)^2y)y \\ &\overset{(i)}{\longleftrightarrow} x(yx)^2(y(yx)^2)y = x((yx)^2y)^2 \overset{(iii)}{\longleftrightarrow} ((yx)^2y)^2x \\ &\overset{(ii)}{\longleftrightarrow} (yx)^4y^2x \overset{(iv)}{\longleftrightarrow} (xy)^4y^2x = py^2x, \end{aligned}$$

which gives $pxy^2 \equiv py^2x$ and hence $xy^2 \equiv y^2x$ as required.

The full proof of the equivalence of identities $(xy)^k \equiv (yx)^k$ and $xy^k \equiv y^kx$, $(k \geq 2)$ for cancellative semigroups is given in Theorem 3.2.

It was conjectured by J. Krempa that the identity $u \equiv v$ implies an identity $a \equiv b$ for cancellative semigroups if and only if for some $p, q \in \mathscr{F} \cup \emptyset$ the words paq and pbq are connected by a sequence of steps as in the above example. We prove this fact in Theorem 2.3.

While considering congruences, we work with pairs of words $(u, v) \in \mathscr{F} \times \mathscr{F}$. Pairs of the type (a, a) we call trivial.

The identity $u \equiv v$ (the pair (u, v)) is called *balanced* if every generator occurs the same number of times in u and v. A cancellative semigroup which satisfies a non-balanced semigroup identity has to satisfy an identity of the type $x^n \equiv x$ which implies that the semigroup is a group (of a finite exponent). So we shall consider only balanced identities.

The *degree* of a balanced identity (pair of words) is the length of u (equal to the length of v). The identity $u \equiv v$ (the pair (u, v)) is called *cancelled* if u, v begin (and end) with different letters. It is easy to show that for any cancellative semigroup S, any semigroup identity satisfied in S can be replaced by a cancelled identity of not higher degree.

We denote by *End* the set of all endomorphisms of the free semigroup $\mathscr{F}$ and speak about *End*-invariant relations instead of fully invariant.

DEFINITION 1.2. A relation on $\mathscr{F}$ is called *End-invariant* if together with every pair (u, v) it contains all pairs (u^e, v^e), $e \in End$.

A relation is called *cancellative* if together with every pair (paq, pbq) it contains (a, b), (for $p, q \in \mathscr{F} \cup \emptyset$).

For any relation $\rho \subset \mathscr{F} \times \mathscr{F}$ we shall consider *the smallest cancellative End-invariant congruence* on $\mathscr{F}$, containing ρ, and denoted by $\rho^\#$. It means that the quotient semigroup is cancellative and satisfies the relation ρ as an identity. In particular, if $\rho = \{(u, v)\}$ where (u, v) is a pair of words from $\mathscr{F}$ then the above congruence will be denoted by $(u, v)^\#$.

In [5] the smallest cancellative congruence containing ρ is described as an infinite sum of relations.

We give here a simple description of the smallest cancellative *End*-invariant congruence containing ρ. This description allows for using computer to show that in a cancellative semigroup one identity implies another.

We describe the poset of all two-variable semigroup identities of degree less or equal to five, and show that if one of them implies another for groups then also for semigroups.

2. Cancellative Congruences

In this section for any relation ρ on $\mathscr{F}$ we describe the smallest cancellative *End*-invariant congruence $\rho^\#$ containing ρ. The existence of such a congruence follows, since the class of cancellative semigroups is closed under forming cartesian products and taking subsemigroups. If ρ consists of trivial pairs, then $\rho^\#$ is equal to diag$(\mathscr{F} \times \mathscr{F})$ and is called trivial.

We need also

DEFINITION 2.1. A relation ρ satisfies Ore conditions, if for every $a, b \in \mathscr{F}$ there exist $a', b' \in \mathscr{F}$ such that $(aa', bb') \in \rho$, and there exist $a'', b'' \in \mathscr{F}$ such that $(a''a, b''b) \in \rho$.

LEMMA 2.2. *Any nontrivial cancellative End-invariant relation ρ on semigroups satisfies Ore conditions.*

PROOF. By using a proper endomorphism we can get a cancelled pair (u, v) of two-variable words in ρ, such that the first letter in u is x and the first letter in v is y. Then

$$u(x, y) = x \cdot u'(x, y), \qquad v(x, y) = y \cdot v'(x, y).$$

For any given a, b, if substitute a, b for x, y then

$$(aa', bb') \in \rho$$

for $a' = u'(a, b)$, $b' = v'(a, b)$, and hence the right Ore condition is satisfied. For the left Ore condition we deal with the last letters.

For a given relation ρ we denote by ρ^{irs} the *End-invariant, reflexive*, and *symmetric* closure of ρ. That is ρ^{irs} is a set containing all *End*-images of pairs $(u, v) \in \rho$ and of pairs (v, u). It contains also all trivial pairs (a, a), $a \in \mathscr{F}$.

$$\rho^{\text{irs}} = \{(u^e, v^e), (v^e, u^e), (a, a); \forall (u, v) \in \rho, e \in End, a \in \mathscr{F}\}.$$

We write it shortly as a sum over $(u, v) \in \rho$:

$$\rho^{\text{irs}} = \cup\{(u, v), (v, u), (x, x)\}^{End}. \tag{1}$$

The smallest *End*-invariant congruence on $\mathscr{F}$, containing a relation ρ is described in [3]. Namely, two words are congruent if and only if they are connected by a ρ^{irs}-sequence.

Our description of the smallest *cancellative End*-invariant congruence containing ρ is also based on conection of words by a ρ^{irs}-sequence.

THEOREM 2.3. *For a given relation ρ, let ρ^* denote a relation consisting of all pairs (a, b) such that for some $p, q \in \mathscr{F} \cup \emptyset$, the words paq and pbq are connected by a ρ^{irs}-sequence. Then $\rho^* = \rho^\#$ is the smallest cancellative End-invariant congruence on $\mathscr{F}$, containing ρ.*

PROOF. Let ρ^* be a relation defined by: $(a, b) \in \rho^*$, if and only if for some $p, q \in \mathscr{F} \cup \emptyset$, paq and pbq are connected by a ρ^{irs}-sequence ($paq \stackrel{\text{irs}}{\Longleftrightarrow} pbq$). It is clear that ρ^* is a cancellative relation and that $\rho^* \subseteq \rho^\#$.

To show that ρ^* is an equivalence relation it is enough to check transitivity, because ρ^{irs} is symmetric and reflexive. Let (a, b) and (b, c) be in ρ^*, that is for some $p, q, r, s \in \mathscr{F}$, $paq \overset{\text{irs}}{\Longleftrightarrow} pbq$ and $rbs \overset{\text{irs}}{\Longleftrightarrow} rcs$. We have to find elements g and h in $\mathscr{F}$ such that $gah \overset{\text{irs}}{\Longleftrightarrow} gch$. By Lemma 2.2 for ρ^{irs} there exist p'', r'', q', s', such that ρ^{irs} contains pairs

$$\text{(i)}\ \ (p''p, r''r), \qquad \text{(ii)}\ \ (qq', ss').$$

Now we denote $g = p''p$, $h = qq'$, then

$$\begin{aligned} gah = p'' \cdot paq \cdot q' &\overset{\text{irs}}{\Longleftrightarrow} p'' \cdot pbq \cdot q' = p''p \cdot b \cdot qq' \\ &\overset{\text{(i)}}{\longleftrightarrow} r''r \cdot b \cdot qq' \overset{\text{(ii)}}{\longleftrightarrow} r''r \cdot b \cdot ss' = r'' \cdot rbs \cdot s' \\ &\overset{\text{irs}}{\Longleftrightarrow} r'' \cdot rcs \cdot s' = r''r \cdot c \cdot ss' \overset{\text{(i)}}{\longleftrightarrow} p''p \cdot c \cdot ss' \overset{\text{(ii)}}{\longleftrightarrow} p''p \cdot c \cdot qq' = gch. \end{aligned}$$

So, ρ^* is transitive and hence the equivalence relation.

We check now that ρ^* is a congruence, that is for every $s, t \in \mathscr{F}$, if $(a, b) \in \rho^*$, then $(sat, sbt) \in \rho^*$. By another words for some p, q we have $paq \overset{\text{irs}}{\Longleftrightarrow} pbq$ and we have to show that there exist g, h such that $g \cdot sat \cdot h \overset{\text{irs}}{\Longleftrightarrow} g \cdot sbt \cdot h$. By Lemma 2.2 for ρ^{irs} we conclude that there are s'', p'', t', q' such that ρ^{irs} contains pairs

$$\text{(i)}\ \ (s''s, p''p), \qquad \text{(ii)}\ \ (tt', qq').$$

If denote $g = s''$, $h = t'$, then

$$\begin{aligned} g \cdot sat \cdot h = s''s \cdot a \cdot tt' &\overset{\text{(i)}}{\longleftrightarrow} p''p \cdot a \cdot tt' \overset{\text{(ii)}}{\longleftrightarrow} p''p \cdot a \cdot qq' = p'' \cdot paq \cdot q' \\ &\overset{\text{irs}}{\Longleftrightarrow} p'' \cdot pbq \cdot q' = p''p \cdot b \cdot qq' \overset{\text{(i),(ii)}}{\longleftrightarrow} s''s \cdot b \cdot tt' = g \cdot sbt \cdot h, \end{aligned}$$

which finishes the proof.

3. Properties of the congruence $(u, v)^{\#}$

We take now a nontrivial balanced pair of words (u, v) as the relation ρ to describe *the smallest cancellative congruence* $(u, v)^{\#}$, such that the quotient semigroup is cancellative and satisfies the identity $u \equiv v$. By Theorem 2.3 a pair (a, b) is in $(u, v)^{\#}$, if and only if for some $p, q \in \mathscr{F} \cup \emptyset$, the words paq and pbq are connected by a $(u, v)^{\text{irs}}$-sequence, where by (1):

$$(u, v)^{\text{irs}} = \{(u, v), (v, u), (x, x)\}^{End}. \tag{2}$$

We need some properties of this congruence.

PROPERTY 1. Two identities $a \equiv b$ and $u \equiv v$ are equivalent on cancellative semigroups if and only if $(a, b)^\# = (u, v)^\#$. These identities are equivalent on groups if and only if the words ab^{-1} and uv^{-1} define the same verbal subgroup in the free group F_∞.

It is clear that if two identities are equivalent on cancellative semigroups, then they are equivalent on groups, and hence we have

PROPERTY 2. If ab^{-1} and uv^{-1} define different verbal subgroups, then the identities $a \equiv b$ and $u \equiv v$ are not equivalent on cancellative semigroups.

The converse statement is an open problem (equivalent to Question 1).

QUESTION 2. Is it possible that $(u, v)^\# \neq (a, b)^\#$, while uv^{-1} and ab^{-1} define the same verbal subgroup in F_∞?

For the next property, we denote by $\overline{u(x_1, \dots, x_n)}$ the word obtained from $u(x_1, \dots, x_n)$ by writing it backward. For example $\overline{xy^2} = y^2x$.

For a pair (a, b) we denote $\overline{(a, b)} := (\overline{a}, \overline{b})$. For a set A we denote $\overline{A} := \{\overline{a}; a \in A\}$.

What will happen to a congruence if we change every pair (a, b) in it for the pair $\overline{(a, b)}$? We shall call a congruence ρ *bar-invariant* if $\overline{\rho} = \rho$.

In the case when $\rho = (u, v)^\#$ we can show that the set $\overline{(u, v)^\#}$ is also a congruence. We call it a bar-congruence.

LEMMA 3.1. $\overline{(u, v)^\#} = (\overline{u}, \overline{v})^\#$.

PROOF. For $e \in End$ we define $\overline{e} \in End$ by: $x_i^{\overline{e}} = \overline{x_i^e}$, then $\overline{End} = End$. For $u = u(x_1, \dots, x_n)$ it holds $\overline{u^e} = \overline{u}^{\overline{e}}$ and hence $\overline{(u^e, v^e)} := (\overline{u^e}, \overline{v^e}) = (\overline{u}^{\overline{e}}, \overline{v}^{\overline{e}})$. So

$$\overline{(u, v)^{\text{irs}}} = \{(\overline{u}, \overline{v}), (\overline{v}, \overline{u}), (x, x)\}^{\overline{End}} = \{(\overline{u}, \overline{v}), (\overline{v}, \overline{u}), (x, x)\}^{End} = (\overline{u}, \overline{v})^{\text{irs}}.$$

Now, a and b are connected by a ρ-step if and only if $\overline{a}$ and $\overline{b}$ are connected by a $\overline{\rho}$-step. Similarly $p\overline{a}q$ and $p\overline{b}q$ are connected by a ρ-sequence if and only if $\overline{q}a\overline{p}$ and $\overline{q}b\overline{p}$ are connected by a $\overline{\rho}$-sequence. To prove $\overline{(u, v)^\#} = (\overline{u}, \overline{v})^\#$ we note that:

(a, b) is in $\overline{(u, v)^\#}$ iff $(\overline{a}, \overline{b}) \in (u, v)^\#$, which is iff for some p, q, $p\overline{a}q$ and $p\overline{b}q$ are connected by a $(u, v)^{\text{irs}}$-sequence, which is iff $\overline{q}a\overline{p}$ and $\overline{q}b\overline{p}$ are connected by a $(\overline{u}, \overline{v})^{\text{irs}}$-sequence, which is iff $(a, b) \in (\overline{u}, \overline{v})^\#$, and hence the statement follows.

It is clear that the congruence $(xy, yx)^\#$ is bar-invariant. As another example we show that $\overline{(xy^2xy, y^3x^2)^\#} = (xy^2xy, y^3x^2)^\#$, which is the same as

$$(x^2y^3, yxy^2x)^\# = (xy^2xy, y^3x^2)^\#. \tag{3}$$

To get $\overline{\rho} = \rho$ it is enough to check $\overline{\rho} \subseteq \rho$, so in our case we check only $(x^2y^3, yxy^2x) \in (xy^2xy, y^3x^2)^\#$.

We take the following pairs in $(xy^2xy, y^3x^2)^\#$:

(i) (xy^2xy, y^3x^2),

(ii) (yx^2yx, x^3y^2),

(iii) $((xy)^2x^2y, (yx)^3x)$ $\quad (= (\mathrm{i})^\alpha, \alpha : x \to x, y \to xy$; cancelled),

(iv) (yx^3yx, x^3yxy) $\quad (= (\mathrm{i})^\alpha, \alpha : x \to xy, y \to x$; cancelled),

(v) (xy^3xy, y^3xyx) $\quad (= (\mathrm{i})^\alpha, \alpha : x \to xy, y \to y)$.

Then for $p = x^3, q = xy$ we get:

$$\begin{aligned} p(x^2y^3)q &= x^3(x^2y^3)xy = x^4(xy^3xy) \\ &\overset{(\mathrm{v})}{\longleftrightarrow} x^4(y^3xyx) = x^3(xy^3xy)x \overset{(\mathrm{v})}{\longleftrightarrow} x^3(y^3xyx)x = (x^3y^2)(yx)^2x \\ &\overset{(\mathrm{ii})}{\longleftrightarrow} (yx^2yx)(yx)^2x = yx^2((yx)^3x) \overset{(\mathrm{iii})}{\longleftrightarrow} yx^2((xy)^2x^2y) = (yx^3yx)yx^2y \\ &\overset{(\mathrm{iv})}{\longleftrightarrow} (x^3yxy)yx^2y = x^3(yxy^2x)xy = p(yxy^2x)q, \end{aligned}$$

which proves the example.

So a natural question arises: Does the following equality always hold $\overline{(u, v)^\#} = (u, v)^\#$? This question can be formulated also as:

QUESTION 3. Are semigroup identities $u \equiv v$ and $\overline{u} \equiv \overline{v}$ always equivalent for cancellative semigroups?

Similar question for groups has a positive answer because $\overline{u(x_1, \ldots, x_n)} = u(x_1^{-1}, \ldots, x_n^{-1})^{-1}$.

We show now that two pairs of different degree can define the same congruence. The following Theorem shows that the pair $((xy)^k, (yx)^k)$ of the degree $2k$ defines the same congruence as the pair (xy^k, y^kx) of the degree $k + 1$.

THEOREM 3.2. *For $k > 0$, $(xy^k, y^kx)^\# = ((xy)^k, (yx)^k)^\#$.*

PROOF. By Theorem 2.3, to show $((xy)^k, (yx)^k) \in (xy^k, y^kx)^\#$ we take $q = x$ and check that $(xy)^kq$ and $(yx)^kq$ are connected by a $(xy^k, y^kx)^{\mathrm{irs}}$-sequence. The sequence will consist of one step, for which we use the pair $(x(yx)^k, (yx)^kx)$, which is equal to $(xy^k, y^kx)^e$ for $e : x \to x, y \to yx$. Namely

$$(xy)^kq = (xy)^kx = x(yx)^k \longleftrightarrow (yx)^kx = (yx)^kq,$$

which gives $((xy)^k, (yx)^k) \in (xy^k, y^kx)^\#$ and hence $((xy)^k, (yx)^k)^\# \subseteq (xy^k, y^kx)^\#$.

To prove $(xy^k, y^k x) \in ((xy)^k, (yx)^k)^\#$ we use the following pairs in $((xy)^k, (yx)^k)^\#$ (we explain later how to obtain them):

(i) $((yx)^{k^2} y^k, ((yx)^k y)^k)$,

(ii) $(x((yx)^k y)^k, ((yx)^k y)^k x)$,

(iii) $((yx)^{k^2}, (xy)^{k^2})$.

Now we can see that for $p = (xy)^{k^2}$, (q empty) the words pxy^k and $py^k x$ are connected by a $((xy)^k, (yx)^k)^\#$-sequence:

$$pxy^k = (xy)^{k^2} xy^k = x(yx)^{k^2} y^k \overset{\text{(i)}}{\longleftrightarrow} x((yx)^k y)^k$$
$$\overset{\text{(ii)}}{\longleftrightarrow} ((yx)^k y)^k x \overset{\text{(i)}}{\longleftrightarrow} (yx)^{k^2} y^k x \overset{\text{(iii)}}{\longleftrightarrow} (xy)^{k^2} y^k x = py^k x,$$

which implies $(xy^k, y^k x)^\# \subseteq ((xy)^k, (yx)^k)^\#$.

Now we show that pairs (i)–(iii) are in $((xy)^k, (yx)^k)^\#$. The first inclusion follows from $((yx)^{kl} y^l, ((yx)^k y)^l) \in ((xy)^k, (yx)^k)^\#$, which can be obtained by induction on l with use of $((yx)^k y^l, y^l (yx)^k) \in ((xy)^k, (yx)^k)^\#$, which follows by induction on l, while for $l = 1$ $((yx)^k y, y(yx)^k) \in ((xy)^k, (yx)^k)^\#$ follows from $(yx)^k y = y(xy)^k \longleftrightarrow y(yx)^k$.

The inclusion for (ii) follows from $(x(xy)^k, (xy)^k x) \in ((xy)^k, (yx)^k)^\#$, by using the endomorphism $g : y \to y(xy)^{k-1} y$, $x \to x$. The inclusion for (iii) is clear. This finishes the proof.

4. Two-variable identities of small degree

Let (u, v) be a pair of two-variable words written through generators x, y, and let σ permutes x and y. It is clear that $(u, v)^\# = (v, u)^\# = (u^\sigma, v^\sigma)^\# = (v^\sigma, u^\sigma)^\#$, so it makes sense to consider only one of the above pairs. We define a standard form for pairs and identities.

We say that a word $u(x, y)$ is of a type $X^k Y^l$ if the first letter in u is x, the exponent sum of x's is k and the exponent sum of y's is l.

We say that a pair (u, v) is of the type $X^k Y^l$ if u is of that type.

In a cancelled balanced pair (u, v) of the type $X^k Y^l$ the word v is of the type $Y^l X^k$. We note that v^σ is then of the type $X^l Y^k$.

Definition 4.1. A cancelled balanced pair (u, v) of the type $X^k Y^l$ is called *standard* if $k < l$ or $k = l$ and u is lexicographically less than, or equal to v^σ. An identity defined by a standard pair is called standard.

Since either (u, v) or (v^σ, u^σ) is standard, we get

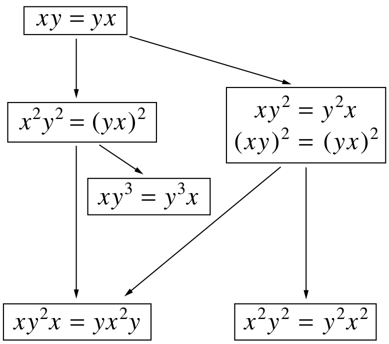
Corollary 4.2. *If (u, v) is any cancelled balanced pair of degree n, then the congruence $(u, v)^\#$ can be defined by a standard pair of degree n.*

QUESTION 4. Is it possible that standard pairs (a, b) and (u, v) of the same degree and of different types define $(a, b)^{\#} = (u, v)^{\#}$?

Two-variable identities of degree ≤ 4

Two semigroup identities are *s-equivalent* if each of them is a consequence of the other in every cancellative semigroup. In this case corresponding pairs define the same congruence. Every identity is s-equivalent to a standard identity. We show that there are seven standard identities of degree ≤ 4, which split into six s-equivalence classes. This classes form a poset with respect to implication of identities in cancellative semigroups.

THEOREM 4.3. *There are six s-equivalence classes of two-variable semigroup identities of degree ≤ 4. The poset of the classes is given below.*



Poset of s-equivalence classes of two-variable semigroup identities of degree ≤ 4

PROOF. The only standard pairs of the degree 2 and 3 are $a := (xy, yx)$ and $b := (xy^2, y^2x)$. To describe congruences of degree 4 we have to consider pairs (u, v) only of the type XY^3, and X^2Y^2. There exists only one pair of the first type: $c := (xy^3, y^3x)$.

The set of possible words u of the type X^2Y^2 is $U = \{x^2y^2, xy^2x, (xy)^2\}$. The set of possible words v is $U^\sigma = \{y^2x^2, yx^2y, (yx)^2\}$. Since the pairs (u, v) have to be cancelled and of the length 4, we have to consider only:

$$(x^2y^2, y^2x^2), \quad (x^2y^2, (yx)^2), \quad (xy^2x, yx^2y), \quad ((xy)^2, (yx)^2).$$

Since by Theorem 3.2, $((xy)^2, (yx)^2)^{\#}$ is also defined by the pair (xy^2, y^2x) of degree 3 we have to consider only pairs:

$$d := (x^2y^2, y^2x^2), \quad e := (x^2y^2, (yx)^2), \quad f := (xy^2x, yx^2y).$$

To show that the six congruences defined by pairs a–f of degree ≤ 4 are different for cancellative semigroups it is enough (by Property 2) to show that the corresponding identities define different verbal subgroups V in the two-generator free group F.

It is clear that

$$V(a) = [F, F], \quad V(b) = [F, F^2], \quad V(c) = [F, F^3], \quad V(d) = [F^2, F^2].$$

For e we write corresponding identity $x^2y^2 = (yx)^2$ in a non-cancelled form as $x^3y^3 = (xy)^3$, then by [4] it defines the verbal subgroup $V(e) = F^3 \cap [F, F]$. For f the corresponding identity $xy^2x = yx^2y$, is equivalent by [11] to 2-engel identity $[x, y, y] = 1$, and hence in the 2-generator group F it defines the verbal subgroup $V(f) = [[F, F], F]$. It is known that all these verbal subgroups are different and hence the congruences are different.

To draw the poset of congruences (s-equivalent classes of identities) we need to chek implications. Since most of implications are obvious, we have to prove only that on cancellative semigroups the identity $x^2y^2 = (yx)^2$ implies both $xy^3 = y^3x$ and $xy^2x = yx^2y$. To prove the first implication we show that $(xy^3, y^3x) \in (x^2y^2, (yx)^2)^\#$.

We define p, q and connect pxy^3q and py^3xq by a $(x^2y^2, (yx)^2)^\#$-sequence. Every step of the sequence uses one of the following pairs in $(x^2y^2, (yx)^2)^\#$:

(i) $(x^2y^2, (yx)^2)$,

(ii) $((xy)^2y, yxy^2x)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to xy, y \to y$; cancelled),

(iii) $(xyxy^4, y^2xy^3x)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to xy, y \to y^2$; cancelled),

(iv) $(y^4x^2, (xy^2)^2)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to y^2, y \to x)$.

Then for $p = y^2, q = x$ we get:

$$\begin{aligned} p(xy^3)q &= y^2xy^3x \xleftrightarrow{\text{(iii)}} xyxy^4 = (xy)^2yy^2 \\ &\xleftrightarrow{\text{(ii)}} yxy^2xy^2 = y(xy^2)^2 \xleftrightarrow{\text{(iv)}} y(y^4x^2) = y^2(y^3x)x = p(y^3x)q, \end{aligned}$$

which by cancellation leads to required $(xy^3, y^3x)^\# \subseteq (x^2y^2, (yx)^2)^\#$.

To prove the second implication we show that $(xy^2x, yx^2y) \in$ $(x^2y^2, (yx)^2)^\#$. We note that $(x^2y^2, (yx)^2)^\#$ contains the following pairs:

(i) $(x^2y^2, (yx)^2)$,

(ii) $(y^2x^2, (xy)^2)$,

(iii) $((xy)^2y, yxy^2x)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to xy, y \to y$; cancelled).

Then for $p = y^2$ we have:

$$p(xy^2x) = y^2(xy^2x) = y(yxy^2x) \xleftrightarrow{\text{(iii)}} y(xy)^2y$$
$$\xleftrightarrow{\text{(ii)}} y(y^2x^2)y = y^2(yx^2y) = p(yx^2y),$$

which finishes the proof.

$$\begin{tikzcd}
{[F,F]} \arrow[d] \arrow[drr] & & \\
F^3 \cap [F,F] \arrow[dr] \arrow[dd] & & {[F,F^2]} \arrow[ddll] \arrow[dd] \\
& {[F,F^3]} & \\
{[[F,F],F]} & & {[F^2,F^2]}
\end{tikzcd}$$

The poset of verbal subgroups defined by a single two-variable semigroup identity of degree ≤ 4 in a two-generator free group F

Two-variable identities of degree ≤ 5

We show that there are 13 standard pairs of degree 5, which give only four new s-equivalence classes, and draw the poset for s-equivalence classes of identities of degree ≤ 5.

THEOREM 4.4. *There are ten s-equivalence classes of two-variable semigroup identities of degree ≤ 5. The poset is given below.*

PROOF. We note that for the degree equal to 5 there are standard pairs only of the type XY^4, and X^2Y^3. There exists only one pair of the first type: (xy^4, y^4x).

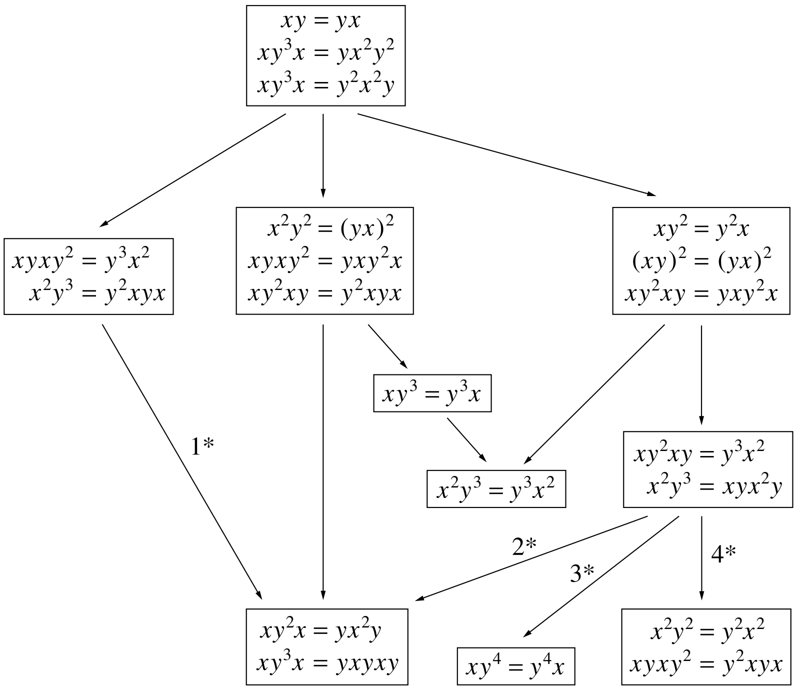
For standard pairs of the type X^2Y^3, the word u is in the set U_{23} below (split with respect to the last letter of the words):

$$U_{23} = \{xy^3x\} \cup \{xyxy^2, xy^2xy, x^2y^3\}.$$

The word v is in

$$U_{32}^{\sigma} = \{yxyxy, yx^2y^2, y^2x^2y\} \cup \{yxy^2x, y^2xyx, y^3x^2\}.$$

Combining possible u and v we can see that there are 12 $(= 3 + 3 \cdot 3)$ different cancelled pairs of the type X^2Y^3. So there are 13 standard pairs of the



Poset of s-equivalence classes of two-variable semigroup identities of degree ≤ 5

degree 5. However we can prove that they define only four new congruences. The following Lemma will finish our proof.

LEMMA 4.5. *There exist only four different congruences of the degree five.*

PROOF. First we show that 7 of the 13 standard pairs of the degree 5 define known congruences, already obtained by using pairs of smaller degrees, namely:

1. $(xy^3x, y^2x^2y)^\# = (xy, yx)^\#$,
2. $(xy^3x, yx^2y^2)^\# = (xy, yx)^\#$,
3. $((xy)^2y, yxy^2x)^\# = (x^2y^2, (yx)^2)^\#$,
4. $(xy^2xy, y(yx)^2)^\# = (x^2y^2, (yx)^2)^\#$,
5. $(xy^3x, (yx)^2y)^\# = (xy^2x, yx^2y)^\#$,
6. $(xy^2xy, yxy^2x)^\# = (xy^2, y^2x)^\#$,
7. $((xy)^2y, y(yx)^2)^\# = (x^2y^2, y^2x^2)^\#$.

PROOF. Equality 1 is proven in [7, p. 132]. We obtain 2 by taking the bar-congruences in equality 1:

$$(xy^3x,\, yx^2y^2)^\# = \overline{(xy^3x,\, y^2x^2y)^\#} = \overline{(xy,\, yx)^\#} = (xy,\, yx)^\#.$$

For following equalities of the type $(a, b)^\# = (u, v)^\#$, we shall check $(a, b) \in (u, v)^\#$ and $(a, b)^\# \ni (u, v)$. To get $(a, b) \in (u, v)^\#$ we define $p,\ q$ and connect paq and pbq by a $(u, v)^\#$-sequence. Every step of the sequence uses some pair in $(u, v)^\#$, which is obtained as an image of (u, v) under some $\alpha \in End$. The pairs and sequences are found by using computer.

3.1. $((xy)^2y,\, yxy^2x) \in (x^2y^2,\, (yx)^2)^\#$.

This follows by applying $\alpha : x \to xy,\, y \to y$ to $(x^2y^2,\, (yx)^2)$ and cancellation.

3.2. $((xy)^2y,\, yxy^2x)^\# \ni (x^2y^2,\, (yx)^2)$.

We use the following pairs in $((xy)^2y,\, yxy^2x)^\#$:

(i) $((xy)^2y,\, yxy^2x)$,

(ii) $((xy^2)^2,\, yxy^3x)$ $\quad(= \text{(i)}^\alpha, \alpha : x \to xy,\, y \to y$; cancelled),

(iii) $((y^2x)^2x,\, xy^2x^2y^2)$ $\quad(= \text{(i)}^\alpha, \alpha : x \to y^2,\, y \to x)$,

(iv) $(y^2xy^3xyx,\, xy^3(xy)^2y)$ $\quad(= \text{(i)}^\alpha, \alpha : x \to y^2,\, y \to xy$; cancelled).

Then for $p = xy^2, q = y^2$ we have:

$$\begin{aligned}
&p(x^2y^2)q = xy^2(x^2y^2)y^2 = (xy^2x^2y^2)y^2\\
&\xleftrightarrow{\text{(iii)}} ((y^2x)^2x)y^2 = y(yxy^2x)xy^2 \xleftrightarrow{\text{(i)}} y((xy)^2y)xy^2 = yxy(xy^2)^2\\
&\xleftrightarrow{\text{(ii)}} yxy(yxy^3x) = y(xy^2)^2yx \xleftrightarrow{\text{(ii)}} y(yxy^3x)yx = y^2xy^3xyx\\
&\xleftrightarrow{\text{(iv)}} xy^3(xy)^2y = xy^2(yx)^2y^2 = p(yx)^2q,
\end{aligned}$$

as required.

So the equality 3 follows.

The equality 4 follows from equality 3 by taking bar-congruences, similarly to as 2 follows from 1.

5.1. $(xy^3x,\, (yx)^2y) \in (xy^2x,\, yx^2y)^\#$.

We apply endomorphism $\alpha : x \to y,\, y \to xy$ to the righthand pair $(xy^2x,\, yx^2y)$. After cancellation it gives $((yx)^2y, xy^3x,) \in (xy^2x,\, yx^2y)^\#$ which implies 5.1.

5.2. $(xy^3x,\, (yx)^2y)^\# \ni (xy^2x,\, yx^2y)$.

We use the following pairs in $(xy^3x, (yx)^2y)^\#$:

(i) $(xy^3x, (yx)^2y)$,
(ii) $((yx)^3y, (xy^2)^2x) \quad (= \text{(i)}^\alpha, \alpha : x \to y, y \to xy)$.

Then for $p = xy^2$ we get:

$$p(xy^2x) = xy^2(xy^2x) = (xy^2)^2x \overset{\text{(ii)}}{\longleftrightarrow} (yx)^3y = ((yx)^2y)xy \overset{\text{(i)}}{\longleftrightarrow} (xy^3x)xy = xy^2(yx^2y) = p(yx^2y),$$

which gives 5.2 and hence 5.

6.1. $(xy^2xy, yxy^2x) \in (xy^2, y^2x)^\#$.
We use the following pairs in $(xy^2, y^2x)^\#$:

(i) (xy^2, y^2x),
(ii) (yx^2, x^2y).

Then for $p = y^2$ we get:

$$p(xy^2xy) = y^2(xy^2xy) = y^2(xy^2)xy \overset{\text{(i)}}{\longleftrightarrow} y^2(y^2x)xy = y^3(yx^2)y \overset{\text{(ii)}}{\longleftrightarrow} y^3(x^2y)y = y^3x(xy^2) \overset{\text{(i)}}{\longleftrightarrow} y^3x(y^2x) = y^2(yxy^2x) = p(yxy^2x).$$

6.2. $(xy^2xy, yxy^2x)^\# \ni (xy^2, y^2x)$.
We use the following pairs in $(xy^2xy, yxy^2x)^\#$:

(i) (xy^2xy, yxy^2x),
(ii) $((yx)^2y^2x, xy(yx)^2y) \quad (= \text{(i)}^\alpha, \alpha : x \to y, y \to xy$; cancelled).

Then for $p = yx, q = xy$ we get:

$$p(xy^2)q = yx(xy^2)xy = yx(xy^2xy) \overset{\text{(i)}}{\longleftrightarrow} yx(yxy^2x) = (yx)^2y^2x \overset{\text{(ii)}}{\longleftrightarrow} xy(yx)^2y = (xy^2xy)xy \overset{\text{(i)}}{\longleftrightarrow} (yxy^2x)xy = yx(y^2x)xy = p(y^2x)q,$$

as required.

7.1. $((xy)^2y, y(yx)^2) \in (x^2y^2, y^2x^2)^\#$.
The lefthand pair is the cancelled image of (x^2y^2, y^2x^2) under $\alpha : x \to xy, y \to y$.

7.2. $((xy)^2y, y(yx)^2)^\# \ni (x^2y^2, y^2x^2)$.
We use the following pairs in $((xy)^2y, y(yx)^2)^\#$:

(i) $((xy)^2y, y(yx)^2)$,

(ii) $((yx)^2x, x(xy)^2)$,

(iii) $((xyx)^2y, y(xyx)^2)$ $\quad$ ($=$ (i)$^\alpha$, $\alpha : x \to x$, $y \to xy$; cancelled),

(iv) $((yx^2)^2, (x^2y)^2)$ $\quad$ ($=$ (i)$^\alpha$, $\alpha : x \to xy$, $y \to x$; cancelled).

Then for $p = xyx = q$ we have:

$$\begin{gathered} p(x^2y^2)q = xyx(x^2y^2)xyx = xyx^3(y(yx)^2) \\ \stackrel{\text{(i)}}{\longleftrightarrow} xyx^3((xy)^2y) = xyx^2(x(xy)^2)y \\ \stackrel{\text{(ii)}}{\longleftrightarrow} xyx^2((yx)^2x)y = ((xyx)^2y)x^2y \stackrel{\text{(iii)}}{\longleftrightarrow} (y(xyx)^2)x^2y = yx(yx^2)^2xy \\ \stackrel{\text{(iv)}}{\longleftrightarrow} yx(x^2y)^2xy = yx^2((xyx)^2y) \stackrel{\text{(iii)}}{\longleftrightarrow} yx^2(y(xyx)^2) = y(x(xy)^2)x^2yx \\ \stackrel{\text{(ii)}}{\longleftrightarrow} y((yx)^2x)x^2yx = (y(yx)^2)x^3yx \\ \stackrel{\text{(i)}}{\longleftrightarrow} ((xy)^2y)x^3yx = xyx(y^2x^2)xyx = p(y^2x^2)q, \end{gathered}$$

as required.

So seven pairs of the degree equal to five give known congruences, which were defined by pairs of smaller degrees.

We have five more pairs of degree five to consider. They define not more than three different congruences because we know by (3) that

8. $(xy^2xy, y^3x^2)^\# = (x^2y^3, yxy^2x)^\#$.

Also we can prove that:

9. $(xyxy^2, y^3x^2)^\# = (x^2y^3, y^2xyx)^\#$.

By bar-equivalence reason we show only that $(xyxy^2, y^3x^2) \in (x^2y^3, y^2xyx)^\#$. We take the following pairs in $(x^2y^3, y^2xyx)^\#$:

(i) $(x^2y^3, y(yx)^2)$,

(ii) (y^2x^3, x^2yxy),

(iii) $(yxyx^3, (x^2y)^2)$ $\quad$ ($=$ (i)$^\alpha$, $\alpha : x \to xy$, $y \to x$; cancelled),

(iv) $(y^4x^3, x(xy^2)^2)$ $\quad$ ($=$ (i)$^\alpha$, $\alpha : x \to yy$, $y \to x$),

(v) (xy^2xy^3, y^2xy^3x) $\quad$ ($=$ (i)$^\alpha$, $\alpha : x \to xyy$, $y \to y$; cancelled).

Then for $p = xy^2, q = xyx$ we get

$$\begin{gathered} p(xyxy^2)q = xy^2((xy)^2y)xyx = xy(yx)^2(y(yx)^2) \\ \stackrel{\text{(i)}}{\longleftrightarrow} xy(yx)^2(x^2y^3) = xy(yxyx^3)y^3 \stackrel{\text{(iii)}}{\longleftrightarrow} xy(x^2y)^2y^3 = xyx^2y(x^2y^3)y \\ \stackrel{\text{(i)}}{\longleftrightarrow} xyx^2y(y(yx)^2)y = xy(x^2y^3)(xy)^2 \end{gathered}$$

$$\xleftrightarrow{\text{(i)}} xy(y(yx)^2)(xy)^2 = xy^3xy(x^2yxy) \xleftrightarrow{\text{(ii)}} xy^3xy(y^2x^3) = xy(y^2xy^3x)x^2$$
$$\xleftrightarrow{\text{(v)}} xy(xy^2xy^3)x^2 = xyx(y^2xy^3x)x \xleftrightarrow{\text{(v)}} xyx(xy^2xy^3)x = xy(x(xy^2)^2)yx$$
$$\xleftrightarrow{\text{(iv)}} xy(y^4x^3)yx = xy^2(y^3x^2)xyx = p(y^3x^2)q,$$

which implies $(xyxy^2, y^3x^2)^\# \subseteq (x^2y^3, y^2xyx)^\#$. By taking bar-congruences we get the equality.

So Lemma is proven, there are not more than three different congruences of the type X^2Y^3 and one of the type XY^4, defined by pairs:

$$g := (xyxy^2, y^3x^2), \qquad i := (x^2y^3, y^3x^2),$$
$$h := (xy^2xy, y^3x^2), \qquad j := (xy^4, y^4x).$$

To show that these pairs are not s-equivalent and hence define different congruences it is enough (by Property 2) to show that the corresponding identities define different verbal subgroups V in the two-generator free group F. It is clear that $V(i) = [F^2, F^3]$, $V(j) = [F, F^4]$.

I. We prove now that $V(g) = [[F, F]F, F^5]$. Because of the known equality $[[F, F]F, F^5] = [[F, F], F][F, F^5]$ it is enough to prove that $g^\# = \{f, (xy^5, y^5x)\}^\#$, since $V(f) = [[F, F], F]$. So we need to check:

1^o. $(xyxy^2, y^3x^2) \in \{(xy^2x, yx^2y), (xy^5, y^5x)\}^\#$,

2^o. $(xyxy^2, y^3x^2)^\# \ni (xy^2x, yx^2y)$,

3^o. $(xyxy^2, y^3x^2)^\# \ni (xy^5, y^5x)$.

1^o. We take the following pairs in $\{(xy^2x, yx^2y), (xy^5, y^5x)\}^\#$:

(i) (xy^2x, yx^2y),

(ii) (xy^5, y^5x),

(iii) (yx^5, x^5y),

(iv) $((xy)^2x, yx^3y)$ $(=$ (i)$^\alpha$, $\alpha : x \to x, y \to xy$; cancelled),

(v) $((yx)^2y, xy^3x)$ $(=$ (i)$^\alpha$, $\alpha : x \to y, y \to xy$; cancelled),

(vi) $(x(xy)^2x^2, yx^5y)$ $(=$ (i)$^\alpha$, $\alpha : x \to xx, y \to xy$; cancelled).

Then for $p = x^2,\ q = yx$ we get:

$$p(xyxy^2)q = x^2(xyxy^2)yx = x^3y(xy^3x)$$
$$\xleftrightarrow{\text{(v)}} x^3y((yx)^2y) = x^2(xy^2x)yxy \xleftrightarrow{\text{(i)}} x^2(yx^2y)yxy = x^2yx(xy^2x)y$$
$$\xleftrightarrow{\text{(i)}} x^2yx(yx^2y)y = (x(xy)^2x^2)y^2 \xleftrightarrow{\text{(vi)}} (yx^5y)y^2 = y(x^5y)y^2$$
$$\xleftrightarrow{\text{(iii)}} y(yx^5)y^2 = y(yx^5y)y \xleftrightarrow{\text{(vi)}} y(x(xy)^2x^2)y = yx^2yx(yx^2y)$$

$$\xleftrightarrow{\text{(i)}} yx^2yx(xy^2x) = yx^2(yx^2y)yx \xleftrightarrow{\text{(i)}} yx^2(xy^2x)yx = (yx^3y)(yx)^2$$
$$\xleftrightarrow{\text{(iv)}} ((xy)^2x)(yx)^2 = x((yx)^2y)xyx$$
$$\xleftrightarrow{\text{(v)}} x(xy^3x)xyx = x^2(y^3x^2)yx = p(y^3x^2)q.$$

2^o. To prove that $(xy^2x, yx^2y) \in (xyxy^2, y^3x^2)^\#$ we use the following pairs in $(xyxy^2, y^3x^2)^\#$:

(i) $(xyxy^2, y^3x^2)$,
(ii) $(yxyx^2, x^3y^2)$,
(iii) $((xy^2)^2, y^3xyx)$ $\quad$ (= (i)$^\alpha$, $\alpha : x \to xy$, $y \to y$; cancelled.)

Then for $p = yxy$, $q = xy^2$ we get:

$$p(xy^2x)q = yxy(xy^2x)xy^2 = y(xyxy^2)x^2y^2$$
$$\xleftrightarrow{\text{(i)}} y(y^3x^2)x^2y^2 = y^4x(x^3y^2) \xleftrightarrow{\text{(ii)}} y^4x(yxyx^2) = y(y^3xyx)yx^2$$
$$\xleftrightarrow{\text{(iii)}} y(xy^2)^2yx^2 = yxy^2x(y^3x^2)$$
$$\xleftrightarrow{\text{(i)}} yxy^2x(xyxy^2) = yxy(yx^2y)xy^2 = p(yx^2y)q.$$

3^o. To prove that $(xy^5, y^5x) \in (xyxy^2, y^3x^2)^\#$ we use the following pairs in $(xyxy^2, y^3x^2)^\#$:

(i) $(xyxy^2, y^3x^2)$,
(ii) $((yx^2)^2, x^3yxy)$ $\quad$ (= (i)$^\alpha$, $\alpha : x \to xy$, $y \to x$; cancelled),
(iii) $((xy^2)^2, y^3xyx)$ $\quad$ (= (i)$^\alpha$, $\alpha : x \to xy$, $y \to y$; cancelled),
(iv) $((y^2x)^2x, x^3y^4)$ $\quad$ (= (i)$^\alpha$, $\alpha : x \to yy$, $y \to x$),
(v) (xy^3xy^2, y^3xy^2x) $\quad$ (= (i)$^\alpha$, $\alpha : x \to xyy$, $y \to y$; cancelled),
(vi) $(y^3xy^3x^2, x^3y^6)$ $\quad$ (= (i)$^\alpha$, $\alpha : x \to yyy$, $y \to x$).

Then for $p = x^4$, $q = y$ we get:

$$p(xy^5)q = x^4(xy^5)y = x^2(x^3y^6) \xleftrightarrow{\text{(vi)}} x^2(y^3xy^3x^2) = x(xy^3xy^2)yx^2$$
$$\xleftrightarrow{\text{(v)}} x(y^3xy^2x)yx^2 = (xy^3xy^2)xyx^2 \xleftrightarrow{\text{(v)}} (y^3xy^2x)xyx^2 = y^3xy(yx^2)^2$$
$$\xleftrightarrow{\text{(ii)}} y^3xy(x^3yxy) = (y^3xyx)x(xy)^2 \xleftrightarrow{\text{(iii)}} (xy^2)^2x(xy)^2 = x((y^2x)^2x)yxy$$
$$\xleftrightarrow{\text{(iv)}} x(x^3y^4)yxy = x^4(y^5x)y = p(y^5x)q,$$

and hence $V(g) = [[F, F]F^5, F]$.

II. We prove now that $V(h) = [[F, F]F^4, F]$. Because of the equality $[[F, F]F^4, F] = [[F, F], F][F^4, F]$ it is enough to prove that $h^\# = \{f, j\}^\#$. So we need to check:

1^{oo}. $(xy^2xy, y^3x^2) \in \{(xy^2x, yx^2y), (xy^4, y^4x)\}^\#$,
2^{oo}. $(xy^2xy, y^3x^2)^\# \ni (xy^2x, yx^2y)$,
3^{oo}. $(xy^2xy, y^3x^2)^\# \ni (xy^4, y^4x)$.

1^{oo}. We use the following pairs in $\{(xy^2x, yx^2y),\ (xy^4, y^4x)\}^\#$:

(i) (xy^2x, yx^2y),
(ii) (xy^4, y^4x),
(iii) (yx^4, x^4y),
(iv) $(yx^4y, x^2y^2x^2)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to y, y \to xx)$.

Then for $q = x^2$ we get:

$$\begin{aligned}(xy^2xy)q &= (xy^2xy)x^2 = (xy^2x)yx^2 \xleftrightarrow{\text{(i)}} (yx^2y)yx^2 = y(x^2y^2x^2)\\ &\xleftrightarrow{\text{(iv)}} y(yx^4y) = y^2(x^4y) \xleftrightarrow{\text{(iii)}} y^2(yx^4) = (y^3x^2)x^2 = (y^3x^2)q.\end{aligned}$$

2^{oo}. To prove that $(xy^2x, yx^2y) \in (xy^2xy, y^3x^2)^\#$ we use the following pairs in $(xy^2xy, y^3x^2)^\#$:

(i) (xy^2xy, y^3x^2),
(ii) (yx^2yx, x^3y^2),
(iii) $((xy)^2x^2y, (yx)^3x)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to x, y \to xy$; cancelled),
(iv) $(x(xy)^2x^3y, (yx)^2yx^4)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to xx, y \to xy$; cancelled),
(v) (yx^3yx, x^3yxy) $\quad (= \text{(i)}^\alpha, \alpha : x \to xy, y \to x$; cancelled),
(vi) $((xy)^3x^2y, (yx)^3xyx)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to xxy, y \to xy$; cancelled).

Then for $p = x^4, q = yxy$ we get:

$$\begin{aligned}p(xy^2x)q &= x^4(xy^2x)yxy = x^4(xy^2xy)xy\\ &\xleftrightarrow{\text{(i)}} x^4(y^3x^2)xy = x(x^3y^2)yx^3y \xleftrightarrow{\text{(ii)}} x(yx^2yx)yx^3y = xy(x(xy)^2x^3y)\\ &\xleftrightarrow{\text{(iv)}} xy((yx)^2yx^4) = xy((yx)^3x)x^2 \xleftrightarrow{\text{(iii)}} xy((xy)^2x^2y)x^2 = x((yx)^3xyx)x\\ &\xleftrightarrow{\text{(vi)}} x((xy)^3x^2y)x = x^2y((xy)^2x^2y)x \xleftrightarrow{\text{(iii)}} x^2y((yx)^3x)x = x(xy^2xy)xyx^3\\ &\xleftrightarrow{\text{(i)}} x(y^3x^2)xyx^3 = xy^2(yx^3yx)x^2 \xleftrightarrow{\text{(v)}} xy^2(x^3yxy)x^2 = xy(yx^3yx)yx^2\\ &\xleftrightarrow{\text{(v)}} xy(x^3yxy)yx^2 = x(yx^3yx)y^2x^2 \xleftrightarrow{\text{(v)}} x(x^3yxy)y^2x^2 = x^4yx(y^3x^2)\\ &\xleftrightarrow{\text{(i)}} x^4yx(xy^2xy) = x^4(yx^2y)yxy = p(yx^2y)q.\end{aligned}$$

3^{oo}. To prove that $(xy^4, y^4x) \in (xy^2xy, y^3x^2)^\#$ we use the following pairs in $(xy^2xy, y^3x^2)^\#$:

(i) (xy^2xy, y^3x^2),

(ii) (yx^2yx, x^3y^2),

(iii) $(x^2y^2x^2y, y^3x^4)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to xx, y \to y)$,

(iv) (yx^3yx, x^3yxy) $\quad (= \text{(i)}^\alpha, \alpha : x \to xy, y \to x$; cancelled),

(v) (xy^3xy, y^3xyx) $\quad (= \text{(i)}^\alpha, \alpha : x \to xy, y \to y$; cancelled).

Then for $p = xy^2, q = x^3$ we get:

$$
\begin{aligned}
p(xy^4)q &= xy^2(xy^4)x^3 = (xy^2xy)y^3x^3 \xleftrightarrow{\text{(i)}} (y^3x^2)y^3x^3 = y^3x^2(y^3x^2)x \\
&\xleftrightarrow{\text{(i)}} y^3x^2(xy^2xy)x = y^3(x^3y^2)xyx \xleftrightarrow{\text{(ii)}} y^3(yx^2yx)xyx = y^4x^2(yx^2yx) \\
&\xleftrightarrow{\text{(ii)}} y^4x^2(x^3y^2) = y(y^3x^4)xy^2 \xleftrightarrow{\text{(iii)}} y(x^2y^2x^2y)xy^2 = yx^2y(yx^2yx)y^2 \\
&\xleftrightarrow{\text{(ii)}} yx^2y(x^3y^2)y^2 = (yx^2yx)x^2y^4 \xleftrightarrow{\text{(ii)}} (x^3y^2)x^2y^4 = x(x^2y^2x^2y)y^3 \\
&\xleftrightarrow{\text{(iii)}} x(y^3x^4)y^3 = xy^3x(x^3y^2)y \xleftrightarrow{\text{(ii)}} xy^3x(yx^2yx)y = (xy^3xy)x(xy)^2 \\
&\xleftrightarrow{\text{(v)}} (y^3xyx)x(xy)^2 = y^3x(yx^3yx)y \xleftrightarrow{\text{(iv)}} y^3x(x^3yxy)y = (y^3x^4)yxy^2 \\
&\xleftrightarrow{\text{(iii)}} (x^2y^2x^2y)yxy^2 = x^2y^2x(xy^2xy)y \\
&\xleftrightarrow{\text{(i)}} x^2y^2x(y^3x^2)y = x(xy^2xy)y^2x^2y \xleftrightarrow{\text{(i)}} x(y^3x^2)y^2x^2y = xy^3(x^2y^2x^2y) \\
&\xleftrightarrow{\text{(iii)}} xy^3(y^3x^4) = xy^2(y^4x)x^3 = p(y^4x)q.
\end{aligned}
$$

So we can see that all ten pairs a–j define different verbal subgroups and hence the four pairs of the degree 5 define different congruences of degree five which finishes the proof of Lemma 2.2.

To finish the proof of Theorem 3.2, that is to draw the poset of s-equivalent classes of identities of degree ≤ 5 we need to check implications. The only non-obvious implications are denoted by * on the picture.

1*. The inclusion $(xy^2x, yx^2y) \in (xyxy^2, y^3x^2)^\#$ is checked in the case 2^o above.

2*. The inclusion $(xy^2x, yx^2y) \in (xy^2xy, y^3x^2)^\#$ is checked in the case 2^{oo}.

3*. The inclusion $(xy^4, y^4x) \in (xy^2xy, y^3x^2)^\#$ is checked in the case 3^{oo}.

4*. To prove that $(x^2y^2, y^2x^2) \in (xy^2xy, y^3x^2)^\#$ we use the following pairs in $(xy^2xy, y^3x^2)^\#$:

(i) (xy^2xy, y^3x^2),

(ii) $(x^2y^2x^2y, y^3x^4)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to xx, y \to y)$,

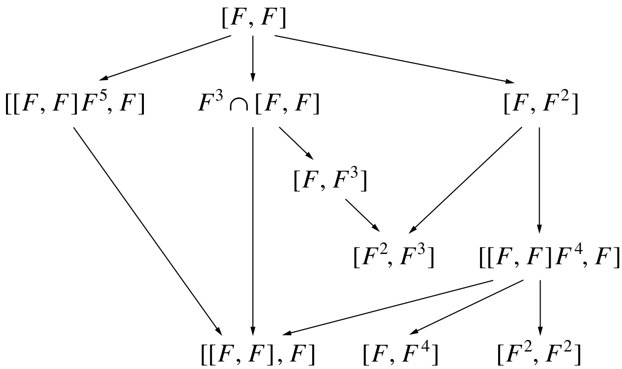
(iii) $(y^2x^2y^2x, x^3y^4)$ $\quad (= \text{(i)}^\alpha, \alpha : x \to yy, y \to x)$,

$$\text{(iv)}\ \ (xy^3xy,\ y^3xyx)\quad (=\text{(i)}^\alpha,\ \alpha : x \to xy,\ y \to y;\ \text{cancelled}),$$
$$\text{(v)}\ \ (yx^3yx,\ x^3yxy)\quad (=\text{(i)}^\alpha,\ \alpha : x \to xy,\ y \to x;\ \text{cancelled}).$$

Then for $p = y^3$, $q = x^3$ we get:

$$\begin{aligned} p(x^2y^2)q &= y^3(x^2y^2)x^3 = y(y^2x^2y^2x)x^2 \overset{\text{(iii)}}{\longleftrightarrow} y(x^3y^4)x^2 = yx^3y(y^3x^2) \\ &\overset{\text{(i)}}{\longleftrightarrow} yx^3y(xy^2xy) = (yx^3yx)y^2xy \overset{\text{(v)}}{\longleftrightarrow} (x^3yxy)y^2xy = x^3y(xy^3xy) \\ &\overset{\text{(iv)}}{\longleftrightarrow} x^3y(y^3xyx) = (x^3y^4)xyx \overset{\text{(iii)}}{\longleftrightarrow} (y^2x^2y^2x)xyx = y^2(x^2y^2x^2y)x \\ &\overset{\text{(ii)}}{\longleftrightarrow} y^2(y^3x^4)x = y^3(y^2x^2)x^3 = p(y^2x^2)q, \end{aligned}$$

which finishes the proof.



The poset of verbal subgroups defined by single two-variable semigroup identities of degree ≤ 5 in a two-generator free group F

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