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CHARACTERISATION OF STRONG SMOOTH STABILITY

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0. Introduction

We begin by recalling Mather's results on C^∞-stability ([5] and [6]). Discussion of stability assumes a topology on $C^\infty(N, P)$; Mather's results use the topology introduced by him in [5] as the Whitney C^∞-topology. Following [8], we will denote it τW^∞.

THEOREM 0.1. *Let N, P be smooth manifolds (without boundary), $f: N \to P$ a proper smooth map. Equip all mapping-spaces with the topology τW^∞. Then the following are equivalent.*

(1) *f is strongly C^∞-stable.*

(2) *f is C^∞-stable.*

(3) *f is locally C^∞-stable.*

(4) *f is infinitesimally stable.*

Precise definitions of the stability notions mentioned above are given in section 4.

Our aim in this paper is to discuss what can be said when f is not proper. Now 0.1 certainly does not hold without some condition on the behaviour of f "at infinity", as Mather was well aware; his counter-examples are to be found in [5] and [6].

We will be concerned with a condition of this kind rather weaker than properness. A smooth map $f : N \to P$ is *quasi-proper* if there is an open neighbourhood V of its discriminant $\Delta(f) = f(\Sigma(f))$ such that $f|_{f^{-1}V} : f^{-1}V \to V$ is proper. (As usual, $\Sigma(f) \subset N$ is the set of points at which the tangent map of f is not of rank $\dim P$. In particular, then, $\Sigma(f) = N$ if $\dim N < \dim P$.) The notion of quasi-properness is introduced in [8]; its interest for us stems from the following result, which is a special case of [8], 4.3.2:

PROPOSITION 0.2. *Let N, P be smooth manifolds (without boundary), $f : N \to P$ a smooth map. If f is strongly C^∞-stable, then f is quasi-proper.*

We give a proof in 6.3.

Thus f quasi-proper is necessary for the statements of 0.1 to be equivalent; it is, however, not sufficient, for we have:

PROPOSITION 0.3.

(1) *f is infinitesimally stable if, and only if, f is locally stable and $f|_{\Sigma(f)}$: $\Sigma(f) \to P$ is proper. In particular, if f is infinitesimally stable, $\Delta(f)$ is closed.*

(2) *Suppose that f is quasi-proper. Then f is infinitesimally stable if, and only if, f is locally stable and $\Delta(f)$ is closed.*

(1) is proved in [6], 5.1. For (2), see 7.1.

PROPOSITION 0.4. *Let N, P be smooth manifolds, $f : N \to P$ a smooth map. The conditions (1)–(4) of 0.1 all hold if, and only if, one of them does, and f is quasi-proper with closed discriminant.*

0.2 and 0.3 yield “only if”; for “if” see [8], 4.3.7.

The condition that the discriminant be closed seems rather unnatural; for example, if f is quasi-proper and locally stable, f is strongly C^r-stable for all $r < \infty$ ([8], 4.3.8). On the other hand, it seems likely that a smooth map with discriminant of positive dimension cannot be C^∞-stable unless the discriminant is closed. Thus many maps which our intuition would suggest are C^∞-stable, are not, according to the definition. It appears, however, that the difficulty lies not so much in our understanding of C^∞-stability as in the topology used.

A rather simple example illustrates this (see also 5.1). Let $f : \mathsf{R}_+ \hookrightarrow \mathsf{R}^2$ be the embedding of the positive half-line into R^2 given by $f(t) = (t, 0)$ for $t \in \mathsf{R}_+ = (0, 1)$. f is quasi-proper and locally stable, but its discriminant (which is its image here) is not closed. Neither is f C^∞-stable. For although any map sufficiently near to f in the topology τW^∞ has as image a non-closed manifold close to the image of f, such images are not necessarily contained in a closed 1-submanifold, whilst the image of f is. The reason for this is that any τW^∞-neighbourhood of f controls only finitely many of the derivatives – say k – of the maps g it contains. Thus such a neighbourhood contains, for example, the graphs of smooth functions on R_+ whose first k derivatives at $t \in \mathsf{R}_+$ all tend to 0 as $t \to 0$, but whose $(k + 1)$-st derivative at $t \in \mathsf{R}_+$ has no limit as $t \to 0$. Such a function does not extend smoothly to $[0, \infty)$; equivalently its graph does not extend smoothly.

It appears that the topology τW^∞ of $C^\infty(\mathsf{R}_+, \mathsf{R}^2)$ is not sufficiently compatible with the smooth structure. We need a topology on the space of smooth maps which gives a generalisation of the following result for C^r-maps:

PROPOSITION 0.5. *Let $r \in \mathbb{N}$ and let N and P be r-manifolds. Let $f \in C^r(N, P)$ and let $U \subset N$ be an open subset. For $h : U \to P$ define $\tilde{h} : N \to P$ as equal to h on U and as f on $N - U$. Then*

$$\mathcal{U}_r = \left\{ h \in C^r(U, P) : \tilde{h} \in C^r(N, P) \right\}$$

is τW^r-open in $C^r(N, P)$, and the map $h \mapsto \tilde{h}$ is τW^r-continuous $\mathcal{U}_r \to C^r(N, P)$.

The τW^r are the Whitney C^r-topologies, implicitly introduced by Whitney in [10], pp. 652–653; their definitions are given in 1.1. τW^∞ is the union of the relative topologies they induce on $C^\infty(N, P)$. Proposition 0.5 is contained in [1], I.4.3.4.4, but see also [8], 3.4.18, for a more detailed proof.

0.5 is definitely false in general if r is replaced by ∞; we will see in 5.2, for example, that

$$\left\{ h \in C^\infty(\mathbb{R}_+, \mathbb{R}) : h \text{ does not extend to a smooth map } \mathbb{R} \to \mathbb{R} \right\}$$

is open and dense in $C^\infty(\mathbb{R}_+, \mathbb{R})$ with respect to τW^∞.

We thus need to change topology. An appropriate topology is what we term *the very strong topology*, and denote τV^∞. This topology was introduced by Cerf ([1], I.4.3.1) in 1962 (it was already implicitly defined by Whitney in [10], p. 654, ll. -10, -9) and was used extensively by Cerf and others (see e.g. the articles in the volume containing [7] and [2]) prior to Mather's work, after which it fell into disuse. This topology gives control "at infinity" of all derivatives. Cerf proved the C^∞-version of 0.5 (see 1.2.2) with respect to the topology τV^∞.

From another result (II.2.2, Théorème 5) in the same paper of Cerf's we also see that our example above is strongly stable with respect to τV^∞. In fact, if $i : N \hookrightarrow P$ is a *proper* embedding (i.e. $i(N)$ is a closed submanifold of P), then the map

$$i^* : C^\infty(P, P) \to C^\infty(N, P);\ g \mapsto g \circ i$$

is a locally trivial topological fibration. This implies that there is a continuous section k from a neighbourhood $\mathcal{U}$ of i in $C^\infty(N, P)$ into the diffeomorphisms of P, such that $i^* k(g) = g$ for $g \in \mathcal{U}$. If i is *non-proper*, as in our example, then i^* is no longer continuous, but we still have a continuous section k as above ([1], II.2.2.2, Corollaire 1).

Reintroduction of the topology τV^∞ is decisive. We will prove:

THEOREM 0.6. *Let $f \in C^\infty(N, P)$. Then the following conditions are equivalent.*

(1) *f is τV^∞-strongly τV^∞-stable.*

(2) *f is quasi-proper and τV^∞-stable.*

(3) *f is quasi-proper and locally stable.*

(4) *f is quasi-proper and τV^∞-quasi-infinitesimally stable.*

Here τV^∞-stable is "C^∞-stable relative to the topology τV^∞" (since we only consider smooth stability, we suppress the prefix C^∞). To make the notation uniform, we should replace C^∞-stable with τW^∞-stable in 0.1 and 0.2, since τW^∞ is the relevant topology there. We will use this notation in the remainder of the paper.

The notion of τV^∞-quasi-infinitesimal stability for f is a weakening of infinitesimal stability, demanding only that smooth vector fields along f near 0 in the τV^∞-sense (and not necessarily every vector field along f, as in infinitesimal stability) can be split into vector fields on N and P (see Definition 4.2). Indeed, (4) is equivalent to a stronger version, in which the splitting is performed τV^∞-continuously.

The statement (1) $\Longrightarrow$ (2) is essentially a τV^∞-version of 0.2, which is presented in 6.3. For the implications (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (1), see 8.1, 7.9, 7.2, 8.9 respectively.

Observe that our theorem also shows that the embedding $f : \mathsf{R}_+ \hookrightarrow \mathsf{R}^2$ discussed above is τV^∞-strongly τV^∞-stable – for it is, as previously observed, quasi-proper and locally stable.

To get this far, we study the topology τV^∞ in some detail. Amongst other things, we will prove (in 3.1.3) that $C^\infty(N, P)$ is a Baire space with respect to τV^∞. Also, in order to be able to use Mather's results from [5] and [6] rather than reproving everything in the τV^∞-context, we will describe (rather technical) methods to detect τV^∞-continuity of maps of mapping-spaces known to be τW^∞-continuous.

1. Mapping-space topologies

In this section we will work in the category of smooth manifolds with boundary, following [5], Ch. 1.; N and P will be such manifolds throughout.

Let P' be a smooth manifold without boundary containing P as a closed subset. Let $x \in N$ and $y \in P$. An r-jet over (x, y) is a class of (smooth) germs $(N, x) \to (P', y)$ with the same derivatives up to order r in x. The space of r-jets $J^r(N, P)$ is a fiber-bundle over $N \times P$ whose typical fiber is the set of r-jets over (x, y); it is clear that the definition is independent of the choice of P'. Observe that over boundary points of P we have included jets that cannot be represented as germs of maps $N \to P$.

1.1. Definitions of the mapping-space topologies. Let $f \in C^\infty(N, P)$, let $k \in \mathbb{N}$, and let D be a closed subset of N. The sets

$$\mathscr{O}_{N,D}(W) = \left\{ g \in C^\infty(N, P) : j^k g(D) \subset W \right\},$$

where W runs through the open neighbourhoods in $J^k(N, P)$ of $j^k f(D)$, form a neighbourhood basis for f in a topology on $C^\infty(N, P)$; we denote this topology τ_D^k.

Since $\tau_D^k \subset \tau_D^{k+1}$, the union over $k \in \mathbb{N}$ of the topologies τ_D^k may be formed and provide the basis of a topology; this will be denoted τ_D^∞.

The union of the topologies τ_K^∞, where K runs through all compact subsets of N, forms a basis for the so called *Thom topology*, τC^∞.

For $k = 0, 1, \ldots, \infty$, the topologies τ_N^k are known as the *Whitney k-topologies*. They will be denoted τW^k in the sequel.

We have another way of describing the Whitney k-topologies for $k \in \mathbb{N}$. Let $\{K_\alpha\}_{\alpha \in \mathbb{N}}$ be a locally finite compact covering of N. For any $k \in \mathbb{N}$ a basis for τW^k is given by sets of the form $\bigcap_\alpha \mathscr{V}_\alpha$, where $\mathscr{V}_\alpha$ is a $\tau_{K_\alpha}^k$-open subset of $C^\infty(N, P)$.

For $k = \infty$ this method does not produce the topology τW^∞ but a stronger topology which we will call the *Very Strong Topology*, τV^∞. To be precise, it has a basis given by $\bigcap_\alpha \mathscr{V}_\alpha$, where $\mathscr{V}_\alpha$ is a $\tau_{K_\alpha}^\infty$-open subset of $C^\infty(N, P)$.

LEMMA 1.1.1 *If $\left\{D_\beta\right\}_{\beta \in \mathbb{N}}$ is a locally finite family of closed subsets of N, then $\bigcap_\beta \mathscr{V}_\beta$ is τV^∞-open if $\mathscr{V}_\beta$ is $\tau_{D_\beta}^\infty$-open for all β. In particular the definition of τV^∞ is independent of the choice of locally finite compact covering of N.*

PROOF. For $f \in \bigcap_\beta \mathscr{V}_\beta$, each $\mathscr{V}_\beta$ contains an f-neighbourhood of the form $\bigcap_{\alpha \in A_\beta} \mathscr{U}_{\beta\alpha}$, where $A_\beta = \left\{\alpha \in \mathbb{N} : K_\alpha \cap D_\beta \neq \emptyset\right\}$ and the $\mathscr{U}_{\beta\alpha}$'s are $\tau_{K_\alpha}^{k_\beta}$-open for some $k_\beta \in \mathbb{N}$. Each α will be a member of only a finite number of A_β's, since $\left\{D_\beta\right\}_{\beta \in \mathbb{N}}$ is locally finite. Hence $\mathscr{U}_\alpha = \bigcap_{\beta, \alpha \in A_\beta} \mathscr{U}_{\beta\alpha}$ is $\tau_{K_\alpha}^\infty$-open for all α, and $\bigcap_{\alpha \in \mathbb{N}} \mathscr{U}_\alpha \subset \mathscr{V}$ is a τV^∞-neighbourhood of f.

Similarly, when k is finite, $\bigcap_\beta \mathscr{V}_\beta$ is τW^k-open if $\mathscr{V}_\beta$ is $\tau_{D_\beta}^k$-open for all β.

1.1.2. The topology $\mathscr{C}^k$ introduced by Cerf in [1], I.4.3.5, coincides for $k \in \mathbb{N}$ with τW^k, while for $k = \infty$ it is the same as the very strong topology, τV^∞. The topology τW^∞ was introduced by Mather in [5], Section 2.

Notice that the topologies τC^∞ and τ_K^∞ for any compact subset K of N have countable base. This is not the case for τW^∞ and τV^∞ when N is non-compact (see e.g. [3], p. 44). For N compact the topologies τC^∞, τW^∞ and τV^∞ coincide.

We will not be much concerned with the Thom topology in this paper. Most (if not all) of the maps of mapping-spaces treated in the following are τC^∞-continuous but, as 6.1 will show, this topology is of little interest in itself in questions of stability of maps with non-compact source. It is included in this exposition to standardize the statements and proofs in Section 3.1.

1.1.3. If $\mathscr{V}$ is τ_D^∞-open and $f, g \in C^\infty(N, P)$ have the same germ at D, then $f \in \mathscr{V} \iff g \in \mathscr{V}$. Indeed, if $f \in \mathscr{V}$ there exists an open subset W of some jet space $J^k(N, P)$, such that $f \in \mathscr{O}_{N,D}(W) \subset \mathscr{V}$. But $j^k g = j^k f$ along D so also $g \in \mathscr{O}_{N,D}(W)$. Hence $f \in \mathscr{V} \Longrightarrow g \in \mathscr{V}$ and vice versa.

1.2. Extensions of smooth maps. Let $\pi_N : J^r(N, P) \to N$ be the natural projection and let M be a submanifold of N of codimension 0. Then, by the above construction, $J^r(M, P)$ is isomorphic as fiber-bundle over M to ${\pi_N}^{-1}M$. Let $s : {\pi_N}^{-1}(M) \to J^r(M, P)$ be the diffeomorphism of the total spaces inducing the isomorphism.

LEMMA 1.2.1.

(1) *Let $M \subset N$ be a closed submanifold of N of codimension* 0. *Then the restriction map*

$$\cdot|_M : C^\infty(N, P) \to C^\infty(M, P); \quad g \mapsto g|_M = g : M \to P$$

is τ_M^∞-continuous.

(2) *Let $f \in C^\infty(N, P)$ and let D be a closed subset of N. Define*

$$\mathscr{F}_D = \left\{ g \in C^\infty(N, P) : g|_{N-D} = f|_{N-D} \right\}.$$

Then

(a) *the relative τW^∞-topology on $\mathscr{F}_D$ coincides with the relative τ_C^∞-topology for any closed subset C of N containing D.*

(b) *If D is compact, the relative τV^∞-topology coincides with the relative τW^∞-topology.*

(3) *In addition to the assumptions in (2), let M be a (not necessarily closed) submanifold of N of codimension* 0, *such that D is contained in the interior of M as subset of N. Define*

$$\mathscr{F}'_D = \left\{ h \in C^\infty(M, P) : h|_{M-D} = f|_{M-D} \right\}.$$

Then restriction to M gives a homeomorphism $\mathscr{F}_D \xrightarrow{\cdot|_M} \mathscr{F}'_D$ in each of the topologies τ_D^∞, τW^∞ or τV^∞.

PROOF. (1): Let $f \in C^\infty(N, P)$, and let $W \subset J^k(M, P)$ be open such that $f|_M \in \mathscr{O}_{M,M}(W)$. Let $W' = s^{-1}W \cup {\pi_N}^{-1}(N - M)$. Then W' is open in $J^k(N, P)$. Clearly $f \in \mathcal{O}_{N,M}(W')$ and if $g \in \mathcal{O}_{N,M}(W')$, then $g|_M \in \mathcal{O}_{M,M}(W)$. This proves the τ_M^∞-continuity.

(2): (a) Let W be an open subset of $J^k(N, P)$ for some k. Then, for $g \in \mathcal{F}_D$, $g \in \mathcal{O}_{N,D}(W)$ if, and only if, $g \in \mathcal{O}_{N,C}(W \cup {\pi_N}^{-1}(N - D))$ for any closed subset C of N containing D. Hence τ_D^∞ coincides with τ_C^∞ and (let C=N) with τW^∞ on $\mathcal{F}_D$.

(b) If D is compact, only finitely many sets from a locally finite cover of N will intersect D non-trivially. If K is a compact subset of N not meeting D then the relative τ_K^∞-topology on $\mathcal{F}_D$ is trivial. Hence any basis set in the relative τ_V^∞-topology on $\mathcal{F}_D$ is a finite intersection of τW^∞-open sets, and the two relative topologies are equal.

(3): Clearly $\cdot|_M$ maps $\mathcal{F}_D$ into $\mathcal{F}'_D$. Let $h \in \mathcal{F}'_D$ and define $\tilde{h} : N \to P$ to be the extension of h by f outside M. Since D lies in the interior of M, any point on the boundary of M in N will be isolated from D, so $\tilde{h}$ equals f in a neighbourhood of such at point. Hence $\tilde{h}$ is smooth, so belongs to $\mathcal{F}_D$, and $\tilde{h}|_M = h$. Thus $\cdot|_M$ is surjective. That it is a homeomorphism is now obvious from the previous arguments.

The inverse of the restriction $\cdot|_M$ in 1.2.1(3) extends any mapping it is applied to smoothly by f. The existence of such an extension is due to the (extremely) good control at infinity of M. As mentioned in the introduction, we cannot replace $\mathcal{F}'_D$ by any τW^∞-neighbourhood of f in $C^\infty(M, P)$ and still expect to extend any map smoothly by f. This, on the contrary, is the main feature of τV^∞ as the following result due to Cerf ([1], I.4.3.4.4) shows.

PROPOSITION 1.2.2. *Let $f \in C^\infty(N, P)$, and let U be an open subset of N. For $h : U \to P$ define $\tilde{h} : N \to P$ as h on U and as f on $N - U$.*

(1) *The set $\mathcal{U} = \{h \in C^\infty(U, P) : \tilde{h}$ is smooth$\}$ is a τV^∞-open neighbourhood of $f|_U$ in $C^\infty(U, P)$.*
(2) *The assignment $h \mapsto \tilde{h}$ is τV^∞-continuous $\mathcal{U} \to C^\infty(N, P)$.*

1.3. Properness. Let X, Y be spaces whose topologies have a countable basis, and let $g : X \to Y$ be a continuous map. We say that $y \in Y$ is an *improper point* for g if $y = \lim_{\alpha\to\infty} g(x_\alpha)$ for some sequence $\{x_\alpha\}_{\alpha\in\mathbb{N}}$ in X without accumulation points. Define the *improper set*, $Z(g)$, of g to be the subset of Y consisting of improper points for g.

DEFINITION 1.3.1 *g is proper if $Z(g) = \emptyset$.*

Properness is equivalent to demanding that the pull-back by g of any compact subset of Y is compact in X. This indicates that properness may be useful when working with the mapping-space topologies. Yet another way of presenting properness of g is by requiring that the extension of g from the one-point compactification of X to the one-point compactification of Y that sends infinity to infinity is continuous.

PROPOSITION 1.3.2.

(1) *Let $g : X \to Y$ be proper. Then $g(C)$ is closed in Y for any closed subset C of X.*

(2) *If X is locally compact, then $Z(g)$ is closed for any continuous $g : X \to Y$.*

PROOF. See [8], 3.2.1 and 3.2.14.

DEFINITION 1.3.3. Let $f \in C^\infty(N, P)$ where N and P are smooth manifolds. f is *quasi-proper* if it satisfies one (and hence both) of the following equivalent conditions

(1) $\Delta(f) \cap Z(f) = \emptyset$.

(2) There exists an open neighbourhood V of $\Delta(f)$ in P, such that $f_V = f : f^{-1}V \to V$ is proper.

The equivalence of (1) and (2) follows since $Z(f)$ is closed by 1.3.2(2). The notion of quasi-properness was introduced in [8], p. 50, where the reader may find a more elaborate treatment of the relation between quasi-properness and strong stability than we give here. It is clear from the definition that properness implies quasi-properness.

1.4. τV^∞-continuity of mapping-space maps. Let $\pi : E \to N$ be a vector bundle and let $\Gamma^\infty(E)$ denote the set of sections of E, i.e. smooth maps $\nu : N \to E$ such that $\pi \circ \nu = \mathrm{id}_N$. $\Gamma^\infty(E)$ has the structure of a $C^\infty(N)$-module by fiberwise multiplication. In this paper a topology on $\Gamma^\infty(E)$ is always induced from a topology on $C^\infty(N, E)$.

PROPOSITION 1.4.1. *The following statements are valid in both of the topologies τW^∞ and τV^∞.*

(1) *The identification*

$$C^\infty(N, P) \times C^\infty(N, Q) = C^\infty(N, P \times Q)$$

is a homeomorphism.

(2) *For manifolds N, P, and Q, the composition mapping*

$$C^\infty_{\mathrm{pr}}(N, P) \times C^\infty(P, Q) \to C^\infty(N, Q); \quad (f, g) \mapsto g \circ f$$

is continuous. Here $C^\infty_{\mathrm{pr}}(N, P)$ is the τW^0-open subset of $C^\infty(N, P)$ consisting of proper maps $N \to P$.

(3) *For any $g \in C^\infty(P, Q)$, the composition mapping*

$$g_* : C^\infty(N, P) \to C^\infty(N, Q); \quad f \mapsto g \circ f$$

is continuous.

(4) *If $E \to N$ and $F \to N$ are vector bundles then the natural map*

$$\Gamma^\infty(E \oplus F) \cong \Gamma^\infty(E) \times \Gamma^\infty(F)$$

is a homeomorphism. If E is isomorphic as vector bundle to F, then also $\Gamma^\infty(E) \cong \Gamma^\infty(F)$ is a homeomorphism.

(5) *$C^\infty(N)$ is a topological ring, and if $E \to N$ is a vector bundle then $\Gamma^\infty(E)$ is a topological $C^\infty(N)$-module.*

(6) *The inversion mapping*

$$\mathrm{Inv} : \mathrm{Diff}^\infty(N, P) \to \mathrm{Diff}^\infty(P, N); \quad f \mapsto f^{-1}$$

is a homeomorphism. Thus $\mathrm{Diff}^\infty(N)$ *is a topological group by (1).*

(7) *The composition map*

$$\Gamma^\infty(TN) \times C^\infty(N, P) \to C^\infty(N, TP); \quad (\xi, f) \mapsto df \circ \xi$$

is continuous (where df is the tangent map $TN \to TP$ of f).

Proof. All the above results are stated and proved in [5], Section 2, in τW^∞. (2) and (6) can be found in [1] as I.4.5.4 Proposition 5 and II.1.4.2 Proposition 2. with respect to τV^∞. We thus consider (1), (3)–(5) and (7) in the topology τV^∞.

(1): Immediate.

(3): Consider the continuous map

$$\Phi : C^\infty(N, P) \to C^\infty(N, N) \times C^\infty(N, P); \quad f \mapsto (\mathrm{id}_N, f).$$

Under the identification from (1) Φ maps into the open subset of proper maps in $C^\infty(N, N \times P)$. Hence the composition

$$\begin{aligned} &C^\infty(N, P) \xrightarrow{\Phi} C^\infty_{\mathrm{pr}}(N, N \times P) \xrightarrow{(\mathrm{id}_N \times g)_*} \\ &\qquad C^\infty(N, N \times Q) = C^\infty(N, N) \times C^\infty(N, Q) \xrightarrow{\text{projection}} C^\infty(N, Q) \end{aligned}$$

is continuous by (2) and coincides with g_*.

(4): Clearly the total space $E \oplus F$ may be embedded as a closed submanifold of the space $E \times F$. If i is the embedding then one sees immediately that

$$i_* : C^\infty(N, E \oplus F) \hookrightarrow C^\infty(N, E \times F)$$

is a homeomorphism onto its image. The image under i_* of $\Gamma^\infty(E \oplus F) \subset C^\infty(N, E \oplus F)$ is contained in $\Gamma^\infty(E) \times \Gamma^\infty(F) \subset C^\infty(N, E) \times C^\infty(N, F) = C^\infty(N, E \times F)$, hence the bijective natural map

$$\Gamma^\infty(E \oplus F) \to \Gamma^\infty(E) \times \Gamma^\infty(F)$$

is a homeomorphism. The last statement of (4) is clear.

(5): Let $\mathsf{R}_N \to N$ denote the trivial bundle over N with fiber R. The multiplicative and additive structures on E are given by the compositions

$$C^\infty(N) \times \Gamma^\infty(E) = \Gamma^\infty(\mathsf{R}_N \times E) \xrightarrow{m_*} \Gamma^\infty(E)$$
$$\Gamma^\infty(E) \times \Gamma^\infty(E) = \Gamma^\infty(E \oplus E) \xrightarrow{a_*} \Gamma^\infty(E)$$

where $m : \mathsf{R}_N \oplus E \to E$ and $a : E \oplus E \to E$ are fiberwise multiplication and addition respectively. Continuity follows from (3).

(7): The map $C^\infty(N, P) \to C^\infty(TN, TP)$; $f \mapsto df$ is easily seen to be continuous. Since $\Gamma^\infty(TN)$ is contained in $C^\infty_{\mathrm{pr}}(N, TN)$, continuity of the map in (7) follows from (2).

2. A technical result

2.1. Setup. Let N, P, Q, R_1, and R_2 be manifolds without boundary, and let $f \in C^\infty_{\mathrm{pr}}(N, P)$ be a proper smooth map. Let $\mathscr{M} \subset C^\infty(N, Q)$, $\mathscr{H} \subset C^\infty(N, R_1)$ and $\mathscr{K} \subset C^\infty(P, R_2)$ be sets of mappings. m_0 is a fixed element of $\mathscr{M}$.

We shall assume that $\mathscr{H}$ and $\mathscr{K}$ are groups with neutral elements h_0 and k_0, and that

$$\cdot_{\mathscr{H}} : \mathscr{M} \times \mathscr{H} \to \mathscr{M} \qquad \text{and} \qquad \cdot_{\mathscr{K}} : \mathscr{K} \times \mathscr{M} \to \mathscr{M}$$

are a right-action by $\mathscr{H}$ and a left-action by $\mathscr{K}$ on $\mathscr{M}$. (In the following we will use $\cdot$ in place of $\cdot_{\mathscr{H}}$ and $\cdot_{\mathscr{K}}$ whenever it is clear from the context which action is refered to.) We require that the group structures and the actions are continuous with respect to both τW^∞ and τV^∞.

The spaces and the actions are subject to the following conditions.

(1) Given a closed subset C and an open subset U of N with $C \subset U$, there exists a τ^∞_C-neighbourhood $\mathscr{U}$ of h_0 in $\mathscr{H}$ such that

$$m_1|_U = m_2|_U \Longrightarrow (m_1 \cdot h)|_C = (m_2 \cdot h)|_C$$

for all $h \in \mathscr{U}$ and $m_1, m_2 \in \mathscr{M}$.

(2) Given an open subset U of N we have

$$h_1|_U = h_2|_U \Longrightarrow (m \cdot h_1)|_U = (m \cdot h_2)|_U$$

for all $h_1, h_2 \in \mathscr{H}$ and $m \in \mathscr{M}$.

(3) Given an open subset U of N we have

$$m_1|_U = m_2|_U \Longrightarrow (k \cdot m_1)|_U = k \cdot m_2|_U$$

for all $k \in \mathscr{K}$ and $m_1, m_2 \in \mathscr{M}$.

(4) Given an open subset U of N we have

$$(k_1 \circ f)|_U = (k_2 \circ f)|_U \Longrightarrow (k_1 \cdot m_0)|_U = (k_2 \cdot m_0)|_U$$

for all $k_1, k_2 \in \mathscr{K}$.

(5) The actions of $\mathscr{H}$ and $\mathscr{K}$ commute, i.e.

$$k \cdot (m \cdot h) = (k \cdot m) \cdot h$$

for all $h \in \mathscr{H}$, $k \in \mathscr{K}$ and $m \in \mathscr{M}$.

(6) There are τV^∞-neighbourhoods $\mathscr{U} \subset C^\infty(N, Q)$ of m_0, $\mathscr{V} \subset C^\infty(N, R_1)$ of h_0 and $\mathscr{W} \subset C^\infty(P, R_2)$ of k_0, such that

$$\mathscr{M} \cap \mathscr{U} = \left\{m \in \mathscr{U} : \forall x \in N \; \exists\, m' \in \mathscr{M} \text{ such that } \widehat{m_x} = \widehat{m'_x}\right\},$$
$$\mathscr{H} \cap \mathscr{V} = \left\{h \in \mathscr{V} : \forall x \in N \; \exists\, h' \in \mathscr{H} \text{ such that } \widehat{h_x} = \widehat{h'_x}\right\} \quad \text{and}$$
$$\mathscr{K} \cap \mathscr{W} = \left\{k \in \mathscr{W} : \forall y \in P \; \exists\, k' \in \mathscr{K} \text{ such that } \widehat{k_x} = \widehat{k'_x}\right\}$$

(where $\widehat{m_x}$ denotes the germ of m at x etc.)

(7) There is a τV^∞-neighbourhood $\mathscr{W}$ of m_0 in $\mathscr{M}$ with the following property: Given a closed subset C and open subset U of N with $C \subset U$ there exists a τV^∞-continuous map

$$r = r_{m_0, C, U} : \mathscr{W} \to \mathscr{M}$$

satisfying

$$r(m)(x) = \begin{cases} m(x), & \text{if } x \in C \\ m_0(x), & \text{if } x \in N - U \text{ or } m(x) = m_0(x) \\ \text{anything}, & \text{otherwise} \end{cases}$$

for all $m \in \mathscr{W}$ and $x \in N$.

We will make use of two particular cases that fit the description above.

LEMMA 2.2. *Suppose* $f \in C^\infty_{\mathrm{pr}}(N, P)$. *Let* $\mathscr{M} = C^\infty_{\mathrm{pr}}(N, P)$, $\mathscr{H} = \mathrm{Diff}^\infty(N)$, $\mathscr{K} = \mathrm{Diff}^\infty(P)$, *and let* $m_0 = f$, $h_0 = \mathrm{id}_N$, $k_0 = \mathrm{id}_P$. $\mathscr{H}$ *and* $\mathscr{K}$ *are groups* *with composition of maps as group-law. Define actions of* $\mathscr{H}$ *and* $\mathscr{K}$ *on* $\mathscr{M}$ *in the following way*

$$m \cdot_{\mathscr{H}} h = m \circ h \qquad k \cdot_{\mathscr{K}} m = k \circ m.$$

Then the conditions (1)–(7) of 2.1 are satisfied.

LEMMA 2.3. *Suppose* $f \in C^\infty_{\mathrm{pr}}(N, P)$. *Let* $\mathscr{M} = \Theta(f)$, $\mathscr{H} = \Theta(N)$, $\mathscr{K} = \Theta(P)$ *and let* m_0, h_0 *and* k_0 *be their* 0*-sections. Define actions of* $\mathscr{H}$ *and* $\mathscr{K}$ *on* $\mathscr{M}$ *in the following way*

$$m \cdot_{\mathscr{H}} h = m - tf(h) \qquad k \cdot_{\mathscr{K}} m = m - \omega f(k).$$

Then the conditions (1)–(7) of 2.1 are satisfied.

For the definitions of $\Theta(f)$, $\Theta(N)$, $\Theta(P)$, tf and ωf, see 4.1. Before proving these two lemmas it is convenient to introduce some notation.

2.3.1. Let $\rho \in C^\infty(N)$. For a closed subset C of N we shall write $C \prec \rho$ when ρ takes values in $I = [0, 1]$ and is equal to 1 on a neighbourhood of C. When $U \subset N$ is open then $\rho \prec U$ means that ρ takes values in I and has support (= closure in N of $\{x : \rho(x) \neq 0\}$) contained in U.

2.3.2. Let $f \in C^\infty(N, P)$, and recall from [5], Section 2, Lemma 3, that we can find a neighbourhood U_Δ of the diagonal Δ in $P \times P$ and a smooth map $\gamma : \mathscr{U}_\Delta \times [0, 1] \to P$, such that

$$\gamma(x, y, 0) = x, \quad \gamma(x, y, 1) = y, \quad \text{and} \quad \gamma(x, x, t) = x$$

for all $(x, y) \in U_\Delta$ and $t \in [0, 1]$. Define

$$\mathscr{W} = \left\{g \in C^\infty(N, P) : (f, g)(N) \subset U_\Delta\right\}.$$

Then $\mathscr{W}$ is τW^0-open. Given a closed subset C and an open subset U of N with $C \subset U$, choose $\rho \in C^\infty(N)$ such that $C \prec \rho \prec U$, and define a map $r = r_{f,C,U} : \mathscr{W} \to C^\infty(N, P)$ by

$$r_{f,C,U}(g) = \gamma \circ (f, g, \rho), \quad g \in \mathscr{W}.$$

Then $r(g)$ coincides with g on the neighbourhood $\rho^{-1}(1)$ of C and coincides with f on $\rho^{-1}(0)$, which is contained in $N - C$. Notice that r is both τW^∞- and τV^∞-continuous by 1.4.1(1) and (3).

PROOF OF 2.2. The group structures and the actions are continuous in both topologies by 1.4.1, so we only need to demonstrate (1)–(7). In (1) we take $\mathscr{U} = \{h \in \mathrm{Diff}^\infty(N) : h(C) \subset U\}$ as our τ^∞_C-neighbourhood. Conditions (2)–(5)

are obvious from the definitions. Since $\mathscr{H} = \mathrm{Diff}^\infty(N)$ and $\mathscr{K} = \mathrm{Diff}^\infty(P)$ are τW^1-open subsets of $C^\infty(N, N)$ and $C^\infty(P, P)$ respectively, then (6) follows by taking $\mathscr{U} = C^\infty_{\mathrm{pr}}(N, P) = \mathscr{M}$, $\mathscr{V} = \mathscr{H}$ and $\mathscr{W} = \mathscr{K}$. In (7), define $\mathscr{W}$ and $r_{m_0,C,U} : \mathscr{W} \to \mathscr{M} = C^\infty_{\mathrm{pr}}(N, P)$ as in 2.3.2.

Proof of 2.3. The group structures and the actions are continuous in both topologies by 1.4.1(5), so we need only to consider the conditions (1)–(7).

In (1) we may take $\mathscr{U} = \mathscr{M}$, while (2)–(5) follow directly from the definitions. In (6), take $\mathscr{U} = C^\infty(N, f^*TP)$, $\mathscr{V} = C^\infty(N, TN)$ and $\mathscr{W} = C^\infty(P, TP)$.

In (7) we simply take $\mathscr{W}$ to be the whole of $\mathscr{M} = \Theta(f)$. Given C and U, choose $\rho \in C^\infty(N)$ such that $C \prec \rho \prec U$, and let

$$r_{m_0,C,U}(m) = \rho m$$

for $m \in \mathscr{W}$. It follows from 1.4.1(1) and (3) that $r_{m_0,C,U}$ is τV^∞-continuous.

The key to passing from Mathers τW^∞-stability results to the corresponding results with respect to τV^∞ is the following proposition.

Proposition 2.4. *In the setup of 2.1, assume that for any open $V \subset P$ and compact $K \subset V$ there is a τW^∞-neighbourhood $\mathscr{U}'$ of m_0 in*

$$\mathscr{M}_{m_0, f^{-1}K} = \left\{ m \in \mathscr{M} : m|_{N - f^{-1}K} = m_0|_{N - f^{-1}K} \right\}$$

and a τW^∞-continuous map $(h', k') : \mathscr{U}' \to \mathscr{H} \times \mathscr{K}$, such that for any $m \in \mathscr{U}'$,

$$\left\{ x \in N : h'(m)(x) \neq h_0(x) \right\} \subset f^{-1}V$$
$$\left\{ y \in P : k'(m)(y) \neq k_0(y) \right\} \subset V$$

and $k'(m) \cdot m \cdot h'(m) = m_0$.

Let D be a closed subset of P. Then there is a τV^∞-neighbourhood $\mathscr{U}$ of m_0 in $\mathscr{M}_{m_0, f^{-1}D}$ and a τV^∞-continuous map $(h, k) : \mathscr{U} \to \mathscr{H} \times \mathscr{K}$ such that

$$k(m) \cdot m \cdot h(m) = m_0$$

for all $m \in \mathscr{U}$.

Proof. (During the proof (1)–(7) will refer to the conditions in 2.1)

The statement is trivial for $D = \emptyset$, so we may assume that D is non-empty. We claim that we may write $D = D_1 \cup \ldots \cup D_l$, where each D_i ($i = 1, \ldots, l$) is a locally finite (relative to P) union of disjoint compact subsets of P. Indeed, from [4], Theorem V 1, we deduce that we can cover P by at most $\dim P + 1$ open subsets, all of whose connected components are relatively compact. By shrinking these open subsets a bit and taking their closures, it follows that P may be covered by at most $\dim P + 1$ closed subsets each of which is a locally finite union of disjoint compact subsets of P. Intersecting these with D we obtain $D_1, \ldots, D_l$ as claimed, and furthermore we see that l may be chosen to be at most $\dim P + 1$.

In the following we let l be the minimal number for which D can be written as above. We shall proceed by induction on $l \geq 1$.

$l = 1$: Write $D = D_1 = \coprod_{\alpha \in \mathsf{N}} K_\alpha$ as the locally finite union of disjoint compact subsets $K_\alpha \subset P$. For every $\alpha \in \mathsf{N}$, choose a relatively compact open subset V_α of P with $K_\alpha \subset V_\alpha$, so that also $\coprod_\alpha \overline{V}_\alpha$ is a locally finite disjoint union of compact subsets of P. We write $U_\alpha = f^{-1} V_\alpha$ and notice that U_α is relatively compact by properness of f.

Fix $\alpha \in \mathsf{N}$. According to the hypothesis of the proposition, there is a τW^∞-continuous mapping $(h'_\alpha, k'_\alpha) : \mathscr{U}'_\alpha \to \mathscr{H} \times \mathscr{K}$ defined on a τW^∞-open neighbourhood $\mathscr{U}'_\alpha$ of m_0 in $\mathscr{M}_{m_0, f^{-1} K_\alpha}$ with

$$\left\{x \in N : h'_\alpha(m)(x) \neq h_0(x)\right\} \subset U_\alpha,$$
$$\left\{y \in P : k'_\alpha(m)(y) \neq k_0(y)\right\} \subset V_\alpha$$

and $k'_\alpha(m) \cdot m \cdot h'_\alpha(m) = m_0$ for all $m \in \mathscr{U}'_\alpha$.

It is clear that a $\tau^\infty_{f^{-1}K_\alpha}$-continuous mapping $r_\alpha : \mathscr{M}_{m_0, f^{-1}D} \to C^\infty(N, Q)$, is defined by the assignment

$$r_\alpha(m)(x) = \begin{cases} m(x), & \text{if } x \in f^{-1} K_\alpha \\ m_0(x), & \text{if } x \in N - f^{-1} K_\alpha \end{cases}$$

for $m \in \mathscr{M}_{m_0, f^{-1}D}$. By definition $r_\alpha(m)$ is independent of m outside the compact subset $f^{-1} K_\alpha$ of N, and hence the topology $\tau^\infty_{f^{-1}K_\alpha}$ coincides with both τW^∞ and τV^∞ on the image of r_α. Every $m \in \mathscr{M}_{m_0, f^{-1}D}$ is equal to elements of $\mathscr{M}$ locally on N, hence by (6) and continuity of r_α we may find a $\tau^\infty_{f^{-1}K_\alpha}$-neighbourhood $\mathscr{U}_\alpha$ of m_0 in $\mathscr{M}_{m_0, f^{-1}D}$, such that $r_\alpha(\mathscr{U}_\alpha) \subset \mathscr{M}$. Shrinking $\mathscr{U}_\alpha$ we may in addition assume that $r_\alpha(\mathscr{U}_\alpha) \subset \mathscr{U}'_\alpha$, so that

$$(h_\alpha, k_\alpha) = (h'_\alpha, k'_\alpha) \circ r_\alpha : \mathscr{U}_\alpha \to \mathscr{H} \times \mathscr{K}$$

is well-defined and $(\tau^\infty_{f^{-1}K_\alpha}, \tau W^\infty)$-continuous.

Now let $\mathscr{U} = \bigcap_{\alpha \in \mathsf{N}} \mathscr{U}_\alpha$. Then $\mathscr{U}$ is τV^∞-open by 1.1.1, and we define a map $(h, k) : \mathscr{U} \to C^\infty(N, R_1) \times C^\infty(P, R_2)$ by the assignments

$$h(m)(x) = \begin{cases} h_\alpha(m)(x), & \text{if } x \in f^{-1} V_\alpha \text{ for some (unique) } \alpha \in \mathsf{N}. \\ h_0(x), & \text{otherwise} \end{cases}$$

and

$$k(m)(y) = \begin{cases} k_\alpha(m)(y), & \text{if } y \in V_\alpha \text{ for some (unique) } \alpha \in \mathsf{N}. \\ k_0(y), & \text{otherwise} \end{cases}$$

for $m \in \mathscr{U}$, $x \in N$ and $y \in P$. By the choice of the V_α, $h(m)$ and $k(m)$ are smooth maps for all $m \in \mathscr{U}$. It is also clear that h and k are $(\tau^\infty_{f^{-1}K_\alpha}, \tau^\infty_{\overline{U}_\alpha})$- and $(\tau^\infty_{f^{-1}K_\alpha}, \tau^\infty_{\overline{V}_\alpha})$-continuous respectively for all α, and hence τV^∞-continuous. Since $h(m)$ is equal to members of $\mathscr{H}$ locally on N, τV^∞-continuity and condition (6) makes it possible to shrink $\mathscr{U}$ so that $h(\mathscr{U}) \subset \mathscr{H}$. Likewise we may assume that $k(\mathscr{U}) \subset \mathscr{K}$.

Let us once more fix an $\alpha \in \mathsf{N}$ and choose an open subset $\tilde{V}_\alpha$ of P containing $\overline{V}_\alpha$ and intersecting V_β trivially for $\beta \neq \alpha$. Applying (1) together with $(\tau^\infty_{f^{-1}K_\alpha}, \tau^\infty_{\overline{U}_\alpha})$-continuity of h, we see that for m in some $\tau^\infty_{f^{-1}K_\alpha}$-neighbourhood of m_0 in $\mathscr{U} \subset \mathscr{M}_{m_0, f^{-1}D}$,

$$m_1|_{f^{-1}\tilde{V}_\alpha} = m_2|_{f^{-1}\tilde{V}_\alpha} \Longrightarrow m_1 \cdot h(m)|_{\overline{U}_\alpha} = m_2 \cdot h(m)|_{\overline{U}_\alpha}$$

for arbitrary $m_1, m_2 \in \mathscr{M}$. If we restrict to $m_1, m_2 \in \mathscr{M}_{m_0, f^{-1}D}$ then m_1 and m_2 agree on $f^{-1}\tilde{V}_\alpha$ exactly when they agree on $f^{-1}K_\alpha$, so we arrive at

$$(1') \qquad m_1|_{f^{-1}K_\alpha} = m_2|_{f^{-1}K_\alpha} \Longrightarrow m_1 \cdot h(m)|_{U_\alpha} = m_2 \cdot h(m)|_{U_\alpha}.$$

Replacing $\mathscr{U}$ with the intersection of all these $\tau^\infty_{f^{-1}K_\alpha}$-neighbourhoods for α running through N we still have that $\mathscr{U}$ is τV^∞-open, that

$$(h, k) : \mathscr{U} \to \mathscr{H} \times \mathscr{K}$$

is τV^∞-continuous, and that (1′) holds for all $\alpha \in \mathsf{N}$, $m \in \mathscr{U}$ and $m_1, m_2 \in \mathscr{M}_{m_0, f^{-1}D}$. We only need to check that $k(m) \cdot m \cdot h(m) = m_0$ for all $m \in \mathscr{U}$.

Let $m \in \mathscr{U}$ and $\alpha \in \mathsf{N}$. Since $h(m)|_{U_\alpha} = h_\alpha(m)|_{U_\alpha} = h'(r_\alpha(m))|_{U_\alpha}$ and $m|_{f^{-1}K_\alpha} = r_\alpha(m)|_{f^{-1}K_\alpha}$, then

$$m \cdot h(m)|_{U_\alpha} \overset{(1')}{=} r_\alpha(m) \cdot h(m)|_{U_\alpha} \overset{(2)}{=} r_\alpha(m) \cdot h'_\alpha(r_\alpha(m))|_{U_\alpha}.$$

Hence, by the assumptions on h'_α and k'_α,

$$\begin{aligned} k(m) \cdot m \cdot h(m)|_{U_\alpha} &\overset{(3)}{=} k(m) \cdot r_\alpha(m) \cdot h'_\alpha(r_\alpha(m))|_{U_\alpha} \\ &= k'(r_\alpha(m)) \cdot r_\alpha(m) \cdot h'_\alpha(r_\alpha(m))|_{U_\alpha} = m_0|_{U_\alpha}. \end{aligned}$$

On the other hand, $h(m)|_{N-f^{-1}D} = h_0|_{N-f^{-1}D}$, so

$$h(m) \cdot m|_{N-f^{-1}D} \overset{(2)}{=} h_0 \cdot m|_{N-f^{-1}D} = m|_{N-f^{-1}D} = m_0|_{N-f^{-1}D},$$

and, since $k(m) \circ f|_{N-f^{-1}D} = k_0 \circ f|_{N-f^{-1}D}$, then

$$k(m) \cdot m|_{N-f^{-1}D} \cdot h(m) \overset{(3)}{=} k(m) \cdot m_0|_{N-f^{-1}D} \overset{(4)}{=} k_0 \cdot m_0|_{N-f^{-1}D} = m_0|_{N-f^{-1}D}.$$

We have thus proved that $k(m) \cdot h(m) \cdot m$ coincides with m_0 on $(N - f^{-1}D) \cup \bigcup_\alpha U_\alpha = N$, as desired.

Induction step. Assume $l > 1$ and write $D = D^\flat \cup D_l$ where $D^\flat = D_1 \cup \ldots \cup D_{l-1}$. Choose a closed $D^\sharp$ such that D_l is contained in the interior $\mathring{D}^\sharp$ of $D^\sharp$, and $D^\sharp$ can be written as a locally finite union of disjoint compact subsets of P. Then by the induction hypothesis applied twice with $D^\flat$ and $D^\sharp$ in place of D we obtain τV^∞-continuous mappings

$$(h^\flat, k^\flat) : \mathscr{U}^\flat \to \mathscr{H} \times \mathscr{K}$$

with $\mathscr{U}^\flat$ a τV^∞-neighbourhood of m_0 in $\mathscr{M}_{m_0, f^{-1}D^\flat}$, and

$$(h^\sharp, k^\sharp) : \mathscr{U}^\sharp \to \mathscr{H} \times \mathscr{K}$$

with $\mathscr{U}^\sharp$ a τV^∞-neighbourhood of m_0 in $\mathscr{M}_{m_0, f^{-1}D^\sharp}$, such that

$$k^\flat(m^\flat) \cdot m^\flat \cdot h^\flat(m^\flat) = m_0 \qquad \text{and} \qquad k^\sharp(m^\sharp) \cdot m^\sharp \cdot h^\sharp(m^\sharp) = m_0$$

for all $m^\flat \in \mathscr{U}^\flat$ and $m^\sharp \in \mathscr{U}^\sharp$.

Choose a closed subset C of N with

$$\overline{N - f^{-1}D^\sharp} \subset \mathring{C} \subset C \subset N - f^{-1}D_l,$$

and let $r = r_{m_0, C, N - f^{-1}D_l} : \mathscr{W} \to C^\infty(N, Q)$ be the τV^∞-continuous map from (7). Let $m \in \mathscr{M}_{m_0, f^{-1}D} \cap \mathscr{W}$. Then $r(m)(x) = m_0(x)$ if either $x \in N - (N - f^{-1}D_l) = f^{-1}D_l$ or $m(x) = m_0(x)$, which is the case when $x \in N - f^{-1}D$. In particular, $r(m)$ agrees with m on $N - f^{-1}D^\flat \subset f^{-1}D_l \cup (N - f^{-1}D)$, so $r(m) \in \mathscr{M}_{m_0, N - f^{-1}D^\flat}$. Consequently, we may choose a τV^∞-neighbourhood $\mathscr{U} \subset \mathscr{M}_{m_0, f^{-1}D} \cap \mathscr{W}$ of m_0 on which $(h^\flat, k^\flat) \circ r$ is well-defined. Using (1) and the continuity of $h^\flat$ we may in addition assume that

$$m_1|_{\mathring{C}} = m_2|_{\mathring{C}} \Longrightarrow m_1 \cdot (h^\flat \circ r)(m)|_{\overline{N - f^{-1}D^\sharp}} = m_2 \cdot (h^\flat \circ r)(m)|_{\overline{N - f^{-1}D^\sharp}}$$

for all $m \in \mathscr{U}$ and $m_1, m_2 \in \mathscr{M}$. Since, for $m \in \mathscr{U}$, $m|_C = r(m)|_C$, this implies that

$$m \cdot (h^\flat \circ r)(m)|_{N - f^{-1}D^\sharp} = r(m) \cdot (h^\flat \circ r)(m)|_{N - f^{-1}D^\sharp},$$

whence

$$k^\flat(r(m)) \cdot r(m) \cdot h^\flat(r(m))|_{N - f^{-1}D^\sharp}$$
$$\overset{(3)}{=} k^\flat(r(m)) \cdot r(m) \cdot h^\flat(r(m))|_{N - f^{-1}D^\sharp} = m_0|_{N - f^{-1}D^\sharp}.$$

This shows us that $m^\sharp := k^\flat(r(m)) \cdot r(m) \cdot h^\flat(r(m))$ belongs to $\mathscr{M}_{m_0, f^{-1}D^\sharp}$ and depends τV^∞-continuously on m. By shrinking $\mathscr{U}$ we may thus assume that

$$h(m) = h^\flat(r(m))h^\sharp(m^\sharp)$$

and

$$k(m) = k^\sharp(m^\sharp)k^\flat(r(m))$$

are defined for $m \in \mathscr{U}$ and depend τV^∞-continuously on m. By condition (5) the map $(h, k) : \mathscr{U} \to \mathscr{H} \times \mathscr{K}$ has the desired properties.

3. The transversality theorem

In this section we prove that $C^\infty(N, P)$ is a Baire space with respect to τV^∞ and go on to prove the Multi-jet Transversality Theorem, also with respect to τV^∞. These results were proved by Mather with respect to τW^∞ ([6], 3.1 and 3.3). More-or-less these results are also stated by Morlet [7]; however, it is not quite clear which topology he intended to use – he remarks on p. 4-03, l. -6, that his topology $\mathscr{C}^\infty$ is that defined by Douady [2], which is τV^∞; on the other hand, his somewhat unclear definitions suggest that it is the trivial topology…

3.1. The Baire space property. We shall first prove that $C^\infty(N, P)$ is a Baire space, i.e. that any countable intersection of open dense subsets is dense, in the very strong topology. In the following N and P will be manifolds without boundary. A few lemmas of a somewhat technical character are extracted from the proof and stated separately to make the arguments more transparent.

In the following d_k will be a choice of a metric on $J^k(N, P)$ that turns $J^k(N, P)$ into a complete metric space. To simplify the notation we define, for $f \in C^\infty(N, P)$ and $k \in \mathbb{N}$,

$$\mathscr{B}^k(f) = \Big\{ g \in C^\infty(N, P) : d_s(j^s g(x), j^s f(x)) < 2^{-k} \\ \text{for all } x \in N, 0 \le s \le k \Big\}.$$

$\mathscr{B}^k(f)$ is easily seen to be a τW^k- and hence a τV^∞- neighbourhood of f.

LEMMA 3.1.1. *Let $f_n \in C^\infty(N, P)$ for $n \in \mathbb{N}$ and assume $f_{n+1} \in \mathscr{B}^n(f_n)$. There exists an $f \in C^\infty(N, P)$ such that, for all $k \in \mathbb{N}$, $j^k f$ is the d_k-uniform limit of the sequence $\{j^k f_n\}_{n \in \mathbb{N}}$ in $C^\infty(N, J^k(N, P))$.*

In particular, $f = \tau C^\infty - \lim_{n\to\infty} f_n$.

PROOF. Let $k \in \mathbb{N}$. For $s_2 > s_1 \geq k$ and $x \in N$ we have

$$\begin{aligned} d_k(j^k f_{s_2}(x), j^k f_{s_1}(x)) &\leq \sum_{i=s_1}^{s_2-1} d_k(j^k f_{i+1}(x), j^k f_i(x)) \\ &\leq \sum_{i=s_1}^{s_2-1} 2^{-i} \qquad \text{(since } f_{i+1} \in \mathscr{B}^i(f_i) \text{ for } i \geq k) \\ &\leq 2^{-s_1+1}. \end{aligned}$$

This shows that $\left\{j^k f_n(x)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $J^k(N, P)$, uniformly in x. By completeness of $J^0(N, P)$ in the metric d_0, the sequence $\left\{j^0 f_n(x)\right\}_{n\in\mathbb{N}}$ has a uniform limit which is the graph of the pointwise limit-map, f, of the sequence of f_n's.

Completeness of the metric d_k implies that, in suitable local coordinates on P, the sequences of partial derivatives of coordinate functions of $\{f_n\}_{n\in\mathbb{N}}$ of degree at most k have uniform local limits. By classical results these limits are the corresponding partial derivatives of the coordinate functions of f. Since this holds for all k, f is smooth.

For the last statement it suffices to observe that for f to be a limit of a sequence $\{f_n\}_{n\in\mathbb{N}}$ in the topology τC^∞ it is only required that the sequence $\left\{j^k f_n\right\}_n$ converge d_k- uniformly to $j^k f$ on any compact subset of N for all k.

LEMMA 3.1.2. *Let $f \in C^\infty(N, P)$, and let $\mathscr{W}$ be a τV^∞-open neighbourhood of f. Then there exists a smaller τV^∞-open neighbourhood $\mathscr{V}$ of f whose τC^∞-closure is contained in $\mathscr{W}$.*

PROOF. Let $\{K_\alpha\}_{\alpha\in\mathbb{N}}$ be a locally finite compact covering of N and let, for all α, $\mathscr{W}_\alpha$ be a $\tau^\infty_{K_\alpha}$-open neighbourhood of f such that $\bigcap_\alpha \mathscr{W}_\alpha \subset \mathscr{W}$. It is clear that we can find a closed neighbourhood $\mathscr{C}_\alpha \subset C^\infty(N, P)$ of f in the topology $\tau^\infty_{K_\alpha}$ such that $\mathscr{C}_\alpha \subset \mathscr{W}_\alpha$. $\mathscr{C}_\alpha$ is in particular τC^∞-closed so $\bigcap_\alpha \mathscr{C}_\alpha$ will be a τC^∞-closed τV^∞-neighbourhood of f contained in $\mathscr{W}$. This proves the lemma.

PROPOSITION 3.1.3. *Let $\mathscr{F}$ be a τC^∞-closed subset of $C^\infty(N, P)$. Then $\mathscr{F}$ is a Baire space with respect to τV^∞.*

In particular, $C^\infty(N, P)$ itself is a Baire space with respect to τV^∞.

PROOF. For all $n \in \mathbb{N}$, let $\mathscr{U}_n \subset C^\infty(N, P)$ be a τV^∞-open set such that $\mathscr{F} \cap \mathscr{U}_n$ is τV^∞-dense in $\mathscr{F}$ and let $\mathscr{U}$ be a τV^∞-open subset of $C^\infty(N, P)$ that intersects $\mathscr{F}$ non-trivially. We must show that $\mathscr{U} \cap \bigcap_n \mathscr{U}_n \cap \mathscr{F} \neq \emptyset$.

Choose $f_1 \in \mathscr{U} \cap \mathscr{U}_1 \cap \mathscr{F}$ (this is possible because $\mathscr{U}$ is open and $\mathscr{U}_1$ is dense in $\mathscr{F}$) and choose by 3.1.2 a τV^∞-open neighbourhood $\mathscr{V}_1$ of f_1 whose τC^∞-closure is contained in $\mathscr{U} \cap \mathscr{U}_1$.

Choose inductively $f_{n+1} \in \overbrace{\mathscr{B}^n(f_n) \cap \mathscr{V}_n \cap \mathscr{F}}^{\text{Open in } \mathscr{F}} \cap \overbrace{\mathscr{U}_{n+1} \cap \mathscr{F}}^{\text{Dense in } \mathscr{F}}$ and (by 3.1.2) a τV^∞-open neighbourhood $\mathscr{V}_{n+1}$ of f_{n+1} whose τV^∞-closure is contained in $\mathscr{B}^n(f_n) \cap \mathscr{V}_n \cap \mathscr{U}_{n+1}$.

Since $f_{n+1} \in \mathscr{B}^n(f_n)$ for all $n \in \mathbb{N}$, Lemma 3.1.1 guarantees the existence of $f = \tau C^\infty - \lim_{n\to\infty} f_n$ which is contained in the τC^∞-closed set $\mathscr{F}$. Since $f_m \in \mathscr{V}_m \subset \mathscr{V}_n$ for all $m, n \in \mathbb{N}$ with $m > n$, f must be contained in the τC^∞-closure of $\mathscr{V}_n$ which in turn is chosen to be a subset of $\mathscr{U}_n$. We therefore only need to check that $f \in \mathscr{U}$. But this follows immediately since the τC^∞-closure of $\mathscr{V}_1$ is contained in $\mathscr{U}$.

3.2. Transversality. As in [6], Ch. 1, we define N^r, respectively $J^k(N, P)^r$, to be the product of N, respectively $J^k(N, P)$, with itself r times and define $\pi_N{}^r : J^k(N, P)^r \to N^r$ to be the r fold Cartesian product of the source map $\pi_N : J^k(N, P) \to N$ with itself. Let $N^{(r)} = \{(x_1, \dots, x_r) \in N^r : x_i \neq x_j,\ 1 \leq i < j \leq r\}$ and define ${}_rJ^k(N, P) = (\pi_N{}^r)^{-1}(N^{(r)})$, the *$r$-fold k-jet bundle*. ${}_rJ^k(N, P)$ is easily seen to be a manifold with the differentiable structure inherited from $J^k(N, P)^r$. For $f \in C^\infty(N, P)$ the k-jet map ${}_rj^k f : N^{(r)} \to {}_rJ^k(N, P)$ is also just the restriction to $N^{(r)}$ of the r-fold Cartesian product of $j^k f$ with itself. We have

PROPOSITION 3.2.1. *Let W be a submanifold of ${}_rJ^k(N, P)$, C a compact subset of W. Then $\mathscr{T}_{W,C} = \{f \in C^\infty(N, P) : {}_rj^k f \pitchfork W \text{ on } C\}$ is τV^∞-open in $C^\infty(N, P)$.*

PROOF. The result follows from [3], Lemma II.4.14, where $\mathscr{T}_{W,C}$ is shown to be even τW^∞-open.

Also the following theorem is valid both with respect to τW^∞ and τV^∞. The τW^∞-case can be found in [3], II.4.13. The proof in the τV^∞-case will build on that result.

MULTI-JET TRANSVERSALITY THEOREM 3.2.2. *Let W be a submanifold of ${}_rJ^k(N, P)$. Define*

$$\mathscr{T}_W = \{f \in C^\infty(N, P) : {}_rj^k f \pitchfork W\}.$$

Then $\mathscr{T}_W$ is residual in $C^\infty(N, P)$, i.e. a countable intersection of open and dense subsets of $C^\infty(N, P)$.

In particular, $\mathscr{T}_W$ is dense in $C^\infty(N, P)$.

PROOF OF 3.2.2. FOR τV^∞. Let $\{C_\alpha\}_{\alpha\in\mathbb{N}}$ be a family of compact sets covering W. Then $\mathscr{T}_W = \bigcap_\alpha \mathscr{T}_{W,C_\alpha}$, and to show that $\mathscr{T}_W$ is τV^∞-residual it suffices to show that each $\mathscr{T}_{W,C_\alpha}$ is open and dense with respect to τV^∞. Openness follows from 3.2.1, so we are left with proving that $\mathcal{T}_{W,C}$ is τV^∞-dense in $C^\infty(N, P)$ whenever C is a compact subset of W. To that end we let $\mathcal{U}$ be an arbitrary non-void τV^∞-open subset of $C^\infty(N, P)$ and aim to prove that $\mathcal{U} \cap \mathcal{T}_{W,C} \neq \emptyset$.

$\pi_N{}^r$ is continuous so $\pi_N{}^r(C)$ is a compact subset of N^r and, with $\mathrm{pr}_i : N^r \to N$ the projection onto the i'th coordinate, we define a compact subset of N by $K = \bigcup_{i=1,\dots,r} \mathrm{pr}_i(\pi_N{}^r(C))$.

Choose $f \in \mathcal{U}$, and let U be an open subset containing K with compact closure $\overline{U}$ in N. Define $\mathcal{W}$ and $r = r_{f,K,U} : \mathcal{W} \to C^\infty(N, P)$ as in 2.3.2. The image of r is contained in the set $\{g \in C^\infty(N, P) : f|_{N-U} = g|_{N-U}\}$ on which τW^∞ and τV^∞ coincide, by 1.2.1(2) and the compactness of $\overline{U}$. In particular, $\mathcal{U} \cap \mathrm{Im}\, r$ is τW^∞-open in $\mathrm{Im}\, r$, and the set $\mathcal{U}' = r^{-1}\mathcal{U} \subset \mathcal{W}$ is τW^∞-open in $C^\infty(N, P)$ by τW^∞-continuity of r. By the τW^∞-version of the transversality theorem there exists a $g \in \mathcal{U}' \cap \mathcal{T}_W$, and since g is equal to $h = r(g)$ on a neighbourhood of K, their multi-jet maps coincide in a neighbourhood of $K^r \cap N^{(r)}$ in $N^{(r)}$. Therefore ${}_r j^k h$ intersects W transversely at every point $x \in K^r \cap N^{(r)}$. On the other hand, if $x \in N^{(r)} - K^r$ then ${}_r j^k h(x) \notin C$ by definition of K. Consequently ${}_r j^k h$ is transverse to W on C, whence h is an element in $\mathcal{U} \cap \mathcal{T}_{W,C}$, proving that this set is non-void.

We have shown that $\mathcal{T}_W$ is τV^∞-residual, so density now follows because $C^\infty(N, P)$ is a τV^∞-Baire space (by 3.1.3).

4. Definitions of stability

In this section N and P are smooth manifolds without boundary.

4.1. For $f \in C^\infty(N, P)$ we let $\Sigma(f)$ be the set of *critical points* of f, i.e. the points in N where the differential-map $df_x : T_xN \to T_{f(x)}P$ is not surjective. $\Delta(f) = f(\Sigma(f))$ is the *discriminant* of f.

In the following we are going to study three particular vector bundles over N and P, namely the tangent bundles $TN \to N$ and $TP \to P$, and for $f \in C^\infty(N, P)$ the bundle $f^*TP \to N$ whose fiber over $x \in N$ is $T_{f(x)}P$. The set of sections $\Theta(N) = \Gamma^\infty(TN)$ in TN are called *vector fields on* N. $\Theta(N)$ will be considered a $C^\infty(N)$-module via fiberwise multiplication and addition. The sections in f^*TP are called the *vector fields along* f; they form the $C^\infty(N)$-module $\Theta(f) = \Gamma^\infty(f^*TP)$. Finally we have the $C^\infty(P)$-module $\Theta(P) = \Gamma^\infty(TP)$ of *vector fields on* P.

For $f \in C^\infty(N, P)$ there are maps $tf : \Theta(N) \to \Theta(f)$ given by $tf(\xi)(x) = df_x(\xi(x))$ for $\xi \in \Theta(N)$, $x \in N$, and $\omega f : \Theta(P) \to \Theta(f)$ given by $\omega f(\eta)(x) = (\eta \circ f)(x)$ for $\eta \in \Theta(P)$, $x \in N$. It follows from 1.4.1(7) that tf is τV^∞- and τW^∞-continuous, while this (in general) is the case for ωf only when f is proper (cf. 1.4.1(2)).

4.2. Definition. Let $f \in C^\infty(N, P)$. f is said to be

(1) *stable* if, for any g in an open neighbourhood $\mathscr{U}$ of f in a prescribed topology on $C^\infty(N, P)$, g is equivalent to f in the sense that there there exist $h_g \in \mathrm{Diff}^\infty(N)$ and $k_g \in \mathrm{Diff}^\infty(P)$ such that $g = k_g \circ f \circ h_g$.

(2) *strongly stable* if f is stable and the assignments $g \mapsto h_g$ and $g \mapsto k_g$ can be chosen to be continuous with respect to given topologies on the spaces of diffeomorphisms in such a way that $h_f = \mathrm{id}_N$ and $k_f = \mathrm{id}_P$.

(3) *quasi-infinitesimally stable* if, for any v in an open neighbourhood $\mathscr{U}$ of the 0-section in some prescribed topology on $\Theta(f)$, there exist $\xi_v \in \Theta(N)$ and $\eta_v \in \Theta(P)$ such that $v = tf(\xi_v) + \omega f(\eta_v)$. (i.e. $tf\Theta(N) + \omega f\Theta(P)$ is a neighbourhood of 0 in $\Theta(f)$.)

(4) *strongly quasi-infinitesimally stable* if f is quasi-infinitesimally stable and the assignments $v \mapsto \xi_v$ and $v \mapsto \eta_v$ can be chosen to be continuous in given topologies on $\Theta(N)$ and $\Theta(P)$ in such a way that $\xi_0 = 0$ and $\eta_0 = 0$.

(5) *infinitesimally stable* if f is quasi-infinitesimally stable (in any topology) with $\mathscr{U} = \Theta(f)$.

It would also make sense to define *strong infinitesimal stablility* in the obvious way, but we do not need this notion here (however, see remark 7.10).

As we already mentioned in the introduction, the notion of stability in (1) is often called C^∞-stability.

4.3. Finally we have the notion of *local stability* of f. It is defined by either one of the following equivalent conditions

(1) For all $y \in P$ and $S \subset f^{-1}(y)$ with at most $p+1$ elements ($p = \dim P$),

$$\widehat{\Theta(f)_S} = tf\widehat{\Theta(N)_S} + \omega f\widehat{\Theta(P)_y}.$$

(2) For all $y \in P$, $S \subset f^{-1}(y)$ with at most $p + 1$ elements, and $k \geq p$,

$$\widehat{\Theta(f)_S} = tf\widehat{\Theta(N)_S} + \omega f\widehat{\Theta(P)_y} + (f^*\mathfrak{m}_y + \mathfrak{m}_S^{k+1})\widehat{\Theta(f)_S}.$$

(3) The r-fold jet-map ${}_rj^k f$ of f is transverse to every orbit in ${}_rJ^k(N, P)$ for $r > p$ and $k \geq p$.

(4) The r-fold jet-map ${}_rj^k f$ of f is transverse to every contact-class in ${}_rJ^k(N, P)$ for $r > p$ and $k \geq p$.

For equivalence of the statements see [6], 4.1 and 4.4.

5. An example

Let $f : \mathsf{R}_+ \hookrightarrow \mathsf{R}^2$ be the embedding of the positive real axis into the plane given by $x \mapsto (x, 0)$, $x \in \mathsf{R}_+$.

PROPOSITION 5.1. *We have:*

(1) *f is quasi-proper.*

(2) *f is locally stable.*

(3) *f is both strongly stable and strongly quasi-infinitesimally stable with respect to τV^∞.*

(4) *f is neither quasi-infinitesimally stable, nor stable with respect to τW^∞.*

PROOF. (1): f is nowhere submersive so $\Sigma(f) = \mathsf{R}_+$ and $\Delta(f) = Im(f)$. Clearly $Z(f) = \{0\}$ avoids $\Delta(f)$.

(2): By the proof of [3], 2.3, any immersion is locally stable.

(3): This follows from II.2.2.2, Corollaire 1, in [1] and is also a consequence of 0.6, since f is quasi-proper and locally stable.

(4): Under the natural identifications $\Theta(f) = C^\infty(\mathsf{R}_+, \mathsf{R}^2)$, $\Theta(N) = C^\infty(\mathsf{R}_+, \mathsf{R})$ and $\Theta(P) = C^\infty(\mathsf{R}^2, \mathsf{R}^2)$, any vector field $\nu = (\nu_1, \nu_2) \in \Theta(f)$ that splits as $\nu = tf\xi + \omega f\eta$ for $\xi \in \Theta(N)$ and $\eta = (\eta_1, \eta_2) \in \Theta(P)$, satisfies $\nu_2 = \eta_2 \circ f$, and hence $\nu_2 : \mathsf{R}_+ \to \mathsf{R}$ extends smoothly to a map $\mathsf{R} \to \mathsf{R}$. By 5.2 below any τW^∞-neighbourhood of $\Theta(f)$ will contain a vector field $\nu = (0, \nu_2)$ where ν_2 cannot be extended to R, hence ν cannot split. Thus f is not τW^∞-quasi-infinitesimally stable.

To prove the lack of τW^∞-stability, let $f_1 : \mathsf{R}_+ \hookrightarrow \mathsf{R}$ be the inclusion, and suppose that $f = (f_1, 0)$ is stable with respect to τW^∞. Then for any $f_2 \in C^\infty(\mathsf{R}_+, \mathsf{R})$ sufficiently close to 0 there exist diffeomorphisms $k = (k_1, k_2)$ of R^2 and h of R_+, such that $k \circ f = (f_1, f_2) \circ h$, i.e. such that

$$k_1(x, 0) = h(x) \qquad \text{and} \qquad k_2(x, 0) = f_2 \circ h(x)$$

for all $x \in \mathsf{R}_+$. Let (y_1, y_2) and z be the canonical coordinates on R^2 and R_+ respectively. Then $\frac{\partial k_1}{\partial y_1}(x, 0) = \frac{dh}{dz}(x)$ and hence

$$\frac{\partial k_2}{\partial y_1}(x, 0) = \frac{df_2}{dz}(h(x))\frac{dh}{dz}(x) = \frac{df_2}{dz}(h(x))\frac{\partial k_1}{\partial y_1}(x, 0)$$

for all $x \in \mathsf{R}_+$. If f_2 is close enough to 0 (with respect to τW^1) then $\frac{df_2}{dz}$ is uniformly bounded and hence $\frac{\partial k_2}{\partial y_1}(0, 0) = 0$ if $\frac{\partial k_1}{\partial y_1}(0, 0) = 0$. But k is a germ of a diffeomorphism at $(0, 0)$ so $\frac{\partial k_2}{\partial y_1}(0, 0)$ and $\frac{\partial k_1}{\partial y_1}(0, 0)$ cannot vanish simultaneously, whence $\frac{\partial k_1}{\partial y_1}(0, 0) \neq 0$. As a consequence, the map $\mathsf{R}_+ \to \mathsf{R}$;

$x \mapsto h(x) = k_1(x, 0)$ extends to a diffeomorphism in a neighbourhood of 0, and hence $f_2 = k_2 \circ (h^{-1}, 0) : \mathsf{R}_+ \to \mathsf{R}$ extends smoothly to a neighbourhood of 0, and so to a smooth map $\mathsf{R} \to \mathsf{R}$. Since f_2 was an arbitrary map close enough to 0 this contradicts Proposition 5.2 below.

PROPOSITION 5.2. *Let*

$$\mathscr{D} = \left\{h \in C^\infty(\mathsf{R}_+, \mathsf{R}) : h \text{ does not extend to a smooth map } \mathsf{R} \to \mathsf{R}\right\}.$$

Then $\mathscr{D}$ is open and dense in $C^\infty(\mathsf{R}_+, \mathsf{R})$ in τW^∞, but open and closed in τV^∞.

PROOF. Define an equivalence relation $\sim$ on $C^\infty(\mathsf{R}_+, \mathsf{R})$ by

$$h_1 \sim h_2 \iff h_1 - h_2 \text{ extends smoothly by 0 to a map } \mathsf{R} \to \mathsf{R}.$$

Each equivalence class is τV^∞-open by 1.2.2, and since both $\mathscr{D}$ and its complement in $C^\infty(\mathsf{R}_+, \mathsf{R})$ is a union of such classes, then $\mathscr{D}$ is both open and closed in τV^∞.

By [9], $\mathscr{D}$ coincides with the set

$$\left\{h \in C^\infty(\mathsf{R}_+, \mathsf{R}) : h^{(r)}(x) \text{ does not converge as } x \to 0_+ \text{ for some } r \in \mathsf{N}\right\}.$$

It is then clear that if $h \in C^\infty(\mathsf{R}_+, \mathsf{R})$, such that $h^{(r)}(x)$ does not converge as $x \to 0_+$ for some r, then $h + (\mathscr{U} \cap C^\infty(\mathsf{R}_+, \mathsf{R})) \subset \mathscr{D}$ if $\mathscr{U}$ is the τW^r-open subset from 0.5 of functions in $C^r(\mathsf{R}_+, \mathsf{R})$ that extend by 0 to a C^r-map $\mathsf{R} \to \mathsf{R}$. Hence $\mathscr{D}$ is indeed τW^∞-open.

To prove density of $\mathscr{D}$ it will suffice to show that any τW^∞-neighbourhood of 0 in $C^\infty(\mathsf{R}_+, \mathsf{R})$ contains a map from $\mathscr{D}$. If r runs through N and $a = \{a_n\}_{n \in \mathsf{N}}$ runs through the set of sequences of positive reals, then the sets

$$\mathscr{V}_{r,a} = \Big\{f \in C^\infty(\mathsf{R}_+, \mathsf{R}) : \\ f^{(k)}\left(\left[\tfrac{1}{n+1}, \tfrac{1}{n}\right] \cup [n, n+1]\right) \subset (-a_n, a_n)\, \forall k < r,\ n \in \mathsf{N}\Big\}$$

form a neighbourhood basis for 0 in τW^∞. Fix such r, a. For each $n \geq 1$, choose a smooth $\phi_n : \mathsf{R}_+ \to \mathsf{R}$ with support in $(\frac{1}{n+1}, \frac{1}{n})$ such that $|\phi_n^{(k)}| < a_n$ for all $k < r$ and $n \in \mathsf{N}$, but with $\sup_{\mathsf{R}_+} |\phi_n^{(r)}| \geq 1$. Then $\sum_{n>1} \phi_n \subset \mathscr{V}_{r,a} \cap \mathscr{D}$.

6. Quasi-properness is necessary for strong stability

The primary task in this section is to show that quasi-properness is a necessary condition for strong stability in each of our two topologies τW^∞ and τV^∞. First of all, however, we present a statement essentially saying, that in order to have any stable maps at all, the topology on $C^\infty(N, P)$ should provide 'good control at infinity'.

PROPOSITION 6.1. *Let N be non-compact and equip $C^\infty(N, P)$ with the topology τC^∞ or any weaker topology. Then $C^\infty(N, P)$ contains no stable maps.*

PROOF. Assume $\mathscr{U}$ is an open subset of $C^\infty(N, P)$ in which all maps are equivalent. Since $\mathscr{U}$ does not restrict the behaviour of its members outside some compact subset K of N, there will be a $g \in \mathscr{U}$ which is single-valued on some open subset of $N - K$ (which is open and non-empty by the non-compactness of N). Thus by equivalence every map in $\mathscr{U}$ is single-valued on some open subset of N, but this cannot be true for every element of an open subset of $C^\infty(N, P)$, and we have a contradiction.

The following shows that even a very mild variant of strong stability implies quasi-properness. Compare with [8], 4.3.2.

PROPOSITION 6.2. *Let $\mathscr{U}$ be a τV^∞-neighbourhood of f in $C^\infty(N, P)$, and let $h : \mathscr{U} \to \mathrm{Diff}^1(N)$ be a $(\tau V^\infty, \tau W^0)$-continuous map such that*

$$\forall g \in \mathscr{U}\ \exists k \in \mathrm{Diff}^1(P) : g = k \circ f \circ h(g).$$

Then f is quasi-proper.

PROOF. We may clearly assume that $h(f) = \mathrm{id}_N$. By definition of quasi-properness (1.3.3) we must show that $Z(f) \cap \Delta(f) = \emptyset$. Suppose that this is not the case. Then there is a non-accumulating sequence $\{x_\alpha\}_{\alpha\in\mathbb{N}}$ in N such that $\{f(x_\alpha)\}_{\alpha\in\mathbb{N}}$ has a limit y in $\Delta(f)$.

By shrinking $\mathscr{U}$ if necessary, we may assume that

$$\lim_{\alpha\to\infty} g(x_\alpha) = \lim_{\alpha\to\infty} f(x_\alpha) = y$$

for all $g \in \mathscr{U}$. Since h is $(\tau V^\infty, \tau W^0)$-continuous, so is $g \mapsto f \circ h(g)$ by the τW^0-version of 1.4.1(2) (see [5], Section 2, Proposition 1), and thus we may also assume that

$$\lim_{\alpha\to\infty} (f \circ h(g))(x_\alpha) = \lim_{\alpha\to\infty} (f \circ h(f))(x_\alpha) = \lim_{\alpha\to\infty} f(x_\alpha) = y.$$

This shows that

$$y = \lim_{\alpha\to\infty} g(x_\alpha) = \lim_{\alpha\to\infty} (k \circ f \circ h(g))(x_\alpha) = k(y)$$

for any $k \in \mathrm{Diff}^1(P)$, with $g = k \circ f \circ h(g)$ as in the statement of the proposition. Since $k(\Delta(f)) = \Delta(g)$, y must belong to $\Delta(g)$, for any $g \in \mathscr{U}$. But by Sard's theorem ([3], II.1.6 and II.1.12), $\Delta(f)$ is a set of measure zero in P, so we can define an arbitrarily small perturbation of f which does not contain y in its discriminant. This contradicts the existence of y; hence f is quasi-proper.

COROLLARY 6.3. *If f is strongly stable with respect to τW^∞ or τV^∞, then f is quasi-proper.*

7. Local stability and quasi-infinitesimal stability.

In this section N and P will be manifolds without boundary.

PROPOSITION 7.1. *Let $f \in C^\infty(N, P)$ be quasi-proper. Then f is infinitesimally stable if, and only if, it is locally stable and $\Delta(f)$ is closed.*

PROOF. By 0.3(1), we only need to prove that for a quasi-proper map, $f|_{\Sigma(f)}$ is proper if and only if $\Delta(f)$ is closed. The "only if"-part follows immediately from 1.3.2(1). For the "if"-part, assume that $y \in P$ is an improper value for $f|_{\Sigma(f)} : \Sigma(f) \to P$. By definition, y is the limit of some (non-accumulating) sequence in $\Sigma(f)$ and hence y is in the closure of $f(\Sigma(f)) = \Delta(f)$. If $\Delta(f)$ is closed, then this implies that $y \in \Delta(f) \cap Z(f)$ in contradiction with quasi-properness of f.

PROPOSITION 7.2. *Let f be quasi-infinitesimally stable with respect to either τW^∞ or τV^∞ on $\Theta(f)$. Then f is locally stable.*

PROOF. Let $y \in P$ and let $S \subset f^{-1}(y)$ be a finite subset. Choose $\rho \in C^\infty(N)$ such that $S \prec \rho$ and ρ has compact support. Let $\nu \in \Theta(f)$. Then we have a continuous map $t \in \mathsf{R} \mapsto t\rho\nu \in \Theta(f)$, since the relative τW^∞- and τV^∞-topologies on the image coincides with the relative $\tau^\infty_{\operatorname{supp}(\rho)}$-topology (1.2.1(2)), which has a countable basis. The assumption of quasi-infinitesimal stability implies that $t\rho\nu \in tf\Theta(N) + \omega f\Theta(P)$ when $t > 0$ is sufficiently close to 0. By R-linearity, $\rho\nu \in tf\Theta(N) + \omega f\Theta(P)$, and taking germs we get

$$\nu\widehat{_S} = \rho\nu\widehat{_S} \in tf\Theta(N)\widehat{_S} + \omega f\Theta(P)\widehat{_y}.$$

Since any element of $\Theta(f)\widehat{_S}$ can be represented as $\nu\widehat{_S}$ for some ν, this proves condition (1) in the definition of local stability.

PROPOSITION 7.3. *Let $f \in C^\infty_{\mathrm{pr}}(N, P)$ be locally stable. Then f is strongly quasi-infinitesimally stable with respect to τW^∞ on $\Theta(f)$.*

To prove 7.3 we will apply Mathers 'continuous' version of the preparation theorem. We will start by recalling the notation of [5], Section 6.

7.4. Let $\mathscr{X}$ be a topological space with topology $\tau_{\mathscr{X}}$, and let $x_0 \in \mathscr{X}$ be a fixed element. For manifolds N and P let $C^{\mathscr{X}}(N, P)$ denote the set of germs $f : (\mathscr{X}, x_0) \to C^\infty_{\mathrm{pr}}(N, P)$ of $(\tau_{\mathscr{X}}, \tau W^\infty)$-continuous mappings. We will identify $C^\infty_{\mathrm{pr}}(N, P)$ with the subset of $C^{\mathscr{X}}(N, P)$ of constant germs.

$C^{\mathscr{X}}(N)$ will be the unital commutative ring of germs $(\mathscr{X}, x_0) \to C^\infty(N)$ of $(\tau_V, \tau W^\infty)$-continuous maps. The ring-structure is inherited from the ring-structure of $C^\infty(N)$. Identify the unital commutative ring $C^\infty(N)$ with the subset of $C^{\mathscr{X}}(N)$ of constant map-germs. The inherited ring-structure from $C^{\mathscr{X}}(N)$ clearly coincides with the usual structure.

We have a ring-homomorphism $\mathrm{ev}_N : C^{\mathscr{X}}(N) \to C^\infty(N)$ that assigns to $\lambda \in C^{\mathscr{X}}(N)$ the constant germ with value $\lambda(x_0)$.

For $f \in C^{\mathscr{X}}(N, P)$, let $f^* : C^{\mathscr{X}}(P) \to C^{\mathscr{X}}(N)$ be the unital ring-homomorphism that takes $\lambda \in C^{\mathscr{X}}(P)$ into the germ at x_0 given by $x \mapsto \lambda(x) \circ f(x)$. This is continuous by the τW^∞-version of 1.4.1(2) and therefore defines an element of $C^{\mathscr{X}}(N)$. $\mathrm{ev}(f^*)$ will be the unital ring-homomorphism $C^\infty(P) \to C^\infty(N)$ sending the constant germ $\lambda \in C^\infty(P)$ to the constant germ $x \mapsto \lambda(x_0) \circ f(x_0)$. Clearly $\mathrm{ev}(f^*) \circ \mathrm{ev}_P = \mathrm{ev}_N \circ f^*$.

Now assume that A is a module over $C^{\mathscr{X}}(P)$ and let $\mathscr{M}_P = \ker \mathrm{ev}_P$, that is, $\mathscr{M}_P$ consists of all germs $\lambda \in C^{\mathscr{X}}(P)$ with $\lambda(x_0) = 0$. We let $\mathrm{ev}\, A = A/\mathscr{M}_P A$ as a $C^\infty(P)$-module. If C is a $C^{\mathscr{X}}(N)$-module and $f \in C^{\mathscr{X}}(N, P)$ we will say that $\alpha : A \to C$ is a $C^{\mathscr{X}}(P)$-module homomorphism over $f^* : C^{\mathscr{X}}(P) \to C^{\mathscr{X}}(N)$ if α is additive and for any $\lambda \in C^{\mathscr{X}}(P)$ and $a \in A$ we have $\alpha(\lambda a) = f^*\lambda\, \alpha(a)$. Such an α induces in the obvious way a $C^\infty(P)$-module homomorphism $\mathrm{ev}\, \alpha : \mathrm{ev}\, A \to \mathrm{ev}\, C$ over $\mathrm{ev}(f^*)$.

We now present one of the statements in [5], Proposition 6.1, as follows:

LEMMA 7.5. *Let $f \in C^{\mathscr{X}}(N, P)$, let A be a finitely generated $C^{\mathscr{X}}(P)$-module, and let B and C be $C^{\mathscr{X}}(N)$-modules with C finitely generated. Let $\alpha : A \to C$ be a $C^{\mathscr{X}}(P)$-module homomorphism over f^* and $\beta : B \to C$ a $C^{\mathscr{X}}(N)$-module homomorphism, such that* $\mathrm{ev}\, \alpha + \mathrm{ev}\, \beta : \mathrm{ev}\, A \oplus \mathrm{ev}\, B \to \mathrm{ev}\, C$ *is surjective. Then $\alpha + \beta : A \oplus B \to C$ is surjective.*

We are going to use 7.5 where the modules are of a certain type, and before starting the proof of 7.3 we will take a closer look at these modules.

7.6. Let $\pi : E \to P$ be a vector bundle over P, and let $g : (\mathscr{X}, x_0) \to C^\infty(N, P)$ be a germ at x_0 of a $(\tau_X, \tau W^\infty)$-continuous map. (We do not demand that g takes values in $C^\infty_{\mathrm{pr}}(N, P)$, so g is not necessarily an element in $C^{\mathscr{X}}(N, P)$.) Let $\Gamma^{\mathscr{X}}(g^*E)$ denote the set of germs at x_0 of continuous maps $\nu : (\mathscr{X}, x_0) \to C^\infty(N, E)$ with $\pi^*(\nu(x))=g(x), \forall x \in \mathscr{X}$. We make $\Gamma^{\mathscr{X}}(g^*E)$ into a $C^{\mathscr{X}}(N)$-module by performing multiplication and addition in the fibers of E. More precisely, $C^{\mathscr{X}}(N) \times \Gamma^{\mathscr{X}}(g^*E)$ is a subset of the set of continuous germs $v = (\rho, v') : (\mathscr{X}, x_0) \to C^\infty(N, \mathsf{R}) \times C^\infty(N, E) = C^\infty(N, \mathsf{R} \times E)$. Composing such a v with the smooth map $m : \mathsf{R} \oplus E \to E$

that performs multiplication within the fibers of E, we get a continuous germ $m \circ v : (\mathscr{X}, x_0) \to C^\infty(N, E)$ by 1.4.1(3). It is clear that $m \circ v$ belongs to $\Gamma^{\mathscr{X}}(g^*E)$, and $m \circ v$ will be the multiplication of $v' \in \Gamma^{\mathscr{X}}(g^*E)$ by $\rho \in C^{\mathscr{X}}(N)$. In a similar fashion we define addition within $\Gamma^{\mathscr{X}}(g^*E)$.

In the case where g is constantly equal to $g(x_0)$ one can show that $\Gamma^{\mathscr{X}}(g^*E)$ is naturally isomorphic as $C^{\mathscr{X}}(N)$-module to the module of continuous germs $(\mathscr{X}, x_0) \to \Gamma^\infty(g(x_0)^*E)$. (We will make no use of this claim.)

We shall need the following two facts from [5], p. 287.

(1) $\Gamma^{\mathscr{X}}(g^*E)$ is finitely generated over $C^{\mathscr{X}}(N)$.

(2) $\operatorname{ev}\Gamma^{\mathscr{X}}(g^*E) = \Gamma^{\mathscr{X}}(g^*E)/\mathscr{M}_N\Gamma^{\mathscr{X}}(g^*E)$ is naturally isomorphic by evaluation in x_0 to $\Gamma^\infty(g(x_0)^*E)$; if $E = TP$, this is $\Theta(g(x_0))$.

Proof of 7.3. Let $\mathscr{X} = \Theta(f)$ and let x_0 be the 0-section in $\Theta(f)$. Let $A = \Gamma^{\mathscr{X}}(\mathrm{id}_P^* TP)$ as $C^{\mathscr{X}}(P)$-module and let $B = \Gamma^{\mathscr{X}}(\mathrm{id}_N^* TN)$ and $C = \Gamma^{\mathscr{X}}(f^*TP)$ as $C^{\mathscr{X}}(N)$-modules in the way described above. From (1) we know that A and C are finitely generated.

Let $\tilde{f} \in C^{\mathscr{X}}(N, P)$ be the constant germ with value f and let $\alpha : A \to C$ be the homomorphism over $\tilde{f}^*$ given by letting ωf operate pointwise with respect to $\mathscr{X}$, that is, $\alpha(\eta')(x) = \omega f(\eta'(x))$ for $\eta' \in A$ and $x \in \mathscr{X}$. In the same manner we define $\beta : B \to C$ to be the $C^{\mathscr{X}}(N)$-homomorphism given by letting tf operate pointwise with respect to $\mathscr{X}$.

From (2) above we know that $\operatorname{ev} A$, $\operatorname{ev} B$, $\operatorname{ev} C$ are naturally isomorphic to $\Theta(P)$, $\Theta(N)$ and $\Theta(f)$ respectively. Under these isomorphism $\operatorname{ev}\alpha : \operatorname{ev} A \to \operatorname{ev} C$ and $\operatorname{ev}\beta : \operatorname{ev} B \to \operatorname{ev} C$ are conjugated to ωf and tf.

f is infinitesimally stable by 0.1, since f is locally stable and proper. Then $\operatorname{ev}\alpha + \operatorname{ev}\beta = \omega f + tf : \Theta(P) \oplus \Theta(N) \to \Theta(f)$ is surjective and so, by 7.5, $\alpha + \beta : A \oplus B \to C$ is also surjective. Let $c \in C$ be the germ at x_0 of the identity $\mathscr{X} = \Theta(f) \to \Theta(f)$. We may then choose $\eta' \in A$ and $\xi' \in B$ such that $\alpha(\eta') + \beta(\xi') = c$. This tells us that for ν in a τW^∞-neighbourhood $\mathscr{U}$ of the 0-section in $\Theta(f)$ ($= \mathscr{X}$) we have $\nu = c(\nu) = \alpha(\eta')(\nu) + \beta(\xi')(\nu) = \omega f(\eta'(\nu)) + tf(\xi'(\nu))$. Letting $\xi = \xi' - \xi'(0)$ and $\eta = \eta' - \eta'(0)$ we get the desired τW^∞-continuous splitting of vector fields along f near the 0-section implemented by the maps ξ and η.

Proposition 7.7. *Let $f \in C^\infty(N, P)$ be proper and locally stable. Then f is strongly quasi-infinitesimally stable with respect to τV^∞.*

Proof. We shall apply 2.4 with the definitions made in 2.3. To check that the assumptions of 2.4 are satisfied, notice that f is strongly quasi-infinitesimally stable with respect to τW^∞ by 7.3. We may thus choose a τW^∞-open $\mathscr{U}' \subset \Theta(f) = \mathscr{M}$ and τW^∞-continuous maps $\xi' : \mathscr{U}' \to \Theta(N) = \mathscr{H}$ and $\eta' :$

$\mathscr{U}' \to \Theta(P) = \mathscr{K}$ with $\xi'(0) = 0$, $\eta'(0) = 0$, such that for any $\nu \in \mathscr{U}$, $\nu = tf\xi'(\nu) + \omega f\eta'(\nu)$, i.e. $(\eta'(\nu) \cdot_{\mathscr{K}} \nu) \cdot_{\mathscr{H}} \xi'(\nu) = m_0$.

Given subsets $K \subset V$ of P with K compact and V open, choose $\rho \in C^\infty(P)$ with $K \prec \rho \prec V$. When restricting to $\mathscr{M}_{f^{-1}K}$ we may replace the maps ξ' and η' by $\rho\xi'$ and $(\rho \circ f)\eta'$ respectively, affecting neither the τW^∞-continuity nor the validity of $(\eta'(\nu) \cdot_{\mathscr{K}} \nu) \cdot_{\mathscr{H}} \xi'(\nu) = m_0$. Clearly ξ' and η' thus defined satisfy the conditions of Proposition 2.4, and the conclusion precisely states that f is strongly quasi-infinitesimally stable with respect to τV^∞.

PROPOSITION 7.8. *Let $f \in C^\infty(N, P)$ be submersive. Then $\Theta(f) = tf(\Theta(N))$. Indeed, there is a τW^∞- and τV^∞-continuous $C^\infty(N)$-module homomorphism $h : \Theta(f) \to \Theta(N)$ such that $tf \circ h$ is the identity on $\Theta(f)$. Moreover, h can be chosen so that $\nu(x) = 0 \iff h(\nu)(x) = 0$ for $x \in N$, $\nu \in \Theta(f)$.*

PROOF. The tangental map $df : TN \to TP$ has constant rank since f is submersive, so $K = \operatorname{Ker} df \subset TN$ is a sub-bundle. Let $L \subset TN$ be another sub-bundle complementary to K, that is, $K \oplus L = TN$. Then df induces an isomorphism $L \to f^*TP$ of vector bundles, and hence the composite map

$$\phi : \Theta(N) = \Gamma^\infty(K) \times \Gamma^\infty(L) \xrightarrow[\cong]{\mathrm{id} \times tf} \Gamma^\infty(K) \times \Theta(f)$$

is a topological isomorphism in both topologies. Define h to be the continuous composition

$$\Theta(f) \xrightarrow{(0,\mathrm{id})} \Gamma^\infty(K) \times \Theta(f) \xrightarrow{\phi^{-1}} \Theta(N).$$

Then h has the desired properties.

THEOREM 7.9. *Let $f \in C^\infty(N, P)$ be quasi-proper and locally stable. Then f is strongly quasi-infinitesimally stable with respect to τV^∞.*

PROOF. We shall describe stepwise a chain of τV^∞-continuous manipulations of vector fields (along f, on N, and on P) starting in a neighbourhood $\mathscr{U}$ of the 0-section in $\Theta(f)$ and eventually producing τV^∞-continuous splittings of the vector fields along f in $\mathscr{U}$. Throughout the proof $\Theta(f)$, $\Theta(N)$, and $\Theta(P)$ are equipped with the topology τV^∞.

Let $V \subset P$ be an open neighbourhood of $\Delta(f)$ such that $f_V : f^{-1}V \to V$ is proper. Let $\sigma \in C^\infty(N)$, $\Sigma(f) \prec \sigma \prec f^{-1}V$. This is possible since $\Sigma(f)$ is closed in N and contained in the open $f^{-1}V$.

The proof works as follows. If ν is a vector field along f, we damp off ν outside $f^{-1}V$. Then we use properness of f_V and 7.7 to split the damped vector field ν_{damped} into vector fields on N and P. It remains to split $\nu - \nu_{\mathrm{damped}}$ which is 0 near $\Sigma(f)$. This can be done from the right by 7.8.

To start with let $\mathscr{U}$ be any open neighbourhood of 0 in $\Theta(f)$. We shall gradually shrink $\mathscr{U}$, when necessary. Throughout we assume that ν is a member of $\mathscr{U}$.

Step 1: Since $\sigma\nu$ is equal to the 0-section outside $\operatorname{supp}\sigma$ which is contained in $f^{-1}V$, the restriction of $\sigma\nu$ in the source to $f^{-1}V$ is continuous by 1.2.1(3), and since the restricted map maps into the open subset f^*TV of f^*TP fibered over $f^{-1}V$, we see that $\nu_1 = \sigma\nu|_{f^{-1}V} : f^{-1}V \to f^*TV \in \Theta(f_V)$ depends continuously on ν.

Step 2: f_V is locally stable since f is, and hence f_V is strongly quasi-infinitesimally stable by 7.7. Let ξ_1 and η_1 be the continuous maps implementing the splitting of vector fields along f_V near 0. We will assume that $\mathscr{U}$ is so small that ξ_1 and η_1 are defined at ν_1, hence $\nu_1 = tf\xi_1(\nu_1) + \omega f\eta_1(\nu_1)$.

Step 3: By 1.2.2 we may assume that $\xi_1(\nu_1)$ can be extended smoothly by the 0-section in $\Theta(N)$ outside $f^{-1}V$ to a vector field ξ_2 on N such that ξ_2 depends continuously on $\xi_1(\nu_1)$ and hence on ν. Likewise may we assume that $\eta_1(\nu_1)$ extends by 0 to a vector field η_2 on P in a way depending continuously on ν.

Step 4: Let $\nu_2 = \nu - tf\xi_2 - \omega f\eta_2$. Then ν_2 depends continuously on ν. For x in the neighbourhood C of $\Sigma(f)$ where $\sigma(x) = 1$ we have the following equalities $\nu(x) = \nu_1(x)$, $\xi_1(\nu_1)(x) = \xi_2(x)$, and $\eta_1(\nu_1)(f(x)) = \eta_2(f(x))$, hence $\nu_2(x) = 0$.

Step 5: Let $N_3 = N - \Sigma(f)$. Since C is a neighbourhood of $\Sigma(f)$ in N, the vector field on N_3 defined by $\nu_3 = \nu_2|_{N_3} : N_3 \to TN_3$ depends continuously on ν_2 and hence on g by 1.2.1(3) and 1.2.1(2).

Step 6: Let $f_3 = f|_{N_3} : N_3 \to P$. Then f_3 is submersive. By 7.8, tf_3 has a continuous right inverse, $\xi_3 : \Theta(f_3) \to \Theta(N)_3$, i.e. $\nu_3 = tf_3\xi_3(\nu_3)$.

Step 7: As in Step 3 we may assume, that $\xi_3(\nu_3)$ extends by 0 on $\Sigma(f)$ to a vector field ξ_4 on N continuously in $\xi_3(\nu_3)$ and hence in ν. Clearly $\nu_2 = tf\xi_4$.

We are now at the end of the chain. Let $\xi(\nu) = \xi_2 + \xi_4$ and $\eta(\nu) = \eta_2$. Then ξ and η depend continuously on ν and

$$tf\xi(\nu) + \omega f\eta(\nu) = tf\xi_4 + (tf\xi_2 + \omega f\eta_2) = \nu_2 + (\nu - \nu_2) = \nu.$$

By construction $\xi(0) = 0$ and $\eta(0) = 0$, and we have the desired τV^∞-continuous splitting of vector fields along f near 0.

7.10. Remark. In 7.3 and 7.7 we saw that locally stable proper maps are strongly quasi-infinitesimally stable with respect to both τW^∞ and τV^∞. Since such maps are also infinitesimally stable (by 0.3(1)) one may ask if we can take the domain $\mathscr{U} \subset \Theta(f)$ from the Definition 4.2(3),(4) of (strong) quasi-infinitesimal stability, on which the splitting maps are defined, to be the whole of $\Theta(f)$. This is indeed possible in the τV^∞-case but requires a slight extension of our arguments in 7.7. It appears that it also holds for τW^∞, but a proof would require a quite substantial reworking of Mather's preparation theorem 7.5.

8. Local stability and strong stability

Throughout N and P are manifolds without boundary.

THEOREM 8.1. *Let $f \in C^\infty(N, P)$ be stable in the topology τV^∞. Then f is locally stable.*

PROOF. The corresponding result for τW^∞ was proved by Mather in [6], 4.1. The proof carries over to the τV^∞-case: By stability of f there is a τV^∞-neighbourhood $\mathscr{U}$ of f consisting of maps that are equivalent to f via diffeomorphisms in N and P. Let $W \subset {}_rJ^k(N, P)$ be an orbit, i.e. W is a subset invariant under the action of (germs of) diffeomorphisms of N and P. By [5], 1.4, W is a submanifold. The set of maps $g \in C^\infty(N, P)$ with ${}_rj^kg$ transverse to W is τV^∞-dense by the Multi-jet Transversality Theorem 3.2.2. Hence there is a $g \in \mathscr{U}$ with ${}_rj^kg$ transversal to W. Since W is an orbit in ${}_rJ^k(N, P)$, ${}_rj^kf$ must also be transverse to W, since f and g are equivalent. Thus f is locally stable by criterion (3) of 4.3.

The proof of sufficiency of local stability (and quasi-properness) for strong stability is more involved and is split into several parts. First we give an explanation of the notation used below.

8.2. For a map $g : N \times I \to P$, where $I = [0, 1]$, will we use the notation g_t for the maps $g(\cdot, t) : N \to P$ defined for $t \in I$. We write (by an innocent abuse of notation) $\frac{\partial g_t}{\partial t}$ for the map $\frac{\partial g}{\partial t}(\cdot, t) : N \to TP$ for each $t \in I$.

Let $\pi_N : N \times I \to N$ and $\pi_P : P \times I \to P$ be the projections and let $g \in C^\infty(N \times I, P)$. Then then there is a map $t'g : \Gamma^\infty(\pi_N^*TN) \to \Gamma^\infty(g^*TP)$ uniquely defined by $t'g(\xi)_t = t(g_t)(\xi_t)$ for $t \in I$ and $\xi \in \Gamma^\infty(\pi_N^*TN)$. Also we define a map $\omega'g : \Gamma^\infty(\pi_P^*TP) \to \Gamma^\infty(g^*TP)$ by $\omega'g(\eta)_t = \omega(g_t)(\eta_t)$ for $t \in I$ and $\eta \in \Gamma^\infty(\pi_P^*TP)$.

The following two lemmas are taken from [5], where they can be found, with proofs, as Lemma 1 and Lemma 2 in Section 7.

LEMMA 8.3. *Let $g : N \times I \to P$ be a smooth map with $g_0 = f$. Suppose $h \in C^\infty(N \times I, N)$ and $k \in C^\infty(P \times I, P)$ are such that $h_0 = \mathrm{id}_N$, $k_0 = \mathrm{id}_P$, h_t and k_t are diffeomorphisms for all $t \in I$, and such that*

$$\frac{\partial g}{\partial t} = t'g(\xi) + \omega'g(\eta),$$

when $\xi \in \Gamma^\infty(\pi_N^*TN)$ and $\eta \in \Gamma^\infty(\pi_P^*TP)$ are defined by

$$\xi_t = -\frac{\partial h_t}{\partial t} \circ h_t^{-1}, \quad \eta_t = \frac{\partial k_t}{\partial t} \circ k_t^{-1}, \qquad t \in I.$$

Then $g_t(x) = k_t \circ f \circ h_t^{-1}(x)$ *for all* $t \in I$ *and* $x \in N$.

LEMMA 8.4. *There exists a* τW^∞*-neighbourhood* $\mathscr{O}_N$ *of* 0 *in* $\Gamma^\infty(\pi_N^* TN)$ *such that for each* $\xi \in \mathscr{O}_N$, *there exists* $\theta_N(\xi) \in C^\infty(N \times I, N)$ *such that* $\theta_N(\xi)_t$ *is a diffeomorphism of* N *for each* $t \in I$, $\theta_N(\xi)_0 = \mathrm{id}_N$, *and*

$$\frac{\partial \theta_N(\xi)_t}{\partial t} \circ \theta_N(\xi)_t^{-1} = \xi_t, \qquad t \in I.$$

Moreover, the mapping $\theta_N : \mathscr{O}_N \to C^\infty(N \times I, N)$ *is* τW^∞*-continuous.*

We now prove

PROPOSITION 8.5. *Let* $f \in C^\infty_{\mathrm{pr}}(N, P)$ *be infinitesimally stable. Let* $D \subset V \subset P$, *where* D *is closed in* P *and* V *open. Then there exist a* τW^∞*-neighbourhood* $\mathscr{U}$ *of* f *in*

$$\mathscr{M}_{f^{-1}D} = \left\{ g \in C^\infty(N, P) : g|_{N - f^{-1}D} = f|_{N - f^{-1}D} \right\}$$

and a τW^∞*-continuous map* $(h, k) : \mathscr{U} \to \mathrm{Diff}^\infty(N) \times \mathrm{Diff}^\infty(P)$ *such that* $(h, k)(f) = (\mathrm{id}_N, \mathrm{id}_P)$, *and for all* $g \in \mathscr{U}$,

(a) $g = k(g) \circ f \circ h(g)^{-1}$,

(b) $\{x \in N : h(g)(x) \neq x\} \subset f^{-1}V$,

(c) $\{y \in P : k(g)(y) \neq y\} \subset V$.

PROOF. Throughout the proof we equip all mapping-spaces with the topology τW^∞.

Let γ and U_Δ be as in 2.3.2, and let $\mathscr{U}'$ be an arbitrary open neighbourhood of f in $C^\infty(N, P)$, such that $(f, g)(N) \subset U_\Delta$ for all $g \in \mathscr{U}'$. Letting $\pi_I : N \times I \to I$ be the projection, we have a continuous mapping (by 1.4.1(1) and (2), since π_N is proper)

$$\Phi : \mathscr{U}' \cap \mathscr{M}_{f^{-1}D} \to \mathscr{X}; \qquad g \mapsto \gamma_*(f \circ \pi_N, g \circ \pi_N, \pi_I),$$

where $\mathscr{X} = \left\{ g' \in C^\infty(N \times I, P) : g'_0 = f \right\}$.

From [5], p. 288, it follows that there is a τW^∞-neighbourhood $\mathscr{Y}$ of $f \circ \pi_N$ in $\mathscr{X}$ and τW^∞-continuous $\xi : \mathscr{Y} \to \Gamma^\infty(\pi_N^* TN)$ and $\eta : \mathscr{Y} \to \Gamma^\infty(\pi_P^* TP)$ satisfying $\xi(f \circ \pi_N) = 0$, $\eta(f \circ \pi_N) = 0$ and for all $g' \in \mathscr{Y}$,

$$\frac{\partial g'}{\partial t} = t' g'(\xi(g')) + \omega' g'(\eta(g')).$$

Choose a function $\rho \in C^\infty(P)$, $D \prec \rho \prec V$. By shrinking $\mathscr{Y}$ we may assume that $\rho \circ g' = 1$ on $f^{-1}D \times I$ for all $g' \in \mathscr{Y}$. Define mappings $\xi' : \mathscr{Y} \to \Gamma^\infty(\pi_N^* TN)$ and $\eta' : \mathscr{Y} \to \Gamma^\infty(\pi_P^* TP)$ by

$$\xi'(g') = (f \circ \pi_N)^* \rho \cdot \xi(g') \quad \text{and} \quad \eta'(g') = \pi_P^* \rho \cdot \eta(g'), \qquad g' \in \mathscr{Y}.$$

By continuity of Φ we may, and do, shrink $\mathscr{U}'$ so that $\operatorname{Im}\Phi \subset \mathscr{Y}$. Let $g' \in \operatorname{Im}\Phi$. Then $\rho \circ g' = \rho \circ f \circ \pi_N$, and since $\left(\frac{\partial g'}{\partial t}\right)_t |_{N-f^{-1}D} = 0$ for all $t \in I$, we have

$$\frac{\partial g'}{\partial t} = (f \circ \pi_N)^*\rho \cdot \frac{\partial g'}{\partial t} = (f \circ \pi_N)^*\rho \cdot \big(t'g'(\xi(g')) + \omega' g'(\eta(g'))\big)$$
$$= t'g'(\xi'(g')) + \omega' g'(\eta'(g')).$$

By continuity of ξ' and η' we may choose $\mathscr{U}'$ so small that, for $g' \in \operatorname{Im}\Phi$, the vector fields $-\xi'(g')$ and $\eta'(g')$ are integrable in the sense of 8.4 and thereby find $h'(g') \in C^\infty(N \times I, N)$ and $k'(g') \in C^\infty(P \times I, P)$ such that $h'(g')_t \in \operatorname{Diff}^\infty(N)$, $k'(g')_t \in \operatorname{Diff}^\infty(P)$ for all $t \in I$, $h'(g')_0 = \mathrm{id}_N$, $k'(g')_0 = \mathrm{id}_P$ and

$$\frac{\partial h'(g')_t}{\partial t} \circ {h'(g')_t}^{-1} = -\xi'_t, \qquad \frac{\partial k'(g')_t}{\partial t} \circ {k'(g')_t}^{-1} = \eta'_t.$$

Integration is continuous, so $g' \mapsto (h'(g'), k'(g'))$ is continuous.

It follows from Lemma 8.3 that

$$g'_t = k'(g')_t \circ f \circ {h'(g')_t}^{-1} \qquad \forall t \in I,$$

so with $i_N \in C^\infty_{\mathrm{pr}}(N, N \times I)$ and $i_P \in C^\infty_{\mathrm{pr}}(P, P \times I)$ defined by $i_N(x) = (x, 1)$, $x \in N$, $i_P(y) = (y, 1)$, $y \in P$, the maps

$$h = i_N^* \circ h' \circ \Phi : \mathscr{U}' \cap \mathscr{M}_{f^{-1}D} \to \operatorname{Diff}^\infty(N)$$

and

$$k = i_P^* \circ k' \circ \Phi : \mathscr{U}' \cap \mathscr{M}_{f^{-1}D} \to \operatorname{Diff}^\infty(P)$$

are continuous with

$$g = k(g) \circ f \circ h(g)^{-1} \quad \text{for} \quad g \in \mathscr{U} = \mathscr{U}' \cap \mathscr{M}_{f^{-1}D}.$$

If $\xi'(g')_t(x) = 0$ for all $t \in I$ for some $x \in N$, $g' \in \operatorname{Im}\Phi$, then $h'(g')_t(x) = x$ for all $t \in I$. In fact, $\beta(t) = h'(g')_t(x)$ is the uniquely determined solution to $\frac{\partial \beta}{\partial t}(t) = -\xi'(\beta(t), t)$ with $\beta(0) = x$, and hence $\beta(t) = x$ for all $t \in I$ by uniqueness. This immediately gives us that

$$\{x \in N : h(g)(x) \neq x\} \subset \operatorname{supp}(f^*\rho) \subset f^{-1}V$$

and that $h(f) = \mathrm{id}_N$ by the construction of h.

The properties of k are proved similarly.

THEOREM 8.6. *Let $f \in C^\infty(N, P)$ be proper and locally stable. Then f is strongly stable in the topology τV^∞.*

PROOF. Let $\mathscr{M} = C^\infty_{\mathrm{pr}}(N, P)$, $m_0 = f$, $\mathscr{H} = \mathrm{Diff}^\infty(N)$, $h_0 = \mathrm{id}_N$, $\mathscr{K} = \mathrm{Diff}^\infty(P)$, and $k_0 = \mathrm{id}_P$ as in 2.2 and define the actions accordingly. By replacing k by k^{-1} we see from 8.5 that the assumptions of 2.4 are fulfilled. The conclusion of 2.4 is exactly that f is strongly stable in τV^∞.

PROPOSITION 8.7. *Let $f \in C^\infty(N, P)$ be submersive. Then there exist a τW^∞-neighbourhood $\mathscr{U}$ of f in $C^\infty(N, P)$ and a τW^∞-continuous mapping $h : \mathscr{U} \to \mathrm{Diff}^\infty(N)$ with $h(f) = \mathrm{id}_N$ and $g \circ h(g) = f$ for all $g \in \mathscr{U}$.*

Moreover, we may suppose that $h(g)(x) = x$ for each $x \in N$ such that $g(x) = f(x)$.

This proposition is a special case of [8], 3.6.1. We give an alternative proof here.

PROOF OF 8.7. Throughout the proof we equip all mapping-spaces with the topology τW^∞. We will use the notation introduced in 7.4 and 7.6.

Let $\mathscr{X} = \{g \in C^\infty(N \times I, P) : g_0 = f\}$ and let $x_0 = f \circ \pi_N$. Let $g \in C^{\mathscr{X}}(N \times I, P)$ be the 'identity', i.e. g is the germ at x_0 of the map $x \in \mathscr{X} \mapsto x \in C^\infty(N \times I, P)$. Regard $C = \Gamma^{\mathscr{X}}(g^*TP)$ as module over $C^{\mathscr{X}}(N \times I)$. From the proof of 7.8 we know that $df : TN \to TP$ induces an isomorphism of vector bundles $L \to f^*TP$, where L is some sub-bundle of $TN \to N$. Clearly this extends to an isomorphism of the bundles $\pi_N^*L \to N \times I$ and $(f \circ \pi_N)^*TP \to N \times I$, hence induces an isomorphism

$$(*) \qquad t'(f \circ \pi_N) : \Gamma^\infty(\pi_N^*L) \to \Gamma^\infty((f \circ \pi_N)^*TP)$$

of $C^\infty(N \times I)$-modules.

Put $B = \Gamma^{\mathscr{X}}(\pi_N^*L)$, and let $\beta : B \to C$ map $\nu \in B$ to the germ at x_0 of $x \mapsto t'(g(x))(\nu(x)) = t'x(\nu(x))$. From 1.4.1(7) follows that this germ is continuous in x, hence β is well-defined. Clearly β is a $C^{\mathscr{X}}(N \times I)$-module homomorphism.

Via the natural isomorphisms from 7.6(2) we have $\Gamma^\infty(\pi_N^*L) = \mathrm{ev}\, B \cong B/\mathscr{M}_{N\times I}B$ and $\Gamma^\infty((f \circ \pi_N)^*TP) = \Gamma^\infty(g(x_0)^*TP) = \mathrm{ev}\, C \cong C/\mathscr{M}_{N\times I}C$, and $\mathrm{ev}\,\beta : \mathrm{ev}\, B \to \mathrm{ev}\, C$ is the ismorphism in (*). Consequently, $C = \beta(B) + \mathscr{M}_{N\times I}C$, and since C is finitely generated by 7.6(1), then $C = \beta(B)$ by Nakayama's lemma [5], p. 281.

Now let ν be the germ at x_0 of $x \mapsto \frac{\partial g(x)_t}{\partial t} \in \Gamma^\infty(g(x)^*TP)$, $t \in I$, $x \in X$. This clearly defines a element of $C = \Gamma^{\mathscr{X}}(g^*TP)$, and we may find $\xi' \in B = \Gamma^{\mathscr{X}}(\pi_N^*L)$ such that $\beta(\xi') = \nu$. Choose a representative $\xi : \mathscr{Y} \subset \mathscr{X} \to \Gamma^\infty(\pi_N^*L) \subset \Gamma^\infty(\pi_N^*TN)$ for ξ', which satisfies $\frac{\partial g}{\partial t} = t'g(\xi(g))$ for all $g \in \mathscr{Y}$. We can then proceed by integration of ξ exactly as in the proof of 8.5 (use $\eta = 0$), obtaining the map h.

To prove the last statement, assume $g(x') = f(x')$ for some $x' \in N$. Continuing with the notation from the proof of 8.5, then $g' = \Phi(g) : N \times I \to P$ will have $g'_t(x') = f(x)$ for all $t \in I$, hence $\left(\frac{\partial g'}{\partial t}\right)_t (x') = 0$ for all t. The lift $\xi = \xi(\frac{\partial g'}{\partial t}) \in \Gamma^\infty(\pi_N^* L)$ will then satisfy $\xi_t(x') = 0$ for all t, and by integrating ξ we see that $h(g)(x') = x'$, as claimed.

THEOREM 8.8. *Let $f \in C^\infty(N, P)$ be submersive. Then there exist a τV^∞-neighbourhood $\mathscr{U}$ of f in $C^\infty(N, P)$ and a τV^∞-continuous mapping $h : \mathscr{U} \to \mathrm{Diff}^\infty(N)$ with $h(f) = \mathrm{id}_N$ and $g \circ h(g) = f$ for all $g \in \mathscr{U}$.*

PROOF. We will apply 2.4 with the data: $\mathscr{M} = C^\infty(N, P)$, $m_0 = f$, $\mathscr{H} = \mathrm{Diff}^\infty(N)$, $h_0 = \mathrm{id}_N$, and with $\mathscr{K}$ the trivial subgroup $\{\mathrm{id}\, P\} \subset \mathrm{Diff}^\infty(P)$; the action of $\mathscr{K}$ on $\mathscr{M}$ is necessarily trivial, and continuous. The proper map denoted f in 2.1 will play no rôle, so, to satisfy the conditions of 2.4, we take this map to be id_N. (To be precise, we define the manifolds N, P, Q, R_1, R_2 in 2.1 to be the manifolds N, N, P, N, N (respectively) from the assumptions of the theorem we are proving.)

Now the assumptions of 2.4 are satisfied by 2.2 in view of the last statement there. The conclusion of 2.4 gives the claim of our theorem.

THEOREM 8.9. *Let $f \in C^\infty(N, P)$ be quasi-proper and locally stable. Then f is strongly stable in the topology τV^∞.*

PROOF. Very much as in the proof of 7.9, we will describe stepwise a chain of compositions of mappings of mapping-spaces eventually producing h and k. Throughout the proof all mapping-spaces are equipped with the topology τV^∞.

By the definition of quasi-properness (1.3.3), there exists an open subset $V \subset P$ such that $f_V : f^{-1}V \to V$ is proper. Choose in addition a closed subset C and an open subset U of N with

$$\Sigma(f) \subset \overset{\circ}{C} \subset C \subset U \subset \overline{U} \subset f^{-1}(V),$$

where $\overset{\circ}{C}$ is the interior of C and $\overline{U}$ is the closure of U in N. Define $\mathscr{W}$ and $r = r_{f,C,U} : \mathscr{W} \to C^\infty(N, P)$ as in 2.3.2.

Let $\mathscr{U} \subset \mathscr{W}$ be a τV^∞-open neighbourhood of f, and let $g \in \mathscr{U}$. To limit the notation, we will not explicitly write down the domains of all the continuous mapping-space mappings that we define below, but simply use the term ‘we may assume...’ to indicate that the succeding statement is valid after a possible shrinking of $\mathscr{U}$.

The idea of the proof is very similar to that of 7.9. Given a map g in $\mathscr{U}$, $r(g)$ restricts continuously to a map $g_1 : f^{-1}V \to V$. By properness and local stability, f_V is strongly stable, hence there exist diffeomorphisms of $f^{-1}V$ and V that conjugate g_1 back to f_V. These diffeomorphisms depend continuously on g_1, and we may thus assume that they extend to diffeomorphisms of N and P respectively. These diffeomorphisms conjugate g to a map (denoted g_2 below) that coincides with f on a neighbourhood of $\Sigma(f)$. Finally we conjugate g_2 to f, using 8.8 and the fact that f is submersive away from $\Sigma(f)$.

Step 1: We may assume that $r(g)(\overline{U}) \subset V$. Since $r(g)$ coincides with f outside $\overline{U} \subset f^{-1}V$, this implies that $r(g)(f^{-1}V) \subset V$. By 1.2.1(3), $g_1 = r(g) : f^{-1}V \to V$ depends continuously on g.

Step 2: f_V is locally stable since f is, and hence f_V is strongly stable by properness and 8.6. Let h_1 and k_1 be the continuous maps from the definition of strong stability taking values in $\mathrm{Diff}^\infty(f^{-1}V)$ and $\mathrm{Diff}^\infty(V)$ respectively. We may assume that h_1 and k_1 are defined in g_1, hence $g_1 = k_1(g_1) \circ f \circ h_1(g_1)^{-1}$.

Let C' be a closed neighbourhood of $\Sigma(f)$ contained in the interior of C. Using that $h_1(f) = \mathrm{id}_N$, we may assume that $h_1(g_1)(C') \subset C$ by continuity (we will need this in Step 4).

Step 3: By 1.2.2(1), we may assume that $h_1(g_1)$ can be extended smoothly by id_N outside $f^{-1}V$ to a map $h_2 \in C^\infty(N, N)$. By the continuity statement in 1.2.2(2) we may assume that h_2 maps into the open subset of diffeomorphisms of N, and that h_2 depends continuously on $h_1(g_1)$, hence on g. Likewise, $k_1(g_1)$ may be extended by id_P outside V to a diffeomorphism k_2 of P, depending continuously on g_1.

Step 4: Let $g_2 = {k_2}^{-1} \circ g \circ h_2$. Then g_2 depends continuously on g. Let C' be as in Step 2. For $x \in C'$, $h_2(x) = h_1(g_1)(x) \in C$, so $g \circ h_2(x) = g_1 \circ [h_1(g_1)](x)$, and

$$g_2(x) = {k_2}^{-1}(g_1 \circ h_2(x)) = [k_1(g_1)^{-1} \circ g_1 \circ h_1(g_1)](x) = f(x).$$

It follows that g_2 coincides with f on C'. It thus remains to find diffeomorphisms that conjugate g_2 to f.

Step 5: Let $N_3 = N - \Sigma(f)$. Since C' is a neighbourhood of $\Sigma(f)$ in N, $g_3 = g_2 : N_3 \to P$ depends continuously on g_2 (by 1.2.1(3)) and hence on g.

Step 6: Let $f_3 = f| : N_3 \to P$. Then f_3 is submersive. Let h_3 be the continuous map from 8.8 taking values in $\mathrm{Diff}^\infty(N_3)$. We may assume that h_3 is defined at g_3, with $h_3(g_3)$ satisfying $g_3 \circ h_3(g_3) = f_3$.

Step 7: By the argument from Step 3, we may assume that $h_3(g_3)$ extends by id_N on $\Sigma(f)$ to a diffeomorphism h_4 of N, continuously in $h_3(g_3)$ and hence in g. Clearly $g_2 \circ h_4 = f$, since $g_2|_{\Sigma(f)} = f|_{\Sigma(f)}$.

We are now at the end of the chain. Let $h(g) = h_2 \circ h_4$ and $k(g) = k_2$. Then (h, k) depends continuously on g, and

$$k(g)^{-1} \circ g \circ h(g) = {k_2}^{-1} \circ g \circ h_2 \circ h_4 = g_2 \circ h_4 = f.$$

By construction $h(f) = \mathrm{id}_N$ and $k(f) = \mathrm{id}_P$, so the maps h and k implement the strong stability of f.

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