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FOCK REPRESENTATION OF THE BINARY SHIFT ALGEBRA

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Abstract

We modify the Fock representation construction of the CAR-algebra to obtain a new proof for that the Binary Shift Algebra is isomorphic to the CAR-algebra.

Introduction

If **S** is a bitstream, i.e. a sequence of 0's and 1's, we consider **S** as the subset of $\mathbb{N}$ given by $i \in \mathbf{S}$ if and only if the i'th term in the sequence is 1.

Let $\mathbf{S} \subset \mathbb{N}$ be a given bitstream and B the corresponding bitstream algebra, i.e. the C^*-algebra generated by a sequence of self-adjoint unitaries that either commute or anticommute in a certain way depending on the bitstream. More specifically, if this sequence is denoted by $\{s_i\}_{i\in\mathbb{N}}$, then s_i and s_j anticommute if $|i - j| \in \mathbf{S}$ and commute otherwise. The notation for the bitstream algebra used here is not common, and in e.g. [7] the bitstream algebra is denoted by $\mathfrak{A}(\mathbf{S})$. Since the bitstream can be thought of as fixed throughout this work, we suppress the dependence and simply denote this algebra by B.

If $C(\mathfrak{D})$ is the algebra of continuous functions on the Cantor set, it is known from a paper by Powers and Price [8] that B will be of the form $M_n(\mathbb{C}) \otimes C(\mathfrak{D})$ if **S** satisfies a certain periodicity condition, and the CAR-algebra otherwise. In this work we will see that by imitating the Fock representation construction of the CAR-algebra we can give an alternative proof for this result.

We will now sketch the approaches in [8] and this work to see their main differences. Whether the family of self-adjoint unitaries above is indexed over $\mathbb{N}$ or $\mathbb{Z}$ does not affect the results, and since many papers (e.g. those dealing with entropy) use the latter, we state all results with respect to this.

In both approaches the bitstream algebra B is considered as an AF-algebra, i.e. $B = \overline{\bigcup_{n=1}^{\infty} B_n}$, where B_n is the finite-dimensional subalgebra generated by $\{s_i\}_{i=1}^{n}$. We denote the center of B_n by $Z(B_n)$. To describe the embeddings $B_n \subset B_{n+1}$ (and thereby the Bratteli diagram corresponding to the AF-algebra

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B) it is essential to know the dimension of $Z(B_n)$ (for all $n \in \mathsf{N}$). In [11] this is done by studying a sequence of matrices with entries in F_2. More specifically, if $n \in \mathsf{N}$, the n'th Toeplitz matrix is given by

$$T_n = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & & a_{n-2} & a_{n-1} \\ a_1 & a_0 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} \\ a_2 & a_1 & a_0 & a_1 & & a_{n-4} & a_{n-3} \\ & \vdots & & & & & \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_1 & a_0 \end{pmatrix},$$

where $a_0 = 0$ and a_i is the i'th term in the bitstream. It is shown that $\dim Z(B_n) = 2^{\mathrm{null}(T_n)}$, so the nullity of T_n determines the dimension of $Z(B_n)$. The sequence $\{\mathrm{null}(T_n)\}_{n\in\mathsf{N}}$, called the center sequence, is then calculated from results on the ranks of the matrices above.

In this work we will avoid all discussions of Toeplitz matrices. The idea is to mimic the Fock representation construction of the CAR-algebra (see e.g. [6] and [1]) to obtain an algebra A, with commutation-relations that depend on the bitstream in such a way that the bitstream algebra occurs as a subalgebra of A. If the bitstream consists only of 1's, the construction of A should be identical to the Fock representation construction of the CAR-algebra. We find that A is an AF-algebra with finite-dimensional subalgebras A_n isomorphic to $M_{2^n}(\mathsf{C})$, i.e. A is isomorphic to the CAR-algebra. If B_n is defined as above, B_n is a subalgebra of A_n. Under the isomorphism above we consider B_n as a subalgebra of $M_{2^n}(\mathsf{C})$. We also obtain a nice description of the commutant of B_n in $M_{2^n}(\mathsf{C})$. Next we find how B_n is decomposed as a direct sum of matrix algebras, and we calculate the center sequence in a quite straightforward manner. With these two results the theorem follows quite easily.

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1. The Hilbert spaces $\mathscr{H}^{\otimes n}$ and $\bigwedge^n \mathscr{H}$

Let $\mathscr{H}$ be a Hilbert space with orthonormal basis $(\xi_i)_{i\in\mathsf{Z}}$. For $n \in \mathsf{Z}^+$ define

$$\mathscr{H}^{\otimes n} = \begin{cases} \mathsf{C} & \text{if } n = 0, \\ \mathscr{H} \otimes \cdots \otimes \mathscr{H} \ (n \text{ copies}) & \text{if } n > 0. \end{cases}$$

Let $\mathbf{S} \subset \mathsf{N}$ be identified with the bitstream $\big(\chi_{\mathbf{S}}(n)\big)_{n\in\mathsf{N}}$ and $\mathbf{S}' \subset \mathsf{Z}^+$ be given by $\chi_{\mathbf{S}'}(0) = 1$ and $\chi_{\mathbf{S}'}(n) = \chi_{\mathbf{S}}(n)\ \forall n \in \mathsf{N}$. The reason for introducing $\mathbf{S}'$ is technical and will soon be clear. Note that, by convention, $\chi_{\mathbf{S}}(0) = 0$.

Let $\mathscr{S}_n$ denote the symmetric group on n letters, and for $n \geq 2$ and $i = 1, \ldots, n-1$ define a unitary operator on $\mathscr{H}^{\otimes n}$ by

$$\mathrm{u_i}\, \xi_{k_1} \otimes \cdots \otimes \xi_{k_i} \otimes \xi_{k_{i+1}} \otimes \cdots \otimes \xi_{k_n} = (-1)^{\chi_{\mathrm{S}'}(|k_i - k_{i+1}|)}\, \xi_{k_1} \otimes \cdots \otimes \xi_{k_{i+1}} \otimes \xi_{k_i} \otimes \cdots \otimes \xi_{k_n}.$$

If $n \in \mathsf{Z}^+$, the group generated by $\{\mathrm{id}_{\mathscr{H}^{\otimes n}}, \mathrm{u_i}\}_{i=1}^{n-1}$ (i.e. the group generated by elements (transpositions) $\mathrm{u_1, u_2, \ldots, u_{n-1}}$ and determined by the group relations $\mathrm{u_i}^2 = \mathrm{I}$, $\mathrm{u_i\, u_{i+1}\, u_i} = \mathrm{u_{i+1}\, u_i\, u_{i+1}}$, and $\mathrm{u_i\, u_j} = \mathrm{u_j\, u_i}$ for $|i - j| > 1$) contains $n!$ elements, each of which can be indexed by a corresponding permutation in $\mathscr{S}_n$ ($\mathscr{S}_0$ is understood to be the trivial group). If u_σ denotes the element in this group corresponding to $\sigma \in \mathscr{S}_n$, then

$$\mathrm{u}_\sigma\, \xi_{k_1} \otimes \cdots \otimes \xi_{k_n} = h_n(\sigma, k_1, \ldots, k_n)\, \xi_{k_{\sigma(1)}} \otimes \cdots \otimes \xi_{k_{\sigma(n)}},$$

where $h_n : \mathscr{S}_n \times \mathsf{Z}^n \to \{\pm 1\}$. h_n is uniquely determined by writing u_σ as a product of $\mathrm{u_i}$'s, because if $1 \leq i < j \leq n$ and $\mathrm{u}_\sigma = \prod \mathrm{u_i}$ is a factorization of u_σ, the number of times $(-1)^{\chi_{\mathrm{S}'}(|k_i - k_j|)}$ contibutes to the sign in

$$\mathrm{u}_\sigma\, \xi_{k_1} \otimes \cdots \otimes \xi_{k_n} = \left(\prod \mathrm{u_i}\right) \xi_{k_1} \otimes \cdots \otimes \xi_{k_n} = \pm \xi_{k_{\sigma(1)}} \otimes \cdots \otimes \xi_{k_{\sigma(n)}}$$

is either odd or even, independent of how u_σ is factorized. Since h_n is unique, the mapping $\sigma \mapsto \mathrm{u}_\sigma$ is indeed a group-isomorphism. We will now give a recursive expression for the sign function h_n. It is clear that h_0 and h_1 are constantly equal to 1, so let $n \geq 2$. If $\sigma \in \mathscr{S}_n$, let $j = \sigma(n)$ and $\sigma' \in \mathscr{S}_{n-1}$ be given by $\sigma'(i) = \sigma(i)$ if $\sigma(i) < j$ and $\sigma'(i) = \sigma(i) - 1$ if $\sigma(i) > j$. Then

$$h_n(\sigma, k_1, \ldots, k_n) = -\prod_{i=j}^{n} (-1)^{\chi_{\mathrm{S}'}(|k_i - k_j|)}\, h_{n-1}(\sigma', k_1, \ldots, \check{k}_j, \ldots, k_n),$$

where $\check{k}_j$ means that k_j is removed. The reason for the minus sign in front of the product is to eliminate the extra minus caused by $(-1)^{\chi_{\mathrm{S}'}(|k_j - k_j|)}$.

For $n \in \mathsf{Z}^+$ define

$$\bigwedge^n \mathscr{H} = \{\xi \in \mathscr{H}^{\otimes n} : \mathrm{u}_\sigma\, \xi = \xi\ \forall \sigma \in \mathscr{S}_n\},$$

so $\bigwedge^n \mathscr{H}$ is a closed subspace of $\mathscr{H}^{\otimes n}$ (note that $\bigwedge^0 \mathscr{H} = \mathsf{C}$ since $\mathscr{S}_0$ is trivial). If $n \in \mathsf{Z}^+$, define an operator $\mathrm{P_n}$ on $\mathscr{H}^{\otimes n}$ by

$$\mathrm{P_n} = \frac{1}{n!} \sum_{\sigma \in \mathscr{S}_n} \mathrm{u}_\sigma\,.$$

If $\xi \in \bigwedge^n \mathscr{H}$, then $\mathrm{P_n}\,\xi = \xi$, and since $\mathrm{u}_\sigma\,\mathrm{P_n} = \mathrm{P_n}\ \forall \sigma \in \mathscr{S}_n$, it follows that $\mathrm{P_n}$ maps $\mathscr{H}^{\otimes n}$ onto $\bigwedge^n \mathscr{H}$ and $\mathrm{P_n}^2 = \mathrm{P_n}$. Calculation of the adjoint of $\mathrm{P_n}$ gives that $\mathrm{P_n^*} = \frac{1}{n!}\sum_{\sigma\in\mathscr{S}_n} \mathrm{u}_\sigma{}^{-1} = \frac{1}{n!}\sum_{\sigma\in\mathscr{S}_n} u_{\sigma^{-1}} = \mathrm{P_n}$, hence $\mathrm{P_n}$ is the orthogonal projection of $\mathscr{H}^{\otimes n}$ onto $\bigwedge^n \mathscr{H}$.

Define the outer product $\wedge$ by

$$\wedge : \mathscr{H} \times \cdots \times \mathscr{H} \to \bigwedge^n \mathscr{H},$$

$$(\eta_1, \ldots, \eta_n) \mapsto \eta_1 \wedge \cdots \wedge \eta_n = \sqrt{n!}\ \mathrm{P_n}\,\eta_1 \otimes \cdots \otimes \eta_n.$$

Observe that $\wedge$ is linear and continuous in each variable, and if $\sigma \in \mathscr{S}_n$, then

$$\begin{aligned} \xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}} &= \sqrt{n!}\ \mathrm{P_n}\,\xi_{k_{\sigma(1)}} \otimes \cdots \otimes \xi_{k_{\sigma(n)}} \\ &= \sqrt{n!}\ \mathrm{P_n}\,h_n(\sigma, k_1, \ldots, k_n)\,\mathrm{u}_\sigma\,\xi_{k_1} \otimes \cdots \otimes \xi_{k_n} \\ &= h_n(\sigma, k_1, \ldots, k_n)\,\xi_{k_1} \wedge \cdots \wedge \xi_{k_n}. \end{aligned}$$

Since $\chi_{\mathbf{S}'}(0) = 1$ this implies that if two k_i's are equal, then the outer product is 0. Because this is a property of the outer product in the Fock-representation construction of the CAR-algebra, the outer product defined here should also behave like this. This is the reason for using $\mathbf{S}'$ instead of $\mathbf{S}$. We conclude this section with some lemmas that will be needed in the sequel.

LEMMA 1.1. *If $\eta \in \mathscr{H}$ and $\sigma \in \mathscr{S}_n$, then*

$$\eta \wedge \xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}} = h_n(\sigma, k_1, \ldots, k_n)\,\eta \wedge \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}.$$

PROOF. Let $\eta = \sum_{i\in\mathsf{Z}} \lambda_i \xi_i \in \mathscr{H}$ and $\sigma \in \mathscr{S}_n$. Let $\sigma' \in \mathscr{S}_{n+1}$ be given by $\sigma'(1) = 1$ and $\sigma'(j+1) = \sigma(j) + 1$ for $1 \le j \le n$. For $i \in \mathsf{Z}$ define $l_i = \big(l_1^{(i)}, l_2^{(i)}, \ldots, l_{n+1}^{(i)}\big) = (i, k_1, \ldots, k_n) \in \mathsf{Z}^{n+1}$. Then

$$\begin{aligned} \eta \wedge \xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}} &= \sum_{i\in\mathsf{Z}} \lambda_i\,\xi_i \wedge \xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}} \\ &= \sum_{i\in\mathsf{Z}} \lambda_i\,\xi_{l^{(i)}_{\sigma'(1)}} \wedge \xi_{l^{(i)}_{\sigma'(2)}} \wedge \cdots \wedge \xi_{l^{(i)}_{\sigma'(n+1)}} \\ &= \sum_{i\in\mathsf{Z}} \lambda_i h_{n+1}(\sigma', l_1^{(i)}, \ldots, l_{n+1}^{(i)})\,\xi_{l_1^{(i)}} \wedge \xi_{l_2^{(i)}} \wedge \cdots \wedge \xi_{l_{n+1}^{(i)}} \\ &= \sum_{i\in\mathsf{Z}} \lambda_i h_n(\sigma, k_1, \ldots, k_n)\,\xi_i \wedge \xi_{k_1} \wedge \cdots \wedge \xi_{k_n} \\ &= h_n(\sigma, k_1, \ldots, k_n)\,\eta \wedge \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}. \end{aligned}$$

LEMMA 1.2. *Let $\mathscr{H}$ be a Hilbert space with orthonormal basis $(\xi_i)_{i\in\mathsf{Z}}$. Then $(\xi_{k_1}\wedge\cdots\wedge\xi_{k_n})_{k\in\mathsf{Z}^n}$ where $k=(k_1,k_2,\ldots,k_n)$ with $k_1<k_2<\cdots<k_n$ is an orthonormal basis for $\bigwedge^n\mathscr{H}$.*

PROOF. Let $k,l\in\mathsf{Z}^n$ where $k=(k_1,k_2,\ldots,k_n)$ with $k_1<k_2<\cdots<k_n$ and $l=(l_1,l_2,\ldots,l_n)$ with $l_1<l_2<\cdots<l_n$. Then

$$\begin{aligned}\big(\xi_{k_1}\wedge\cdots\wedge\xi_{k_n}\,,\,\xi_{l_1}\wedge\cdots\wedge\xi_{l_n}\big)&=n!\big(\xi_{k_1}\otimes\cdots\otimes\xi_{k_n}\,,\,\mathrm{P_n}\,\xi_{l_1}\otimes\cdots\otimes\xi_{l_n}\big)\\&=\sum_{\sigma\in\mathscr{S}_n}\big(\xi_{k_1}\otimes\cdots\otimes\xi_{k_n}\,,\,\mathrm{u}_\sigma\,\xi_{l_1}\otimes\cdots\otimes\xi_{l_n}\big)\\&=\sum_{\sigma\in\mathscr{S}_n}h_n(\sigma,l_1,\ldots,l_n)\prod_{i=1}^{n}\big(\xi_{k_i}\,,\,\xi_{l_{\sigma(i)}}\big)\\&=\begin{cases}1&\text{if }k=l,\\0&\text{if }k\neq l\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\overline{\mathrm{lin}}\,\{\xi_{k_1}\wedge\cdots\wedge\xi_{k_n}:k=(k_1,\ldots,k_n)\in\mathsf{Z}^n,k_1<\cdots<k_n\}\\&\quad=\overline{\mathrm{lin}}\,\{\xi_{k_1}\wedge\cdots\wedge\xi_{k_n}:k=(k_1,\ldots,k_n)\in\mathsf{Z}^n\}\\&\quad=\overline{\mathrm{lin}}\,\{\mathrm{P_n}\,\xi_{k_1}\otimes\cdots\otimes\xi_{k_n}:k=(k_1,\ldots,k_n)\in\mathsf{Z}^n\}\\&\quad=\mathrm{P_n}\,\overline{\mathrm{lin}}\,\{\xi_{k_1}\otimes\cdots\otimes\xi_{k_n}:k=(k_1,\ldots,k_n)\in\mathsf{Z}^n\}\\&\quad=\mathrm{P_n}\,\mathscr{H}^{\otimes n}=\bigwedge^{n}\mathscr{H}.\end{aligned}$$

For the next result the classical theory applies verbatim (see [6]).

LEMMA 1.3. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces, $n\in\mathsf{Z}^+$, and $\mathrm{L}\in\mathscr{B}(\mathscr{H}_1^{\otimes n},\mathscr{H}_2)$. If L satisfies $\mathrm{L}\,\mathrm{u}_\sigma=\mathrm{L}\ \forall\sigma\in\mathscr{S}_n$, then $\mathrm{L}\,\eta_1\wedge\cdots\wedge\eta_n=\sqrt{n!}\,\mathrm{L}\,\eta_1\otimes\cdots\otimes\eta_n$.*

PROOF.

$$\begin{aligned}\mathrm{L}\,\eta_1\wedge\cdots\wedge\eta_n&=\sqrt{n!}\,\mathrm{L}\,\mathrm{P_n}\,\eta_1\otimes\cdots\otimes\eta_n\\&=\frac{\sqrt{n!}}{n!}\,\mathrm{L}\Big(\sum_{\sigma\in\mathscr{S}_n}\mathrm{u}_\sigma\,\eta_1\otimes\cdots\otimes\eta_n\Big)\\&=\frac{\sqrt{n!}}{n!}\sum_{\sigma\in\mathscr{S}_n}\mathrm{L}\,\eta_1\otimes\cdots\otimes\eta_n\\&=\sqrt{n!}\,\mathrm{L}\,\eta_1\otimes\cdots\otimes\eta_n.\end{aligned}$$

2. The operators $\mathrm{A_n}(\xi)$ and $\mathrm{a_n}(\xi)$

Let $\mathscr{H}$ be a Hilbert space with orthonormal basis $(\xi_i)_{i\in\mathbb{Z}}$. For $\xi \in \mathscr{H}$ and $n \in \mathbb{Z}^+$ define an operator $\mathrm{A_n}(\xi)$ by

$$\mathrm{A_n}(\xi) : \mathscr{H}^{\otimes n} \to \bigwedge^{n+1} \mathscr{H}, \quad \eta_1 \otimes \cdots \otimes \eta_n \mapsto \frac{1}{\sqrt{n!}}\, \xi \wedge \eta_1 \wedge \cdots \wedge \eta_n.$$

If $m \in \mathbb{N}$ and $\{\lambda_i\}_{i=1}^m \subset \mathbb{C}$, then

$$\begin{aligned}
&\left\| \mathrm{A_n}(\xi) \sum_{i=1}^m \lambda_i\, \eta_1^{(i)} \otimes \cdots \otimes \eta_n^{(i)} \right\| \\
&\qquad = \left\| \frac{\sqrt{(n+1)!}}{\sqrt{n!}}\, \mathrm{P_{n+1}} \sum_{i=1}^m \lambda_i\, \xi \otimes \eta_1^{(i)} \otimes \cdots \otimes \eta_n^{(i)} \right\| \\
&\qquad \le \sqrt{n+1} \left\| \sum_{i=1}^m \lambda_i\, \xi \otimes \eta_1^{(i)} \otimes \cdots \otimes \eta_n^{(i)} \right\| \\
&\qquad = \sqrt{n+1}\, \|\xi\| \left\| \sum_{i=1}^m \lambda_i\, \eta_1^{(i)} \otimes \cdots \otimes \eta_n^{(i)} \right\|,
\end{aligned}$$

so $\mathrm{A_n}(\xi)$ extends to a bounded operator on $\mathscr{H}^{\otimes n}$ with $\|\mathrm{A_n}(\xi)\| \le \sqrt{n+1}\ \|\xi\|$. If $\sigma \in \mathscr{S}_n$, then

$$\begin{aligned}
\mathrm{A_n}(\xi)\, \mathrm{u}_\sigma\, \xi_{k_1} \otimes \cdots \otimes \xi_{k_n} &= \frac{1}{\sqrt{n!}}\, h_n(\sigma, k_1, \ldots, k_n)\, \xi \wedge \xi_{k_{\sigma(1)}} \wedge \cdots \wedge \xi_{k_{\sigma(n)}} \\
&= \frac{1}{\sqrt{n!}}\, \xi \wedge \xi_{k_1} \wedge \cdots \wedge \xi_{k_n} \\
&= \mathrm{A_n}(\xi)\, \xi_{k_1} \wedge \cdots \wedge \xi_{k_n},
\end{aligned}$$

where the second equality follows from Lemma 1.1. It now follows from Lemma 1.3 that

$$\mathrm{A_n}(\xi)\, \eta_1 \wedge \cdots \wedge \eta_n = \sqrt{n!}\, \mathrm{A_n}(\xi)\, \eta_1 \otimes \cdots \otimes \eta_n = \xi \wedge \eta_1 \wedge \cdots \wedge \eta_n.$$

For $n \in \mathbb{Z}^+$ let $\mathrm{a_n}(\xi) = \mathrm{A_n}(\xi)\,\big|_{\bigwedge^n \mathscr{H}}$, so

$$\mathrm{a_n}(\xi) : \bigwedge^n \mathscr{H} \to \bigwedge^{n+1} \mathscr{H},$$

$$\eta_1 \wedge \cdots \wedge \eta_n \mapsto \xi \wedge \eta_1 \wedge \cdots \wedge \eta_n$$

and $\|\mathrm{a_n}(\xi)\| \le \sqrt{n+1}\,\|\xi\|$ (we will later see that this norm is independent of n when $\xi = \xi_i$). The adjoint of $\mathrm{a_n}(\xi)$ will be denoted by $\mathrm{a_n^*}(\xi)$.

PROPOSITION 2.1. *If $i, j \in \mathsf{Z}$, then*

$$\mathrm{a_{n+1}}(\xi_\mathrm{i})\,\mathrm{a_n}(\xi_\mathrm{j}) - (-1)^{\chi_{S'}(|i-j|)}\,\mathrm{a_{n+1}}(\xi_\mathrm{j})\,\mathrm{a_n}(\xi_\mathrm{i}) = 0$$

and

$$\mathrm{a_n^*}(\xi_\mathrm{i})\,\mathrm{a_n}(\xi_\mathrm{j}) - (-1)^{\chi_{S'}(|i-j|)}\,\mathrm{a_{n-1}}(\xi_\mathrm{j})\,\mathrm{a_{n-1}^*}(\xi_\mathrm{i}) = (\xi_j, \xi_i)\,\mathrm{id}_{\bigwedge^n \mathscr{H}}\,.$$

The first equation is valid for $n \in \mathsf{Z}^+$ and the second for $n \in \mathsf{N}$.

PROOF. It is sufficient to show that the equations are valid for an orthonormal basis for $\bigwedge^n \mathscr{H}$ (see Lemma 1.2), so let $n \in \mathsf{N}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathsf{Z}^n$ with $k_1 < k_2 < \cdots < k_n$. Then

$$\mathrm{a_{n+1}}(\xi_\mathrm{i})\,\mathrm{a_n}(\xi_\mathrm{j})\,\chi_{k_1} \wedge \cdots \wedge \chi_{k_n} = \xi_i \wedge \xi_j \wedge \chi_{k_1} \wedge \cdots \wedge \chi_{k_n}$$

and

$$\begin{aligned}\mathrm{a_{n+1}}(\xi_\mathrm{j})\,\mathrm{a_n}(\xi_\mathrm{i})\,\chi_{k_1} \wedge \cdots \wedge \chi_{k_n} &= \xi_j \wedge \xi_i \wedge \chi_{k_1} \wedge \cdots \wedge \chi_{k_n} \\ &= (-1)^{\chi_{S'}(|i-j|)}\,\xi_i \wedge \xi_j \wedge \chi_{k_1} \wedge \cdots \wedge \chi_{k_n},\end{aligned}$$

so $\mathrm{a_{n+1}}(\xi_\mathrm{i})\,\mathrm{a_n}(\xi_\mathrm{j}) = (-1)^{\chi_{S'}(|i-j|)}\,\mathrm{a_{n+1}}(\xi_\mathrm{j})\,\mathrm{a_n}(\xi_\mathrm{i})$ (in the case $n = 0$ exchange $\chi_{k_1} \wedge \cdots \wedge \chi_{k_n}$ with $1 \in \bigwedge^0 \mathscr{H}$ in the computations above).

To show the second equation we need an expression for $\mathrm{a_n^*}(\xi_\mathrm{i})$. We will use the following notation: If an element is marked with an $\check{}$, then this element is to be omitted, and if $i \in \mathsf{Z}$, $k = (k_1, \ldots, k_n) \in \mathsf{Z}^n$, then δ_{i,k_m} shall equal 1 if there exists an m such that $i = k_m$ and 0 otherwise. If $\delta_{i,k_m} = 0$, then all expressions where k_m is present are set to zero. Let $h'_n : \mathsf{Z}^n \to \{\pm 1, 0\}$ be given by

$$h'_n(k_1, \ldots, k_n) = \begin{cases} h_n(\sigma, k_1, \ldots, k_n) & \text{if there exists } \sigma \in \mathscr{S}_n \text{ such that} \\ & k_{\sigma(1)} < k_{\sigma(2)} < \cdots < k_{\sigma(n)}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that h'_n is well-defined since a permutation that strictly orders a given set of integers must be unique.

Let $k \in \mathsf{Z}^{n+1}$, $l \in \mathsf{Z}^n$ where $k = (k_1, k_2, \ldots, k_{n+1})$ with $k_1 < k_2 < \cdots <$

k_{n+1} and $l = (l_1, l_2, \ldots, l_n)$ with $l_1 < l_2 < \cdots < l_n$. Then

$$\begin{aligned}
&\big(\chi_{k_1} \wedge \cdots \wedge \chi_{k_{n+1}}, \mathrm{a_n}(\xi_\mathrm{i})\, \psi_{l_1} \wedge \cdots \wedge \psi_{l_n}\big)\\
&\quad = \big(\chi_{k_1} \wedge \cdots \wedge \chi_{k_{n+1}}, \xi_i \wedge \psi_{l_1} \wedge \cdots \wedge \psi_{l_n}\big)\\
&\quad = \begin{cases} h'_{n+1}(i, k_1, \ldots, \check{k}_m, \ldots, k_{n+1}) & \text{if } k_1 = l_1, k_2 = l_2, \ldots, k_m = i \\ & k_{m+1} = l_m, \ldots, k_{n+1} = l_n, \\ 0 & \text{otherwise} \end{cases}\\
&\quad = \Big(\delta_{i,k_m} h'_{n+1}(i, k_1, \ldots, \check{k}_m, \ldots, k_{n+1})\\
&\qquad\qquad \chi_{k_1} \wedge \cdots \wedge \check{\chi}_{k_m} \wedge \cdots \wedge \chi_{k_{n+1}}, \psi_{l_1} \wedge \cdots \wedge \psi_{l_n}\Big),
\end{aligned}$$

so if $n \in \mathsf{Z}^+$ (for $n = 0$ exchange $\psi_{l_1} \wedge \cdots \wedge \psi_{l_n}$ with 1 above), then

$$\begin{aligned}
&\mathrm{a_n^*}(\xi_\mathrm{i})\, \chi_{k_1} \wedge \cdots \wedge \chi_{k_{n+1}}\\
&\quad = \delta_{i,k_m} h'_{n+1}(i, k_1, \ldots, \check{k}_m, \ldots, k_{n+1}) \chi_{k_1} \wedge \cdots \wedge \check{\chi}_{k_m} \wedge \cdots \wedge \chi_{k_{n+1}}.
\end{aligned}$$

With this expression at hand, we are ready to verify the second equation in the proposition. Assume first that $i = j$. If $\delta_{i,k_m} = 1$, then

$$\mathrm{a_n^*}(\xi_\mathrm{i})\, \mathrm{a_n}(\xi_\mathrm{i})\, \chi_{k_1} \wedge \cdots \wedge \chi_{k_n} = 0$$

and

$$\begin{aligned}
&\mathrm{a_{n-1}}(\xi_\mathrm{i})\, \mathrm{a_{n-1}^*}(\xi_\mathrm{i})\, \chi_{k_1} \wedge \cdots \wedge \chi_{k_n}\\
&\quad = \mathrm{a_{n-1}}(\xi_\mathrm{i})\, h'_n(i, k_1, \ldots, k_n) \chi_{k_1} \wedge \cdots \wedge \check{\chi}_{k_m} \wedge \cdots \wedge \chi_{k_n}\\
&\quad = h'_n(i, k_1, \ldots, k_n)^2 \chi_{k_1} \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \chi_{k_n}\\
&\quad = \chi_{k_1} \wedge \cdots \wedge \chi_{k_n},
\end{aligned}$$

so $\mathrm{a_n^*}(\xi_\mathrm{i})\, \mathrm{a_n}(\xi_\mathrm{i}) + \mathrm{a_{n-1}}(\xi_\mathrm{i})\, \mathrm{a_{n-1}^*}(\xi_\mathrm{i}) = \mathrm{id}_{\bigwedge^n \mathscr{H}}$. The case $\delta_{i,k_m} = 0$ is similar, so assume that $i < j$. Then

$$\begin{aligned}
&\mathrm{a_n^*}(\xi_\mathrm{i})\, \mathrm{a_n}(\xi_\mathrm{j})\, \chi_{k_1} \wedge \cdots \wedge \chi_{k_n}\\
&\quad = \mathrm{a_n^*}(\xi_\mathrm{i})\, h'_{n+1}(j, k_1, \ldots, k_n) \chi_{k_1} \wedge \cdots \wedge \xi_j \wedge \cdots \wedge \chi_{k_n}\\
&\quad = \delta_{i,k_m} h'_{n+1}(j, k_1, \ldots, k_n) h'_{n+1}(i, k_1, \ldots, \check{k}_m, \ldots, j, \ldots, k_n)\\
&\qquad\qquad \chi_{k_1} \wedge \cdots \wedge \check{\chi}_{k_m} \wedge \cdots \wedge \xi_j \wedge \cdots \wedge \chi_{k_n}
\end{aligned}$$

and

$$
\begin{aligned}
&\mathrm{a_{n-1}}(\xi_j)\,\mathrm{a^*_{n-1}}(\xi_i)\,\chi_{k_1}\wedge\cdots\wedge\chi_{k_n}\\
&\quad=\mathrm{a_{n-1}}(\xi_j)\,\delta_{i,k_m}h'_n(i,k_1,\ldots,\check{k}_m,\ldots,k_n)\chi_{k_1}\wedge\cdots\wedge\check{\chi}_{k_m}\wedge\cdots\wedge\chi_{k_n}\\
&\quad=\delta_{i,k_m}h'_n(i,k_1,\ldots,\check{k}_m,\ldots,k_n)h'_n(j,k_1,\ldots,\check{k}_m,\ldots,k_n)\\
&\qquad\qquad\chi_{k_1}\wedge\cdots\wedge\check{\chi}_{k_m}\wedge\cdots\wedge\xi_j\wedge\cdots\wedge\chi_{k_n}.
\end{aligned}
$$

By observing that $h'_{n+1}(j,k_1,\ldots,k_n)h'_{n+1}(i,k_1,\ldots,\check{k}_m,\ldots,j,\ldots,k_n)=(-1)^{\chi_{S'}(|i-j|)}h'_n(i,k_1,\ldots,\check{k}_m,\ldots,k_n)h'_n(j,k_1,\ldots,\check{k}_m,\ldots,k_n)$, we get the second equation. The case $i>j$ is similar.

3. The antisymmetric Fock space and the CAR-algebra

Let $\mathscr{H}$ be an infinite-dimensional, separable Hilbert space. Define the Full Fock space of $\mathscr{H}$ as

$$EXP(\mathscr{H})=\bigoplus_{n\geq 0}\mathscr{H}^{\otimes n}$$

and the bitstream Fock space of $\mathscr{H}$ as

$$\mathscr{F}(\mathscr{H})=\bigoplus_{n\geq 0}\bigwedge^n\mathscr{H},$$

so $\mathscr{F}(\mathscr{H})$ can be identified with a closed subspace of $EXP(\mathscr{H})$. (If the bitstream consists only of 1's, then $\mathscr{F}(\mathscr{H})$ is the antisymmetric Fock space of $\mathscr{H}$.) Proposition 2.1 gives that

$$\big(\mathrm{a^*_n}(\xi_i)\,\mathrm{a_n}(\xi_i)\big)^2=\mathrm{a^*_n}(\xi_i)\big(\mathrm{id}_{\bigwedge^{n+1}\mathscr{H}}-\mathrm{a^*_{n+1}}(\xi_i)\,\mathrm{a_{n+1}}(\xi_i)\big)\,\mathrm{a_n}(\xi_i)=\mathrm{a^*_n}(\xi_i)\,\mathrm{a_n}(\xi_i),$$

so $\mathrm{a^*_n}(\xi_i)\,\mathrm{a_n}(\xi_i)$ is a projection ($\neq 0$), hence $\|\mathrm{a_n}(\xi_i)\|=1$.

Let $\eta\in\mathscr{F}(\mathscr{H})$, so $\eta=(\eta_n)_{n\in\mathsf{Z}^+}$ where $\eta_n\in\bigwedge^n\mathscr{H}$, and for $i\in\mathsf{Z}$, define the operators

$$
\begin{aligned}
&\mathrm{a}(\xi_i),\quad \mathrm{a}^*(\xi_i):\mathscr{F}(\mathscr{H})\to\mathscr{F}(\mathscr{H}),\\
&\qquad\qquad\mathrm{a}(\xi_i)\,\eta=(\mathrm{a_n}(\xi_i)\,\eta_n),\quad \mathrm{a}^*(\xi_i)\,\eta=(\mathrm{a^*_n}(\xi_i)\,\eta_{n+1}).
\end{aligned}
$$

Since $\|\mathrm{a_n}(\xi_i)\|$ is independent of n, we get that

$$
\begin{aligned}
\|\mathrm{a}(\xi_i)\,\eta\|^2=\sum_n\|\mathrm{a_n}(\xi_i)\,\eta_n\|^2&\leq\sum_n\|\mathrm{a_n}(\xi_i)\|^2\,\|\eta_n\|^2\\
&\leq\sum_n\|\eta_n\|^2=\|\eta\|^2,
\end{aligned}
$$

so $a(\xi_i) \in \mathscr{B}\big(\mathscr{F}(\mathscr{H})\big)$. If $\eta, \eta' \in \mathscr{F}(\mathscr{H})$, then

$$\big(a(\xi_i)\,\eta\,,\,\eta'\big) = \sum_n \big(a_n(\xi_i)\,\eta_n\,,\,\eta'_{n+1}\big) = \sum_n \big(\eta_n\,,\,a_n^*(\xi_i)\,\eta'_{n+1}\big)$$
$$= \big(\eta\,,\,a^*(\xi_i)\,\eta'\big),$$

so $a(\xi_i)^* = a^*(\xi_i)$. Furthermore, $a(\xi_i)$ satisfies the same commutation-relations as $a_n(\xi_i)$: Since $a_0^*(\xi_i)\,a_0(\xi_j) = (\xi_j, \xi_i)\,\mathrm{id}_{\bigwedge^0 \mathscr{H}}$, it follows from Proposition 2.1 that

$$a(\xi_i)\,a(\xi_j) - (-1)^{\chi_{S'}(|i-j|)}\,a(\xi_j)\,a(\xi_i) = 0$$

and

$$a^*(\xi_i)\,a(\xi_j) - (-1)^{\chi_{S'}(|i-j|)}\,a(\xi_j)\,a^*(\xi_i) = (\xi_j, \xi_i)\,\mathrm{id}_{\mathscr{F}(\mathscr{H})}\,.$$

Define

$$CAR_0(\mathscr{H}) = C^*(a(\xi_i) : i \in \mathbb{Z}),$$

so $CAR_0(\mathscr{H})$ is a C^*-subalgebra of $\mathscr{B}\big(\mathscr{F}(\mathscr{H})\big)$.

Remark. If $\mathbf{S} = 111\cdots$, $CAR_0(\mathscr{H})$ as defined here will equal the usual CAR-algebra, and in the next section we will see that $CAR_0(\mathscr{H}) \cong \bigotimes_1^\infty M_2(\mathbb{C})$ which justifies the notation. In the Fock representation construction of the CAR-algebra the canonical map $a : \mathscr{H} \to \mathscr{B}\big(\mathscr{F}(\mathscr{H})\big)$, $\xi \mapsto a(\xi)$ is an isometry, so $CAR(\mathscr{H}) = C^*(a(\xi) : \xi \in \mathscr{H}) = C^*(a(\xi_i) : i \in \mathbb{Z})$. Hence, if $\mathbf{S} = 111\ldots$, $\|a(\xi)\| = \|\xi\|$, but it is easy to see that this need not be the case when $\mathbf{S} \neq 111\ldots$.

4. The AF-algebras *A* and *B*

For $i \in \mathbb{Z}$ let $x_i = a(\xi_i)$ and $s_i = a(\xi_i) + a^*(\xi_i)$. If $i, j \in \mathbb{Z}$, then s_i is a self-adjoint operator, and the commutation-relations give that

$$s_i s_j = \begin{cases} (-1)^{\chi_S(|i-j|)} s_j s_i & \text{if } i \neq j, \\ \mathrm{id}_{\mathscr{F}(\mathscr{H})} & \text{if } i = j, \end{cases}$$

so $\{s_i\}_{i\in\mathbb{Z}}$ is a family of self-adjoint, unitary operators satisfying $s_i s_j = (-1)^{\chi_S(|i-j|)} s_j s_i$.

For notational reasons the Hilbert spaces will from now on have an orthonormal basis indexed over $\mathbb{N}$ instead of over $\mathbb{Z}$. The results below have corresponding proofs for $\mathbb{N}$ and $\mathbb{Z}$, and where there are differences they will be commented on. So for $i \in \mathbb{N}$, $x_i = a(\xi_i)$ and $s_i = x_i + x_i^*$.

For $n \in \mathbb{N}$, define

$$A_n = C^*(x_i : 1 \leq i \leq n), \qquad A = C^*(x_i : i \in \mathbb{N})$$

and

$$B_n = C^*(s_i : 1 \le i \le n), \qquad B = C^*(s_i : i \in \mathbb{N}),$$

so $A = CAR_0(\mathscr{H})$. Observe that A and B are AF-algebras with $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ and $B = \overline{\bigcup_{n=1}^{\infty} B_n}$. (If the index set is $\mathbb{Z}$, let $A_n = C^*\big(x_i : \frac{1-n}{2} \le i \le \frac{n}{2}\big)$ and $A = C^*(x_i : i \in \mathbb{Z})$ etc.)

The commutation-relations will now read:

$$x_i^2 = 0, \qquad x_i x_j = (-1)^{\chi_S(|i-j|)} x_j x_i \quad (i \ne j)$$

and

$$x_i^* x_i + x_i x_i^* = 1, \qquad x_i^* x_j = (-1)^{\chi_S(|i-j|)} x_j x_i^* \quad (i \ne j).$$

PROPOSITION 4.1. *If $n \in \mathbb{N}$, then $A_n \cong M_{2^n}(\mathbb{C})$.*

PROOF. For $1 \le i \le n$ let

$$u_i = \prod_j (x_j x_j^* - x_j^* x_j),$$

where the product is taken over those $j < i$ which satisfy $x_i x_j = -x_j x_i$ (if no such j's exist, set $u_i = 1$). Since

$$\begin{aligned}(x_j x_j^* - x_j^* x_j)^2 &= x_j x_j^* x_j x_j^* + x_j^* x_j x_j^* x_j \\ &= x_j(1 - x_j x_j^*) x_j^* + x_j^*(1 - x_j^* x_j) x_j \\ &= 1,\end{aligned}$$

then

$$(x_j x_j^* - x_j^* x_j)(x_k x_k^* - x_k^* x_k) = \begin{cases} (x_k x_k^* - x_k^* x_k)(x_j x_j^* - x_j^* x_j) & \text{if } j \ne k, \\ 1 & \text{if } j = k. \end{cases}$$

From this it follows that each u_i is a self-adjoint unitary operator and that $u_i u_j = u_j u_i$.

We now use the u_i's to construct n pairwise commuting systems of 2×2 matrix units. This construction is standard and yields the result (see e.g. the discussion following Definition 1.2 in [12] and Lemma 7.16 in [6]).

From this result the following theorem is immediate.

THEOREM 4.2. *If $\mathscr{H}$ is an infinite-dimensional, separable Hilbert space, then*

$$CAR_0(\mathscr{H}) \cong \bigotimes_1^{\infty} M_2(\mathbb{C}).$$

5. Description of B_n and B'_n

By the isomorphism in the proof of Proposition 4.1, $B_n \subset A_n$ is identified with a subalgebra of $M_{2^n}(\mathbb{C})$. By abuse of notation we will also denote this algebra by B_n, and the elements in $M_{2^n}(\mathbb{C})$ corresponding to s_i an u_i $(1 \le i \le n)$ by s_i and u_i, respectively. Since B_n is generated by $\{s_i\}_{i=1}^n$ and the s_i's either commute or anticommute, it is clear that

$$B_n = \operatorname{lin}\{1, s_{i_1} \cdots s_{i_k} : 1 \le i_1 < \cdots < i_k \le n\},$$

hence B_n is a 2^n-dimensional subalgebra of $M_{2^n}(\mathbb{C})$. In what follows we will give a closer description of this subalgebra.

Let $a, b, c, d \in M_2(\mathbb{C})$ be given by

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let $(a_i)_{i \in \mathbb{N}} \subset M_2(\mathbb{C})$ be defined by

$$a_i = \begin{cases} a & \text{if } \chi_{\mathbf{S}}(i) = 0, \\ c & \text{if } \chi_{\mathbf{S}}(i) = 1. \end{cases}$$

For $n \in \mathbb{N}$ the isomorphism in the proof of Proposition 4.1 gives that in $M_{2^n}(\mathbb{C})$ we have that

$$\begin{aligned} s_1 &= b \otimes a \otimes \cdots \otimes a \otimes a \cdot u_1 \quad (n-1\ a\text{'s}) \\ s_2 &= a \otimes b \otimes \cdots \otimes a \otimes a \cdot u_2 \\ &\vdots \\ s_n &= a \otimes a \otimes \cdots \otimes a \otimes b \cdot u_n\,, \end{aligned}$$

where

$$\begin{aligned} u_1 &= a \otimes a \otimes a \otimes \cdots \otimes a \otimes a \\ u_2 &= a_1 \otimes a \otimes a \otimes \cdots \otimes a \otimes a \\ u_3 &= a_2 \otimes a_1 \otimes a \otimes \cdots \otimes a \otimes a \\ &\vdots \\ u_n &= a_{n-1} \otimes a_{n-2} \otimes a_{n-3} \otimes \cdots \otimes a_1 \otimes a. \end{aligned}$$

Hence, if $1 \le i \le n$, then $s_i = a_{i-1} \otimes a_{i-2} \otimes \cdots \otimes a_1 \otimes b \otimes a \otimes \cdots \otimes a$.

REMARK. If $1 \le i < j \le n$, then

$$\begin{aligned} s_i s_j &= a_{i-1} \otimes \cdots \otimes a_1 \otimes b \otimes a \otimes \cdots \otimes a \otimes \cdots \otimes a \\ &\qquad \cdot a_{j-1} \otimes \cdots \otimes a_{j-i+1} \otimes a_{j-i} \otimes a_{j-i-1} \otimes \cdots \otimes b \otimes \cdots \otimes a \\ &= (-1)^{\chi_{\mathbf{S}}(|j-i|)} s_j s_i, \end{aligned}$$

because $ba_{j-i} = (-1)^{\chi_S(|j-i|)} a_{j-i} b$ and the rest of the terms commute.

If B_n' denotes the commutant of B_n in $M_{2^n}(\mathbb{C})$, define the map κ to be linear and

$$\kappa : B_n \to B_n', \; s_i \mapsto \overline{s_i},$$

where $\overline{s_i}$ means s_i "read backwards", i.e. $\overline{s_i} = a \otimes \cdots \otimes a \otimes b \otimes a_1 \otimes \cdots \otimes a_{i-2} \otimes a_{i-1}$.

Let i, j be such that $i + j > n + 1$, and set $k = n - i$. Then

$$\begin{aligned}\overline{s_i} s_j &= a \otimes \cdots \otimes a \otimes b \otimes a_1 \otimes \cdots \otimes a_{j-1-k} \otimes \cdots \otimes a_{i-1} \\ &\qquad \cdot a_{j-1} \otimes \cdots \otimes a_{j-k} \otimes a_{j-1-k} \otimes a_{j-2-k} \otimes \cdots \otimes b \otimes \cdots \otimes a \\ &= s_j \overline{s_i}.\end{aligned}$$

Since it is clear that $\overline{s_i}$ and s_j commute when $i+j \leq n+1$, κ really maps B_n into B_n'. It is clear that κ is an injective $*$-homomorphism, so $\dim B_n' \geq \dim B_n$. We will soon see that $\dim B_n' = \dim B_n$, from which it follows that κ is surjective.

6. The state ω_Ω

Let Ω denote the vector $1 \in \bigwedge^0 \mathscr{H} \subset \mathscr{F}(\mathscr{H})$, and for $n \in \mathbb{N}$ define

$$\mathscr{K}_n = B_n \Omega.$$

Since $\{\Omega, s_{i_1} \cdots s_{i_k} \Omega : 1 \leq i_1 < \cdots < i_k \leq n\}$ is an orthonormal basis for $\mathscr{K}_n$, it follows that $\mathscr{K}_n$ is a 2^n-dimensional subspace of $\mathscr{F}(\mathscr{H})$.

Remark. $\mathscr{K}_n = A_n \Omega$, because if $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq i \leq n$, then $x_{i_1} \cdots x_{i_k} \Omega = s_{i_1} \cdots s_{i_k} \Omega$, $x_i^* \Omega = 0$, and $x_i^* \left(x_{i_1} \cdots x_{i_k} \Omega\right) = \pm x_{i_1} \cdots \check{x}_{i_m} \cdots x_{i_k} \Omega$ if there exists $m \in \{1, \ldots, k\}$ such that $i = i_m$, and 0 otherwise.

Lemma 6.1.

$$\overline{\bigcup_{n=1}^{\infty} \mathscr{K}_n} = \mathscr{F}(\mathscr{H}).$$

Proof. Let $\varepsilon > 0$ and $\eta \in \mathscr{F}(\mathscr{H})$, so $\eta = (\eta_n)_{n \in \mathbb{Z}^+}$, where $\eta_n \in \bigwedge^n \mathscr{H}$. Then there exists $\eta' = (\eta_n')_{n \in \mathbb{Z}^+} \in \mathscr{F}(\mathscr{H})$ and $N \in \mathbb{Z}^+$ such that $\|\eta - \eta'\| < \frac{\varepsilon}{2}$ and $\eta_n' = 0$ for $n > N$. Since $s_{k_1} \cdots s_{k_n} \Omega = \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}$ if $1 \leq k_1 < k_2 < \cdots < k_n$ ($k_1 < k_2 < \cdots < k_n$ if the index set is $\mathbb{Z}$), and these vectors constitute a basis for $\bigwedge^n \mathscr{H}$, then for each $i \in \{0, \ldots, N\}$ there exists $n_i \in \mathbb{N}$ and $b^{(i)} \in B_{n_i}$ with $b_j^{(i)} = 0$ for $j \neq i$ such that $\|\eta_i' - b_i^{(i)} \Omega\| < \frac{\varepsilon}{2\sqrt{N+1}}$. Then $\sum_{i=0}^N b^{(i)} \Omega \in \mathscr{K}_m$ where $m = \max_i n_i$ and $\|\eta' - \sum_{i=0}^N b^{(i)} \Omega\|^2 =$

$\sum_{i=0}^{N} \|\eta_i' - b_i^{(i)}\Omega\|^2 < \sum_{i=0}^{N} \frac{\varepsilon^2}{4(N+1)} = \frac{\varepsilon^2}{4}$. Hence $\|\eta - \sum_{i=0}^{N} b^{(i)}\Omega\| \le \|\eta - \eta'\| + \|\eta' - \sum_{i=0}^{N} b^{(i)}\Omega\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so $\eta \in \overline{\bigcup_{n=1}^{\infty} \mathscr{K}_n}$.

COROLLARY 6.2. *Ω is cyclic for B (and hence for A).*

PROOF. $\overline{B\Omega} = \overline{\bigcup_{n=1}^{\infty} B_n\Omega} = \mathscr{F}(\mathscr{H})$.

LEMMA 6.3. *If $n \in \mathbb{N}$, then Ω is separating for B_n on $\mathscr{K}_n$.*

PROOF. Since Ω is cyclic for B_n on $\mathscr{K}_n$, Ω is separating for B_n'. This gives that the linear map $B_n' \to \mathscr{K}_n,\ b \mapsto b\,\Omega$ is injective, so $\dim B_n' = \dim B_n'\Omega$. Hence $\dim \mathscr{K}_n = \dim A_n\Omega \ge \dim B_n'\Omega = \dim B_n' \ge \dim B_n = \dim \mathscr{K}_n$, where the last inequality follows from that κ is injective. Since $\mathscr{K}_n$ is finite dimensional, this gives that $B_n'\Omega = \mathscr{K}_n$, so Ω is cyclic for B_n' on $\mathscr{K}_n$, hence Ω is separating for $B_n'' = B_n$.

COROLLARY 6.4. *κ is a $*$-isomorphism.*

PROOF. The only thing left to prove is that κ is surjective, but that follows from the inequalities in the previous lemma.

Define the state ω_Ω on A by $\omega_\Omega(x) = \big(x\,\Omega\,,\ \Omega\big)$. Recall that by Proposition 4.1 $A_n \cong M_{2^n}(C)$, so $\omega_\Omega\big|_{A_n}$ can be regarded as a state on the full matrix algebra $M_{2^n}(\mathbb{C})$.

PROPOSITION 6.5. *ω_Ω is a pure state on A. Moreover, if $n \in \mathbb{N}$ and* Tr *is the usual trace on $M_{2^n}(\mathbb{C})$ with* $\operatorname{Tr}(1) = 2^n$, *then $\omega_\Omega\big|_{A_n} = \phi_n$, where*

$$\phi_n : M_{2^n}(\mathbb{C}) \to \mathbb{C},\ \ x \mapsto \operatorname{Tr}\Bigg(\bigotimes_{1}^{n}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} x\Bigg).$$

PROOF. It is enough to show that if $n \in \mathbb{N}$, then $\omega_\Omega\big|_{A_n}$ is a pure state on A_n, because $A = \overline{\bigcup_{n=1}^{\infty} A_n}$. By the commutation-relations we have

$$A_n = \operatorname{lin}\{1, x_{j_s}^*\cdots x_{j_1}^* x_{i_1}\cdots x_{i_r} : 1 \le i_1 < i_2 < \cdots < i_r \le n \quad \text{and} \\ 1 \le j_1 < j_2 < \cdots < j_s \le n \quad (\text{either } r \text{ or } s \text{ can be } 0)\}.$$

We evaluate ω_Ω on the basisvectors:

$$\begin{aligned}\omega_\Omega(x_{j_s}^*\cdots x_{j_1}^* x_{i_1}\cdots x_{i_r}) &= \big(x_{i_1}\cdots x_{i_r}\Omega\,,\ x_{j_1}\cdots x_{j_s}\Omega\big)\\ &= \big(\xi_{i_1}\wedge\cdots\wedge\xi_{i_r}\,,\ \xi_{j_1}\wedge\cdots\wedge\xi_{j_s}\big)\\ &= \begin{cases}1 & \text{if } i_1 = j_1, i_2 = j_2, \ldots, i_r = j_s,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Let ϕ_n be defined as above. Since $\bigotimes_1^n \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a 1-dimensional projection, ϕ_n is a pure state on $M_{2^n}(\mathbb{C})$. We will show that if $n \in \mathbb{N}$, then $\omega_\Omega\big|_{A_n} = \phi_n$, and to do so we inductively use the systems of matrix units from the proof of Proposition 4.1:

$$\phi_1(x_1^* x_1) = \operatorname{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e_{11}^{(1)}\right) = 1, \quad \phi_1(x_1^*) = \operatorname{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e_{12}^{(1)}\right) = 0$$

$$\text{and} \quad \phi_1(x_1) = \operatorname{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e_{21}^{(1)}\right) = 0,$$

so $\omega_\Omega\big|_{A_1} = \phi_1$. Assume that $\omega_\Omega\big|_{A_{n-1}} = \phi_{n-1}$ for an $n \geq 2$. We evaluate ϕ_n on the basisvector $x = x_{j_s}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_r}$ where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_s \leq n$ (either r or s can be 0). Assume first that $i_r, j_s \neq n$. Since $x \in A_{n-1} \cong M_{2^{n-1}}(\mathbb{C})$, then x is identified with $x_{j_s}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_r} \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in M_{2^n}(\mathbb{C})$, so

$$\begin{aligned}
\phi_n(x) &= \operatorname{Tr}\left(\bigotimes_1^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_{j_s}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_r} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \operatorname{Tr}\left(\left(\bigotimes_1^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_{j_s}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_r}\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \operatorname{Tr}\left(\bigotimes_1^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_{j_s}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_r}\right) \operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \phi_{n-1}(x).
\end{aligned}$$

If $i_r = j_s = n$, observe that $\bigotimes_1^n \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \prod_{i=1}^n x_i^* x_i$, so $x_n \bigotimes_1^n \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) x_n^* = \left(\prod_{i=1}^{n-1} x_i^* x_i\right) x_n x_n^* x_n x_n^* = \left(\prod_{i=1}^{n-1} x_i^* x_i\right) x_n x_n^* = \bigotimes_1^{n-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so

$$\begin{aligned}
\phi_n(x) &= \operatorname{Tr}\left(\bigotimes_1^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_n^* \cdots x_{j_1}^* x_{i_i} \cdots x_n\right) \\
&= \operatorname{Tr}\left(\left(\bigotimes_1^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) x_{j_{s-1}}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_{r-1}}\right) \\
&= \phi_{n-1}(x_{j_{s-1}}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_{r-1}}).
\end{aligned}$$

Assume at last that $j_s = n, i_r \neq n$ (the case $j_s \neq n, i_r = n$ is similar).

Observe that $x_n^* = e_{12}^{(n)} u_n = \left(\bigotimes_1^{n-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right) u_n$, so

$$\begin{aligned}
\phi_n(x) &= \operatorname{Tr}\left(\bigotimes_1^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_n^* x_{j_{s-1}}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_r}\right) \\
&= \operatorname{Tr}\left(\bigotimes_1^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(\bigotimes_1^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) u_n x_{j_{s-1}}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_r}\right) \\
&= \operatorname{Tr}\left(\bigotimes_1^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_n x_{j_{s-1}}^* \cdots x_{j_1}^* x_{i_i} \cdots x_{i_r}\right) \operatorname{Tr}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0
\end{aligned}$$

From all this we get that $\phi_n(x_{j_s}^* \cdots x_{j_1}^* x_{i_1} \cdots x_{i_r}) = \omega_\Omega(x_{j_s}^* \cdots x_{j_1}^* x_{i_1} \cdots x_{i_r})$, so the result follows by induction.

COROLLARY 6.6. *A acts irreducibly on $\mathscr{F}(\mathscr{H})$.*

PROOF. Since $\Omega \in \mathscr{F}(\mathscr{H})$ is a cyclic unit vector which is such that $\omega_\Omega(x) = (x\,\Omega\,,\ \Omega)$, the triple $(\mathrm{id}_A, \mathscr{F}(\mathscr{H}), \Omega)$ satisfies the conditions in the GNS-construction. Since ω_Ω is a pure state on A, id_A is an irreducible representation, and the corollary follows.

7. The center of B_n

We start this section with some definitions and explain the notation that will be used.

A *word* in B_n is an element of the form $w = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$, where the vector $(k_1, k_2, \ldots, k_n) \in \mathsf{F}_2^n$. If at least one of the k_i's is 1, the word w is called *non-trivial*, and if all the k_i's are 0, we define w to be 1 (the *trivial* word). If w occures (e.g. in a computation) where the sign is not important, i.e. we can replace w by $-w$ without affecting the result, we do so if it is convenient. If $C \subset B_n$ is a family of words, then C is called an *independent* family (of words) if none of the words in C can be written as a product (up to a sign) of the other words in C. Note that if C is an independent family, then $1 \in C$ implies $C = \{1\}$. If $C = \{w_i\}_{i=1}^m \neq \{1\}$ and $w_i = s_1^{k_1^{(i)}} \cdots s_n^{k_n^{(i)}}$, then C is an independent family if and only if $\{(k_1^{(i)}, \ldots, k_n^{(i)})\}_{i=1}^m$ is a set of linearly independent vectors in F_2^n.

If $k = (k_1, \ldots, k_n)$ is a vector in F_2^n, then let $\overline{k}$ denote the reversed vector $(k_n, k_{n-1}, \ldots, k_1) \in \mathsf{F}_2^n$.

A bitstream $\mathbf{S} \subset \mathsf{N}$ is called *mirror-periodic* if the sequence $\left(\chi_{\mathbf{S}}(|n|)\right)_{n\in\mathsf{Z}}$ is periodic. Observe that $\mathbf{S}$ is mirror-periodic if and only if there exists $m \in \mathsf{N}$ such that $\chi_{\mathbf{S}}(j) = \chi_{\mathbf{S}}(|m - j|)\ \forall j \in \mathsf{Z}^+$.

LEMMA 7.1. *If $n \in \mathbb{N}$, then $\omega_\Omega\big|_{B_n} = \mathrm{tr}\big|_{B_n}$, where* tr *is the normalized trace on $M_{2^n}(\mathbb{C})$. Morover, if w is a non-trivial word in B_n, then $\omega_\Omega(w) = 0$.*

PROOF. Since B_n is generated by the words it contains (see section 5), it is enough to check the equality for these elements. It is clear that $\omega_\Omega(1) = \mathrm{tr}\,1$, so let $w = s_1^{k_1} \cdots s_n^{k_n}$ be a non-trivial word in B_n. Let j be such that $k_j = 1$ and $k_i = 0$ for $i > j$. Since $w = s_1^{k_1} \cdots s_{j-1}^{k_{j-1}} s_j = b_1 \otimes b_2 \otimes \cdots \otimes b_{j-1} \otimes b \otimes a \otimes \cdots \otimes a$, where $b_1, b_2, \ldots, b_{j-1} \in M_2(\mathbb{C})$, it follows that

$$\mathrm{tr}\, w = \mathrm{tr}\, b_1\, \mathrm{tr}\, b_2 \cdots \mathrm{tr}\, b_{j-1}\, \mathrm{tr}\, b\, \mathrm{tr}\, a \cdots \mathrm{tr}\, a = 0.$$

From Proposition 6.5 we get that

$$\omega_\Omega(w) = \mathrm{Tr}\left(\bigotimes_1^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} b_1 \otimes b_2 \otimes \cdots \otimes b_{j-1} \otimes b \otimes a \otimes \cdots \otimes a \right) = 0.$$

The next lemma can be found in [8]. We include the proof for completeness.

LEMMA 7.2. *Denote by $Z(B_n)$ the center of B_n. Every element in $Z(B_n)$ can be written as a linear combination of words in $Z(B_n)$.*

PROOF. Let $v \in Z(B_n)$. We may assume that $v \neq 0$, so there exists a family of words $\{w_i\}_{i=1}^m$ such that $v = \sum_i \lambda_i w_i$, where each $\lambda_i \neq 0$. Assume there exists a j such that $w_j \notin Z(B_n)$, i.e. there exists a word $w \in B_n$ such that $w w_j = -w_j w$. Then $v = w^* v w = \sum_i \lambda_i w^* w_i w = \sum_i \pm\lambda_i w_i$ with minus for $i = j$, so $\lambda_j = -\lambda_j$, hence $\lambda_j = 0$.

LEMMA 7.3. *Let $D_n(\mathbb{C})$ denote the subalgebra of $M_n(\mathbb{C})$ consisting of the diagonal matrices. If $n \in \mathbb{N}$, then there exist $c_n, d_n \in \mathbb{Z}^+$ such that*

$$B_n \cong M_{2^{d_n}}(\mathbb{C}) \otimes D_{2^{c_n}}(\mathbb{C}).$$

PROOF. If $Z(B_n) = \mathbb{C}1$, then B_n is isomorphic to a full matrix algebra, so in this case n must be even since $\dim B_n = 2^n$. Hence $c_n = 0$ and $d_n = \frac{n}{2}$. If $Z(B_n) \neq \mathbb{C}1$, then Lemma 7.2 gives that there exists an independent family of words, $C = \{w_j\}_{j=1}^m$, generating $Z(B_n)$. For $j = 1, \ldots, m$ define

$$q_j^+ = \frac{1}{2}(1 + \lambda_j w_j) \qquad \text{and} \qquad q_j^- = \frac{1}{2}(1 - \lambda_j w_j),$$

where $\lambda_j \in \mathbb{C}$ equals 1 if w_j is self-adjoint and i otherwise (if w_j is not self-adjoint, then $w_j^* = -w_j$). This gives rise to 2^m orthogonal central-projections of the form $q_1 q_2 \cdots q_m$, where each q_i is either q_j^+ or q_j^-, so let $\{p_i\}_{i=1}^{2^m}$ be the family consisting of these projections. Since $Z(B_n)$ is a 2^m-dimensional commutative algebra, it follows that the p_i's are the minimal projections in $Z(B_n)$. Hence, to each p_i there exists a corresponding $n_i \in \mathbb{N}$ such that

$$B_n = \bigoplus_{i=1}^{2^m} M_{n_i}(\mathbb{C}) \otimes \mathbb{C}_{\mu_i} \subset M_{2^n}(\mathbb{C}),$$

where $\mu_i \in \mathbb{N}$ is the multiplicity of the representation of $p_i B_n$ ($\cong M_{n_i}(\mathbb{C})$) in $\mathscr{B}(p_i \mathbb{C}^{2^n})$ and $\mathbb{C}_{\mu_i} = \mathbb{C}1 \subset M_{\mu_i}(\mathbb{C})$. The commutant of $\bigoplus_{i=1}^{2^m} M_{n_i}(\mathbb{C}) \otimes \mathbb{C}_{\mu_i}$ is $\bigoplus_{i=1}^{2^m} \mathbb{C}_{n_i} \otimes M_{\mu_i}(\mathbb{C})$, so the fact that B_n is isomorphic to its commutant (Corollary 6.4) gives that $\bigotimes_{i=1}^{2^m} \mathbb{C}_{n_i} \otimes M_{\mu_i}(\mathbb{C})$ has dimension 2^n, hence $\sum_{i=1}^{2^m} \mu_i^2 = 2^n$. Since B_n contains $1 \in M_{2^n}(\mathbb{C})$, we get that $\sum_{i=1}^{2^m} \mu_i n_i = 2^n$, and by calculating the dimension of B_n we also get that $\sum_{i=1}^{2^m} n_i^2 = 2^n$. Morover, if p_i is a minimal projection in $Z(B_n)$, then $p_i = q_1 \cdots q_m = \frac{1}{2}(1 \pm \lambda_1 w_1) \ldots \frac{1}{2}(1 \pm \lambda_m w_m)$, which equals $\frac{1}{2^m}1$ plus a linear combination of products of the form $\prod_{i \in I} w_i$, where $I \subset \{1, \ldots, m\}$. Since C is an independent family, Lemma 7.1 implies that $\operatorname{tr} \prod_{i \in I} w_i = 0$ for each $I \subset \{1, \ldots, m\}$, so $\operatorname{tr} p_i = \frac{1}{2^m}$. This yields that the product $\mu_i n_i$ is independent of i, so $\sum_{i=1}^{2^m} \mu_i n_i = 2^m \mu_i n_i = 2^n$. Thus $2^n = \sum_{i=1}^{2^m} \mu_i^2 = \sum_{i=1}^{2^m} \frac{2^{2n-2m}}{n_i^2}$, so $\sum_{i=1}^{2^m} \frac{1}{n_i^2} = 2^{2m-n}$. Now the Cauchy-Schwarz inequality gives us that

$$2^{2m} = \left(\sum_{i=1}^{2^m} n_i \frac{1}{n_i} \right)^2 \leq \left(\sum_{i=1}^{2^m} n_i^2 \right) \left(\sum_{i=1}^{2^m} \frac{1}{n_i^2} \right) = 2^n \cdot 2^{2m-n} = 2^{2m}.$$

Equality in the Cauchy-Schwarz inequality implies that there exists a constant c such that $(n_1, \ldots, n_{2^m}) = c(\frac{1}{n_1}, \ldots, \frac{1}{n_{2^m}})$, so all the n_i's are equal. It follows that $c_n = m$ and $d_n = \frac{n-m}{2}$, which concludes the proof.

Remark 1. It follows from the proof of this lemma that if $n \in \mathbb{N}$, then there exists an independent family $C = \{w_i\}_{i=1}^{c_n}$ when $c_n \neq 0$ and $C = \{1\}$ when $c_n = 0$ which generates $Z(B_n)$, and it is clear that any independent family in $Z(B_n)$ consisting of c_n words, generates $Z(B_n)$.

Remark 2. If $J : x\,\Omega \mapsto x^*\,\Omega$, then $J\,p_i\,J = p_i$ and $J\,B_n\,J = B_n'$, see [2, Ch. I §5]. From this it follows that $\mu_i = n_i$ above.

Remark 3. For a shorter and quite different proof of this result (which also includes Lemma 7.5) see [3].

Corollary 7.4. *If $n \in \mathbb{N}$, then $n = 2d_n + c_n$.*

Proof. $2^n = \dim B_n = \dim(M_{2^{d_n}}(\mathbb{C}) \otimes D_{2^{c_n}}(\mathbb{C})) = (2^{d_n})^2 \cdot 2^{c_n}$.

In what follows we will describe the behaviour of the sequence $(c_n)_{n=1}^{\infty}$ in order to understand the AF-algebra $B = \overline{\bigcup_{n=1}^{\infty} B_n}$.

LEMMA 7.5. *If $n \in \mathbb{N}$, then $c_{n+1} = c_n \pm 1$. Morover $c_{n+1} = c_n + 1 \iff d_{n+1} = d_n$.*

PROOF. Let $n \in \mathbb{N}$. It follows from Lemma 7.3 that $d_{n+1} \geq d_n$, so assume first that $d_{n+1} = d_n$. Then Corollary 7.4 gives that $2d_{n+1} + c_{n+1} = 2d_n + c_n + 1$, so $c_{n+1} = c_n + 1$ (this calculation also gives the reverse implication). If $d_{n+1} > d_n$, then the same corollary gives that $2 \leq 2(d_{n+1} - d_n) = n+1-c_{n+1}-(n-c_n)$, so $c_{n+1} \leq c_n - 1$. From this it follows that $c_n \geq 1$, so there exists an independent family of words, $C = \{w_i\}_{i=1}^{c_n}$, that generates $Z(B_n)$. Since $c_{n+1} \leq c_n - 1$, there exists $j \in \{1, \dots, c_n\}$ such that $w_j \notin Z(B_{n+1})$. If $w_i \in C \setminus Z(B_{n+1})$ $(i \neq j)$, this means that w_i anticommutes with s_{n+1}, and since the same is true for w_j it follows that $w_j w_i \in Z(B_{n+1})$. Hence, by replacing the words $w_i \in C \setminus Z(B_{n+1})$ $(i \neq j)$ with $w_j w_i$, we get that $c_{n+1} \geq c_n - 1$. (It is clear that the family obtained by this replacement is independent.)

LEMMA 7.6. *$c_{n+1} = c_n + 1 \iff Z(B_n) \subset Z(B_{n+1})$.*

PROOF. Assume that $c_{n+1} = c_n + 1$. If $c_n = 0$, the implication is trivial, so we may assume that $c_n \geq 1$. Let $C = \{w_i\}_{i=1}^{c_n}$ be an independent family of words generating $Z(B_n)$, and assume that $Z(B_n) \not\subset Z(B_{n+1})$. The same argument as in the proof of Lemma 7.5 gives that $w_{c_n} \notin Z(B_{n+1})$ and $\{w_i\}_{i=1}^{c_n-1} \subset Z(B_{n+1})$ (by modifying and rearranging some of the w_i's if necessary). Now, since $c_{n+1} = c_n + 1$, there must exist two words $w', w'' \in Z(B_{n+1})$ such that $\{w', w''\} \cup \{w_i\}_{i=1}^{c_n-1}$ is an independent family of words generating $Z(B_{n+1})$. This, however, is impossible, since then at least one of the words $w', w'', w'w''$ does not contain s_{n+1}, i.e. is contained in $Z(B_n)$. This violates the fact that C generates $Z(B_n)$. The reverse implication is immediate from Lemma 7.5.

LEMMA 7.7. *If $s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$ is a word in B_n, then $s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n} \in Z(B_n)$ if and only if $s_1^{k_n} s_2^{k_{n-1}} \cdots s_n^{k_1} \in Z(B_n)$.*

PROOF. By symmetry it is enough to show one of the implications. Let $w = s_1^{k_1} \cdots s_n^{k_n}$ be a word in $Z(B_n)$ and $1 \leq i \leq n$. Observe that

$$s_i s_1^{k_n} s_2^{k_{n-1}} \cdots s_n^{k_1} = \left(\prod_{j=1}^{n} (-1)^{\chi_S(|i-j|) k_{n+1-j}} \right) s_1^{k_n} s_2^{k_{n-1}} \cdots s_n^{k_1} s_i,$$

so we must show that $\prod_{j=1}^{n}(-1)^{\chi_{\mathbf{S}}(|i-j|)k_{n+1-j}} = 1$. Since $w \in Z(B_n)$, then

$$ws_{n+1-i} = s_{n+1-i}w = \left(\prod_{j=1}^{n}(-1)^{\chi_{\mathbf{S}}(|n+1-i-j|)k_j}\right)ws_{n+1-i},$$

hence $\prod_{j=1}^{n}(-1)^{\chi_{\mathbf{S}}(|n+1-i-j|)k_j} = 1$. Now the lemma follows by the substitution $k = n+1-j$.

The next proposition describes the sequence $(c_n)_{n=1}^{\infty}$ completely. To make the notation easier we define $B_0 = \mathbb{C}1$, so $c_0 = 0$. Since $c_1 = 1$, Lemma 7.5 is also valid for $n = 0$.

Proposition 7.8. *Let $\mathbf{S} \subset \mathbb{N}$ be a given bitstream. Then there exists a strictly increasing sequence $(n_r)_{r\in I}$ of even integers, where $I = \{1, 2, \ldots, N\}$ ($N \in \mathbb{N}$ and set $n_{N+1} = \infty$) or $I = \mathbb{N}$, such that $n_1 = 0$ and if $m = \frac{n_{r+1}-n_r}{2}$, then*

$$c_{n_r+j} = \begin{cases} j & \text{if } 0 \le j < m, \\ 2m - j & \text{if } m \le j \le 2m. \end{cases}$$

Furthermore, $\mathbf{S}$ is mirror-periodic if and only if I is finite.

Proof. It is enough to show that if $n \in \mathbb{Z}^+$ is such that $c_n = 0$, then c_{n+j} will behave like in the proposition for all $j \in \mathbb{N}$ until $c_{n+j} = 0$. The result then follows by induction. Let $n \in \mathbb{N}$ be such that $c_n = 0$. By Lemma 7.5 there exists $k = (k_1, \ldots, k_{n+1}) \in \mathbb{F}_2^{n+1}$ such that the word $w_1 = s_1^{k_1}\cdots s_{n+1}^{k_{n+1}}$ generates $Z(B_{n+1})$. Lemma 7.7 implies that $k = \overline{k}$, and if $k_1 = k_{n+1} = 0$, then $w_1 \in Z(B_n)$, so $k_1 = 1$. By Lemma 7.5 again, there exists $m \in \mathbb{N} \cup \{\infty\}$ such that $c_{n+j} = j$ for $1 \le j \le m$ and $c_{n+m+1} = m - 1$. For $1 \le i \le m$ let

$$w_i = s_i^{k_1}\cdots s_{n+i}^{k_{n+1}}.$$

We claim that $\{w_i\}_{i=1}^{j}$ generates $Z(B_{n+j})$ for $1 \le j \le m$, so let $1 \le r < m$ be such that $\{w_i\}_{i=1}^{r}$ generates $Z(B_{n+r})$. Since $c_{n+r+1} = r+1$, Lemma 7.6 gives that $Z(B_{n+r}) \subset Z(B_{n+r+1})$, so $w_1 = s_1^{k_1}\cdots s_{n+1}^{k_{n+1}}s_{n+2}^{0}\cdots s_{n+r+1}^{0} \in Z(B_{n+r+1})$. Lemma 7.7 then imply that $s_{r+1}^{k_{n+1}}\cdots s_{n+r+1}^{k_1} \in Z(B_{n+r+1})$, hence, since $k = \overline{k}$, $w_{r+1} \in Z(B_{n+r+1})$. Because $\{w_i\}_{i=1}^{r+1}$ is an independent family of $r+1$ words in $Z(B_{n+r+1})$, it generates $Z(B_{n+r+1})$, and the claim follows by induction. If $m = 1$ (which implies $c_{n+m+1} = 0$) or $m = \infty$, we are done, so assume $1 < m < \infty$. Since $[w_i, s_{n+m+1}] = [w_{i-1}, s_{n+m}] = 0$ for $2 \le i \le m$, it follows that $\{w_i\}_{i=2}^{m}$ generates $Z(B_{n+m+1})$. Hence $w_1 \notin Z(B_{n+m+1})$, i.e. w_1 anticommutes with s_{n+m+1}, so if $j \in \{1, \ldots, m\}$, then w_j anticommutes with s_{n+m+j}. From this we get that $w_1, \ldots, w_j \notin Z(B_{n+m+j})$.

Because $[w_i, s_{n+m+j}] = [w_{i-j}, s_{n+m}] = 0$ for $j < i \leq m$, it follows that $\{w_i\}_{j+1}^m \subset Z(B_{n+m+j})$ (by induction, where $\{w_i\}_{m+1}^m = \mathbb{C}1$). Lemma 7.6 and 7.5 now implies that $\{w_i\}_{j+1}^m$ generates $Z(B_{n+m+j})$.

Assume **S** is mirror-periodic with period p. If $j \in \mathbb{N}$, then

$$\begin{aligned} s_1 s_{p+1} s_j &= (-1)^{\chi_S(|p+1-j|)}(-1)^{\chi_S(|1-j|)} s_j s_1 s_{p+1} \\ &= (-1)^{\chi_S(|1-j|)}(-1)^{\chi_S(|1-j|)} s_j s_1 s_{p+1} \\ &= s_j s_1 s_{p+1}, \end{aligned}$$

so $s_1 s_{p+1} \in Z(B_n)$ for $n \geq p+1$, hence I is finite.

Assume I is finite. By the first part of the proposition there exists a word $w_1 = s_1^{k_1} \cdots s_n^{k_n}$, where n is odd, $k = (k_1, \ldots, k_n)$ satisfies $k = \overline{k}$, and $k_1 = 1$, such that $w_1 \in Z(B_{n+j})$ for all $j \in \mathbb{Z}^+$. If $n = 1$, $w_1 = s_1$, so in this case **S** consists of only 0's. Since this **S** is mirror-periodic we may assume that $n > 1$. For $j \in \mathbb{Z}$ define $l_j = \big(\chi_S(|j|), \chi_S(|j+1|), \ldots, \chi_S(|j+n-1|)\big) \in \mathbb{F}_2^n$ and $l_j' \in \mathbb{F}_2^{n-1}$ as the vector obtained from l_j by deleting its last entry. Let $A \in M_{n-1}(\mathbb{F}_2)$ be given by

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \\ k_1 & k_2 & k_3 & \cdots & k_{n-1} \end{pmatrix},$$

and let $j \geq 1-n$. By calculating the sign in $s_{j+n} w_1 = \pm w_1 s_{j+n}$ and using that $w_1 \in Z(B_{n+j})$ for all $j \in \mathbb{Z}^+$, we get that $\sum_{i=1}^n \chi_S(|j+n-i|)k_i = 0$. Since $k = \overline{k}$, this yields that $0 = \sum_{i=1}^n \chi_S(|j-1+n+1-i|)k_{n+1-i} = \sum_{i=1}^n \chi_S(|j-1+i|)k_i = l_j \cdot k$ for all $j \geq 1-n$ (where the middle equality follows by substituting $n+1-i$). If $j < 1-n$, then $l_j \cdot k = \overline{l_j} \cdot \overline{k} = l_{-j-(n-1)} \cdot \overline{k} = l_{1-n-j} \cdot k = 0$, because $k = \overline{k}$, so $l_j \cdot k = 0$ for all $j \in \mathbb{Z}$. Since $k_n = k_1 = 1$ this implies that $l_{j+1}' = A l_j'$ for all $j \in \mathbb{Z}$, and since A is invertible over $\mathbb{F}_2$, there exists $m \in \mathbb{N}$ such that $A^m = 1$. From this it follows that $l_{j+m}' = l_j'$ for all $j \in \mathbb{Z}$, so **S** is mirror-periodic.

Remark 1. The calculation used to show that **S** is mirror-periodic implies that I is finite can also be found in [4], and the proof of the reverse implication is due to [10].

Remark 2. If $B_n = C^*(s_i : \frac{1-n}{2} \leq i \leq \frac{n}{2})$, the proof of Proposition 7.8 is the same, but we must exchange $\{w_i\}_{i=1}^j$ with $\{s_{-\frac{n}{2}+i}^{k_1} \cdots s_{\frac{n}{2}+i}^{k_{n+1}}\}_{\frac{1-j}{2} \leq i \leq \frac{j}{2}}$ and so on.

REMARK 3. We see from the proof of Proposition 7.8 that for a given sequence $(n_r)_{r\in I} \subset \mathbb{N}$, where $I = \{1, 2, \dots, N\}$ or $I = \mathbb{N}$ which satisfies $n_1 = 0$, n_r even, and $n_r < n_{r+1}$, there exists a bitstream $\mathbf{S} \subset \mathbb{N}$ giving rise to this sequence. This is Theorem 6.6. in [9].

THEOREM 7.9. *Let $\mathbf{S} \subset \mathbb{N}$ be a bitstream. Then there exists a family of self-adjoint, unitary operators, $\{s_j\}_{j\in\mathbb{Z}}$, such that $s_i s_j = (-1)^{\chi_{\mathbf{S}}(|i-j|)} s_j s_i$, and if $B = C^*(s_i : i \in \mathbb{Z})$, then*

$$B \cong \begin{cases} M_{2^n}(\mathbb{C}) \otimes \bigotimes_1^\infty D_2(\mathbb{C}) & \text{if } \mathbf{S} \text{ is mirror-periodic,} \\ \bigotimes_1^\infty M_2(\mathbb{C}) & \text{if } \mathbf{S} \text{ is not mirror-periodic.} \end{cases}$$

PROOF. Let $I \subset \mathbb{N}$ be the index set given by Proposition 7.8. If $I = \{1, \dots, N\}$, there exists $n \in \mathbb{N}$ such that $c_{n+j} = j$ for all $j \in \mathbb{Z}^+$, so $d_{n+j} = d_n$ for all $j \in \mathbb{Z}^+$ by Lemma 7.5. Lemma 7.3 now implies that $B \cong M_{2^{d_n}}(\mathbb{C}) \otimes \bigotimes_1^\infty D_2(\mathbb{C})$. If $I = \mathbb{N}$, Lemma 7.3 and Proposition 7.8 implies that $B_n \cong M_{2^{d_n}}(\mathbb{C})$ for infinitly many n's, so $B \cong \bigotimes_1^\infty M_2(\mathbb{C})$.

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